UKACC International Conference on Control 2012
Cardiff, UK, 3-5 September 2012

A Relaxation-based Approach for the Orthogonal Procrustes Problem with Data Uncertainties

Shakil Ahmed
Department of Electrical
and Electronic Engineering,
Imperial College London, UK
shakil.ahmed08@imperial.ac.uk

Imad M. Jaimoukha
Department of Electrical
and Electronic Engineering,
Imperial College London, UK
i.jaimouka@imperial.ac.uk

***Abstract*—The orthogonal Procrustes problem (OPP) deals with matrix approximations. The solution of this problem gives an orthogonal matrix to best transform one data matrix to another, in a Frobenius norm sense. In this work, we use semidefinite relaxation (SDR) to find the solutions of different OPP formulations. For the standard problem formulation, this approach yields an exact solution, i.e. no relaxation gap. We also address uncertainties in the data matrices and formulate a min-max robust problem. The robust problem, being non-convex, turns out to be a difficult optimization problem; however, it is relatively straight forward to approximate it into a convex optimization problem using SDR. Our preliminary results on robust problem show that the solution of the relaxed uncertain problem does not guarantee zero relaxation gap, and as a result, we cannot always find a solution, which satisfies the orthogonality constraint. In such cases we use orthogonalization, which gives the nearest orthogonal matrix from the SDR based solution. All these relaxed formulations, can be easily converted into a semidefinite program (SDP), for which polynomial time efficient algorithms exists. For the nominal problems, the presented approach may not be computationally efficient than other existing methods. In this work, our main contribution is to demonstrate that the SDR approach provides a unified framework to solve not only the standard OPP but can also solve the problems with uncertainties in the data matrices, which other existing approaches cannot handle.**

I. INTRODUCTION

Orthogonal Procrustes problem (OPP) is a well known mathematical problem [1], [2]. It deals with finding a geometrical transformation that involves rotations or reflections with orthogonality constraint. In simple words, given two arbitrary real matrices A and B of the same dimension, OPP finds an orthogonal matrix X, which can best transform one matrix to the other, such that the Frobenius norm of the error $AX-B$ is minimized. There are many formulations of this problem, which can address rotations, reflections and translations having applications in various areas, such as image processing, computer vision, statistics, satellites and aerospace. In this work, we are mainly interested in the formulations, which can address rotations and reflections.

In image processing and machine learning, OPP is used in pose estimation, which involves estimation of a camera or some object position and orientation, either relative to a model reference frame, or at a previous time using a camera or a range sensor [3]. This application involves both rotation and translation. In satellites and aerospace applications, OPP is used for rigid body attitude determination and involve only rotations. In these applications, information of some vector quantities, such as the earth magnetic field, sun and star direction, object position, is obtained from both a sensor and a mathematical model to estimate the rigid body attitude. In statistics, OPP is used for principal component analysis, which is a mathematical procedure that uses an orthogonal transformation to convert a set of observations of possibly correlated variables into a set of values of linearly uncorrelated variables called principal components. For other variants of the Procrustes problems and their applications, please see [2].

Many solutions of the OPP exist in literature, suiting different applications. One mostly used solution of the orthogonal Procrustes problem is based on the singular values decomposition (SVD) [1], [4]. Different variants of the SVD based solutions are used in many applications. Few customized algorithms were also developed to meet some application specific requirements. For example, different solutions proposed for the Wahba problem [5], which is a subclass of OPP for rotations, and used in satellite attitude determination. Although, some proposed algorithms are based on the SVD approach [6], most of the algorithms used in practical applications are based on customized solutions for high computational speed, such as QUEST [7], ESOQ, ESOQ2 [8], [9].

Semidefinite relaxation (SDR) is considered a powerful and computationally efficient approximation technique for difficult optimization problems [10], such as non-convex and robust problems with quadratic cost and constraints. In this work, we demonstrate that this relaxation approach provides a unified framework to solve different formulations of the Procrustes problem and some of its more sophisticated extensions. We present SDR based convex formulations for the standard OPP and its variant for rotations, which needs to deal with the additional nonlinear constraint of $det(X)=1$. The relaxed formulations, however, result in no gap, giving an exact solution for the nominal problems. We also show that the SDR framework is much more general and can handle many extensions of this problem, which the other existing approaches cannot deal with. In this regard, our main contribution is to use the relaxation approach for the OPP with uncertain data matrices, which is a comparatively less addressed topic because of its mathematical complexity [11], [12]. We formulate and solve a relaxed version of the robust min-max problem. For the relaxed robust problem, the gap is not always zero. For such cases, we present a method to extract the nearest possible solution from the optimal, satisfying the constraints of the original problem.

The paper is organized as follows. In Section II, an orthogonal Procrustes problem is presented along with an existing solution based on the singular value decomposition. The application of the SDR is discussed in Section III, both for the standard problem and its extension for rotations. Section IV discusses the robust formulation and its solution. Section V presents numerical simulations to support the presented results and to show the effectiveness of the robust formulation.

Notation: For a matrix, $A \succ 0$ ($A \succeq 0$) means that A is positive definite (semidefinite). The Frobenius norm of a matrix A is $\|A\|_F = \sqrt{\text{tr}(A^TA)}$. I_n denotes the identity matrix of size n, while $0_{n\times m}$ represents a matrix of n rows and m columns with all zero entries. The set of singular value (eigenvalue) of a A is represented by $\sigma(A)$ ($\lambda(A)$). Operator $\text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ represents a matrix of size $n \times n$, having only diagonal elements $\lambda_1, \lambda_2, \ldots, \lambda_n$. Operator $\text{trace}(A)$ is sum of all the diagonal entries of a matrix A.

II. THE ORTHOGONAL PROCRUSTES PROBLEM

The orthogonal Procrustes problem (OPP) is mathematically defined as

$$\begin{aligned} \min_{X} \quad & \|AX - B\|_F^2 \\ \text{subject to} \quad & XX^T = I, \end{aligned} \tag{1}$$

where $A, B \in \mathbb{R}^{m\times n}$ are given data matrices, $m \geq n$ and $X \in \mathbb{R}^{n\times n}$ is the unknown orthogonal matrix, which belongs to an orthogonal group of order n, i.e. $\mathbb{X} =: \{X : XX^T = I, \det(X) = \pm 1\}$.

One important subclass of the orthogonal Procrustes problem includes an additional nonlinear constraint $\det(X) = +1$ (see, for example, Wahba problem [5]). This problem deals specifically with rotations and has a wide range of applications. In these applications, we are only interested in $X \in \mathbb{R}^{3\times 3}$ i.e. the solution now belongs to $SO(3)$, a special orthogonal group of order 3, defined as $\mathbb{X}_+ =: \{X : X \in \mathbb{R}^{3\times 3}, XX^T = I, \det(X) = +1\}$.

One can find many solutions of this problem in the literature. Most solutions are generally application specific, satisfying some special requirements, such as computational efficiency, numerical stability. We will present here one existing approach, which is considered numerically the most robust, and is based on the singular value decomposition (SVD) [1], [6].

A. An SVD based solution

To find an SVD based solution, first we write the objective function of (1) as

$$\begin{aligned} \|AX - B\|_F^2 &= \text{tr}\left((AX - B)(AX - B)^T\right) \\ &= \text{tr}\left(AXX^TA^T + BB^T - AXB^T - BX^TA^T\right). \end{aligned} \tag{2}$$

To simplify the expression, we use the constraint $XX^T = I$ in (2) and write $AXX^TA^T = AA^T$. Further, neglecting the constant terms, i.e. $AA^T + BB^T$, we can write following maximization problem, which is equivalent to (1) in its argument.

$$\begin{aligned} \max_{X} \quad & \text{tr}(BX^TA^T) \\ \text{subject to} \quad & XX^T = I. \end{aligned} \tag{3}$$

To solve this problem, we use the permutation property of the trace operator [16] and write the cost function as

$$\begin{aligned} \text{tr}(BX^TA^T) &= \text{tr}(X^TA^TB), \\ &= \text{tr}(X^TU\Sigma V^T) \\ &= \text{tr}(V^TX^TU\Sigma) \\ &\leq \text{tr}(\Sigma) = \sum_i \sigma_i. \end{aligned} \tag{4}$$

Here $U\Sigma V^T$ is the singular value decomposition of the term A^TB, where U, V are unitary matrices. The inequality in (4) becomes an equality when $V^TX^TU = I$, i.e. $X = UV^T$, which is the required solution.

For problems with additional constraint $\det(X) = +1$ (rotations), to maximize (4), we use few properties of the determinant operator [16] and write as

$$\begin{aligned} \det(V^TX^TU) &= \det(U^TVX) \\ &= \det(U^TV)\det(X) \\ &= \det(U^TV) = \pm 1. \end{aligned} \tag{5}$$

If $\det(U^TV) = 1$, the maximum is attained for $V^TX^TU = I$, while if $\det(U^TV) = -1$, the maximum is attained for $V^TX^TU = \text{diag}(1, 1, \ldots, 1, -1)$.

A unified solution for both problems can be given as

$$X = U\,\text{diag}(1, 1, \ldots, \det(U^TV))V^T. \tag{6}$$

III. A RELAXATION APPROACH FOR OPP

We addressed the orthogonal Procrustes problem (1) using semi-definite relaxation approach [12], [13]. We will present a relaxed formulation of the standard OPP (1) as well as the OPP for rotations.

A. Relaxation of the standard OPP

To derive a semi-definite relaxation of the standard OPP, we use (2) and simplify the expression using the constraint $XX^T = I$. By introducing a linear objective, we can write the following optimization problem, which is equivalent to (1),

$$\begin{aligned} \min_{X,M} \quad & \text{tr}(M) \\ \text{subject to} \quad & M - AA^T - BB^T + AXB^T + BX^TA^T \succeq 0, \\ & XX^T = I. \end{aligned} \tag{7}$$

As the orthogonality constraint $XX^T = I$ is not convex, we relax it to a convex quadratic inequality $XX^T \preceq I$. Using this relaxation, we write an approximate problem, which has a linear cost and linear matrix inequality constraints, as

$$\begin{aligned} \min_{X,M} \quad & \text{tr}(M) \\ \text{subject to} \quad & \mathscr{M} \succeq 0, \\ & \mathscr{X} \succeq 0, \end{aligned} \tag{8}$$

where $\mathscr{M} = \begin{bmatrix} M+AXB^T+BX^TA^T & A & B \\ A^T & I & 0 \\ B^T & 0 & I \end{bmatrix}$ and $\mathscr{X} = \begin{bmatrix} I & X \\ X^T & I \end{bmatrix}$ are Schur complement [16] for the first and the relaxed second constraint in (7).

We have the following result regarding the gap between the original problem and its relaxation.

Theorem 1: There is no gap between problem (1) and its relaxation (8) and the SDR solution of (8) is the optimal solution of (1).

Proof: The proof is evident from (4). To elaborate this point, assume that the gap between (1) and (8) is zero, i.e. $\mathrm{tr}(M) = AA^T + BB^T - 2BX^TA^T$. Now the minimum value of $\mathrm{tr}(M)$ will be obtained when BX^TA^T is maximum. From (4), we know that the maximum value of BX^TA^T is achieved when $X = UV^T$, where UV^T is the singular value decomposition of A^TB and the obtained X satisfies the orthogonality constraint. Hence, the solution of the SDR is obtained by pushing XX^T towards I, resulting in no gap between (8) and (1). ■

B. Relaxation of the OPP for rotations

The OPP formulation for rotations needs to handle the additional nonlinear constraint $\det(X) = 1$, which cannot be directly handled in the SDR framework. The OPP for rotations is of much interest from practical point of view for many applications. In such applications, we are only interested in the data matrices, where $n = 3$, e.g. applications such as finding camera orientation or rigid body attitude. In such cases, the rows of A and B represent information of same quantities obtained from different sources.

One approach to handle rotation problem in the SDP framework is proposed by [15]. We refer to [17, Proposition 4.1] for more details. We present the main point in a simplified form.

Consider a symmetric matrix $Z \in \mathbb{R}^{4\times 4}$, then an exact SDP representation of the convex hull of $SO(3)$ is given by the following expression, where the orthogonal matrix $X \in SO(3)$ is represented in terms of the elements of the matrix Z as

$$X(Z) = \begin{bmatrix} z_{11}+z_{22}-z_{33}-z_{44} & 2z_{23}-2z_{14} & 2z_{24}+2z_{13} \\ 2z_{23}+2z_{14} & z_{11}-z_{22}+z_{33}-z_{44} & 2z_{34}-2z_{12} \\ 2z_{24}-2z_{13} & 2z_{34}+2z_{12} & z_{11}-z_{22}-z_{33}+z_{44} \end{bmatrix}, \tag{9}$$

if and only if the matrix Z satisfies the following constraints

$$\begin{aligned} Z &\succeq 0, \\ \mathrm{trace}(Z) &= 1. \end{aligned} \tag{10}$$

In the above expression $z_{ij} = Z(i,j)$. To understand this fact, consider that positive semidefinite 4×4 matrices with both trace and rank 1 are given as

$$Z = \frac{1}{a^2+b^2+c^2+d^2} \begin{bmatrix} a^2 & ab & ac & ad \\ ab & b^2 & bc & bd \\ ac & bc & c^2 & cd \\ ad & bd & cd & d^2 \end{bmatrix}. \tag{11}$$

The image of this rank 1 matrix under the linear map (9) is precisely the group $SO(3)$ [17]. This parameterization is known as the *Cayley transform*.

Now embedding the trace constraint, i.e. $z_{11}+z_{22}+z_{33}+z_{44} = 1$ within the definition of Z, we can write an exact SDP representation for the rotation problem as

$$\begin{aligned} \min_{Z,M} \quad & \mathrm{tr}(M) \\ \text{subject to} \quad & \mathscr{M} \succeq 0, \\ & Z \succeq 0, \end{aligned} \tag{12}$$

where

$$\mathscr{M} = \begin{bmatrix} M+AX(Z)B^T+BX(Z)^TA^T & A & B \\ A^T & I & 0 \\ B^T & 0 & I \end{bmatrix}. \tag{13}$$

The new $X(Z)$ with embedded trace constraint is given as

$$X(Z) = \begin{bmatrix} 2z_{11}+2z_{22}-1 & 2z_{23}-2z_{14} & 2z_{24}+2z_{13} \\ 2z_{23}+2z_{14} & 2z_{11}+2z_{33}-1 & 2z_{34}-2z_{12} \\ 2z_{24}-2z_{13} & 2z_{34}+2z_{12} & -2z_{22}-2z_{33}+1 \end{bmatrix}. \tag{14}$$

The solution of this problem gives Z satisfying the constraints given in (10). This Z is then used to calculate X, which is the optimal solution of the rotation problem. Later on, we will make use of this transformation to solve the rotation problem with data uncertainties.

IV. THE OPP WITH DATA UNCERTAINTIES

Depending upon application, the data matrices A and B are generally obtained from different sources, e.g. some camera, sensors, mathematical models. This input information has always some sort of uncertainty, such as noise, sensor and modelling errors. The level of uncertainty, however depends upon the quality of measurement or mathematical modelling. In some applications the level of uncertainty can be high. It is a well known fact that these uncertainties can severely affect the accuracy of the obtained solution. The error in the solution will be large in the worst case uncertainties. To overcome the issue of sensitivity of the solution to data uncertainties, in this section, we will formulate and solve a robust optimization problem. In the formulated robust min-max problem, we obtain a solution, which could minimize the chosen cost function under the worst case uncertainties. The solution of the robust OPP gives better estimate of the unknown matrix X, with large data uncertainties.

A. Uncertainty Representation in the Data Matrices

We consider the following data uncertainty structure:

$$\begin{bmatrix} \tilde{A} & \tilde{B} \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} + E\Delta \begin{bmatrix} F_1 & F_2 \end{bmatrix}, \tag{15}$$

where $\tilde{A}$ and $\tilde{B}$ are uncertain data matrices, A and B represent the nominal data, E, F_1 and F_2 are known matrices and Δ is the uncertainty matrix such that $\Delta\Delta^T \preceq I$.

Perturbation model of this form is common in robust estimation, filtering and control [18], [19]. By a suitable selection of E, F_1, F_2 and Δ, this model can represent both structured and unstructured uncertainty. For example, a norm-bounded full Δ will represent unstructured uncertainty, while a norm-bounded diagonal Δ with a suitable choice of other matrices will represent structured uncertainty. The suitable choice of E, F_1 and F_2 specify both the components of A and B affected by the uncertainty Δ and also the amount of uncertainty, e.g. as a percentage of input data.

In this work, we use this general uncertainty structure to formulate the robust problem. The choice of the constant matrices will define it to be structured or unstructured. Further, we will consider a ball uncertainty, i.e. $\Delta\Delta^T \preceq I$.

B. The Robust Problem

For the robust problem formulation, we will follow the min-max approach, i.e. to minimize the objective function under the worst case uncertainties. Using the uncertainty model (15), we define the following robust problem:

$$\begin{aligned} \min_{X} \max_{\Delta} \quad & \|\tilde{A}X - \tilde{B}\|_F^2 \\ \text{subject to} \quad & XX^T = I, \\ & \Delta\Delta^T \preceq I, \end{aligned} \tag{16}$$

where $\tilde{A} = A + E\Delta F_1$ and $\tilde{B} = B + E\Delta F_2$.

To solve the min-max problem, we use SDR approach, following the same steps as used while solving the nominal problem (8). The main challenge here is to handle uncertainty in the max problem. By expanding the cost of (16), we get

$$\begin{aligned} J(X,\Delta) =& \|\tilde{A}X - \tilde{B}\|_F^2 \\ =& \operatorname{tr}((A+E\Delta F_1)XX^T(A+E\Delta F_1)^T - \\ & (A+E\Delta F_1)X(B+E\Delta F_2)^T - \\ & (B+E\Delta F_2)X^T(A+E\Delta F_1)^T + \\ & (B+E\Delta F_2)(B+E\Delta F_2)^T). \end{aligned} \tag{17}$$

The constraint $XX^T = I$ is used to further simplify the cost. To transform this problem into a tractable LMI formulation, we first replace the cost $J(X,\Delta)$ in (17) with a linear objective function trace(M), and write the following equivalent problem

$$\begin{aligned} \min_{X,M,\Delta} \quad & \operatorname{tr}(M) \\ \text{subject to} \quad & M - J(X,\Delta) \succeq 0 \ \forall\, \Delta : \Delta\Delta^T \preceq I, \\ & XX^T = I. \end{aligned} \tag{18}$$

We can further simplify the optimization problem by relaxing the first constraint and making it independent of Δ. For this we use following identity, where the left hand side is equal to the right hand side:

$$\begin{aligned} & M - J(X,\Delta) = \lambda E(I - \Delta\Delta^T)E^T + \\ & \begin{bmatrix} I & E\Delta \end{bmatrix} \begin{bmatrix} M - J_n(X) - \lambda EE^T & T_2(X) \\ (T_2(X))^T & \lambda I - J_\Delta(X) \end{bmatrix} \begin{bmatrix} I \\ \Delta^T E^T \end{bmatrix} \end{aligned} \tag{19}$$

where $J_n(X), T_2(X)$ and $J_\Delta(X)$ are defined as

$$\begin{aligned} J_n(X) &= AA^T + BB^T - AXB^T - BX^TA^T, \\ T_2(X) &= BX^TF_1^T + AXF_2^T - AF_1^T - BF_2^T, \\ J_\Delta(X) &= F_1F_1^T + F_2F_2^T - F_1XF_2^T - F_2X^TF_1^T. \end{aligned}$$

Further we define the matrix in the second term on right hand side of (19) as

$$\mathscr{T}(M,X,\lambda) = \begin{bmatrix} M - J_n(X) - \lambda EE^T & T_2(X) \\ (T_2(X))^T & \lambda I - J_\Delta(X) \end{bmatrix}. \tag{20}$$

The right hand side of (19) is either zero or positive because $(I - \Delta\Delta^T) \succeq 0$ and we impose $\lambda \geq 0$ and $\mathscr{T} \succeq 0$, ensuring that M is an upper bound on $J(X,\Delta)$. Finally we write a relaxation of (16) as

$$\begin{aligned} \min_{X,M,\lambda} \quad & \operatorname{tr}(M) \\ \text{subject to} \quad & \mathscr{T}(M,X,\lambda) \succeq 0, \\ & \mathscr{X} \succeq 0, \\ & \lambda \geq 0, \end{aligned} \tag{21}$$

where $\mathscr{X}$ is the same as defined in (8).

Remark 1: Unlike Theorem 1, the gap between the robust problem (16) and its semidefinite relaxation (21) is not necessarily zero.

One possible reason of the non-zero gap may be that the relaxed maximization for all $\Delta : \Delta\Delta^T \preceq I$, is achieved at an X, which lies inside $XX^T = I$, i.e. the obtained X does not satisfy the orthogonality constraint of the original problem.

C. Orthogonalization of X

When the gap between the robust problem and its semidefinite relaxation is not zero, the X does not satisfy the orthogonality constraint. For such cases, we find the nearest orthogonal X in the Frobeneous norm sense. Such an X can be obtained by solving following optimization problem:

$$\begin{aligned} \min_{X_o} \quad & \|X_o - X\|_F^2 \\ \text{subject to} \quad & X_oX_o^T = I, \end{aligned} \tag{22}$$

where X_o is the required orthogonal matrix. This problem is same as the nominal Procrustes problem (1) with $A = I$ and $B = X$. The required matrix $X_o = U_xV_x^T$, where $U_x\Sigma_xV_x^T = X$ is the singular value decomposition of X.

D. Effect on the robust performance

It can be argued that X_o is not the optimal solution of (21), however it may be considered a suitable solution of (16), because this solution not only satisfies the orthogonality constraint and also results in a minimum cost variation. To evaluate this, let $e_1 = \tilde{A}X_o - \tilde{B}$ and $e_2 = \tilde{A}X - \tilde{B}$, then $\|e_1 - e_2\| = \|\tilde{A}(X_o - X)\| \leq \|\tilde{A}\|\|X_o - X\|$. A minimum value of $\|X_o - X\|$ will ensure that orthogonal X_o will result in a minimum cost variation from the SDR solution. However, this new solution may not have the same properties from the robustness perspective. Some analysis of the performance is presented in the simulation section, however, the robustness properties of X_o needs further analysis.

E. Robust problem for rotations

Using the same transformation as discussed in Section III-B, we can write the robust problem (21) for rotations as

$$\begin{aligned} \min_{Z,M,\lambda} \quad & \mathrm{tr}(M) \\ \text{subject to} \quad & \mathscr{T}(M,Z,\lambda) \succeq 0, \\ & Z \succeq 0, \\ & \lambda \geq 0, \end{aligned} \tag{23}$$

where $\mathscr{T}(M,Z,\lambda)$ is the same as (20) with X replaced by $X(Z)$.

However, it is observed that the proposed transformation also does not always work for problems with uncertainties in the data matrices. So for the cases where X is not orthogonal, we still need to use orthogonalization as discussed above, to obtain a solution of the rotation problem.

Finally, based on the numerical simulations performed for the robust problem, we present following remark.

Remark 2: It is observed in the numerical simulations that limiting the size of the maximum uncertainty in the robust problem (21) can significantly reduce the number of cases, where the solution X is not orthogonal. This observation also applies for rotation OPP (23), where the number of cases when $\mathrm{rank}(Z) \neq 1$, reduce significantly, reducing the likelihood of occurrence of the non-orthogonal X.

V. SIMULATION RESULTS

This section presents numerical simulations to evaluate the performance of the presented relaxation approaches and to support different discussions and results.

A. Analysis of the SDR for the standard OPP

Firstly, we compare the solution of (8) with the SVD solution. For this comparison, we generate random A and B matrices of the size 10×10 using MATLAB command `randn`. Results of the simulation is given in Figure 1. The plot compares and cost $\|AX - B\|_F^2$ using both the SVD and the SDR based solutions. Gap between the relaxed and the original problem is also shown, which is zero, supporting Theorem 1. The last subplot shows $\det(X)$, which is ± 1 indicating that the random problems are either rotations or reflections.

B. Analysis of the SDR for rotations OPP

The performance of the OPP for rotations is shown in Figure 2. First subplot presents the relaxation gap, which is zero, validating the exactness of (12) for rotations. Other subplots show some other parameters of the Z and X matrices. It can be observed that the parameters, such as trace and rank of Z, and determinant of X are as desired for all random cases.

C. Robust performance evaluation

In this section, we evaluate the performance of the solutions obtained by solving the approximate formulations of the standard and the robust problems for a set of bounded uncertainties in the input matrices. For this analysis, we considered a structured uncertainty description with a suitable choice of

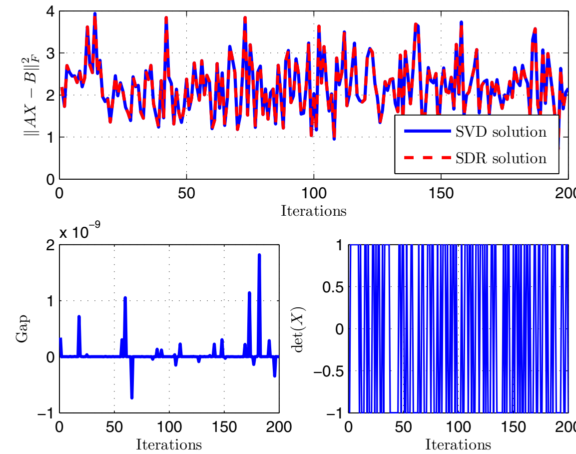

Fig. 1. Analysis of the semidefinite relaxation of the standard OPP (8).

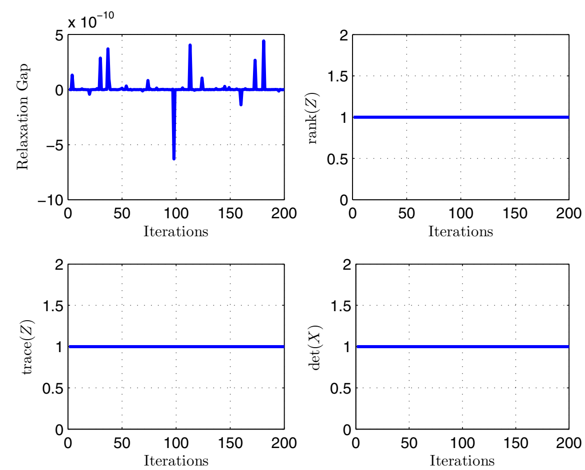

Fig. 2. Analysis of the semidefinite relaxation of OPP for rotations (12).

E, F_1, and F_2 matrices. To obtained data for this test, first we generated a random A matrix using MATLAB's `randn` command, and an orthogonal matrix X using `orth` command, both in $\mathbb{R}^{3\times 3}$. Using this orthogonal matrix, we calculate the matrix B. This A, B pair represents an exact data set, i.e. matrix A can be exactly transformed to B using X. In this pair, we added uniformly distributed random error within a range of 30% of the size of the elements of the true matrices, to obtain a data set with errors. We then solved both the nominal problem (4) and the robust problem (21). A large number of tests were performed by adding unifromly distrubuted random error in the nominal data within the set uncertainty bounds. The cost value was evaluated for both the nominal and the robust solution. The histogram of the test results is given in Figure 3, where $x-$axis represents the cost value, while $y-$axis represents the frequency of occurrance of the tests. It can be observed that the dispersion of the cost value using the nominal solution is much larger than the robust solution. This benefit is more obvious for the worst case scenarios. However, for nominal cases the robust solution suffers from an offset as compared to the nominal solution.

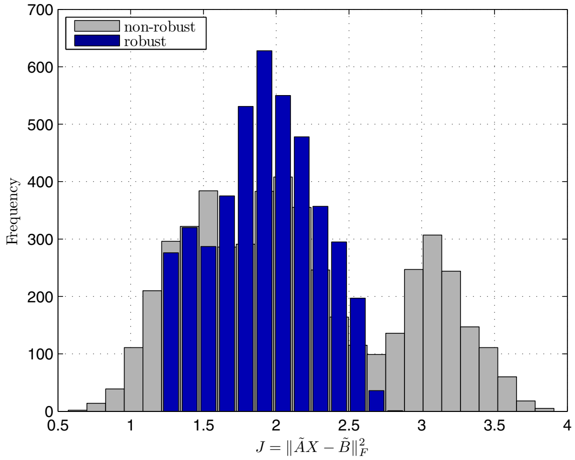

Fig. 3. Evaluation of the robust performance.

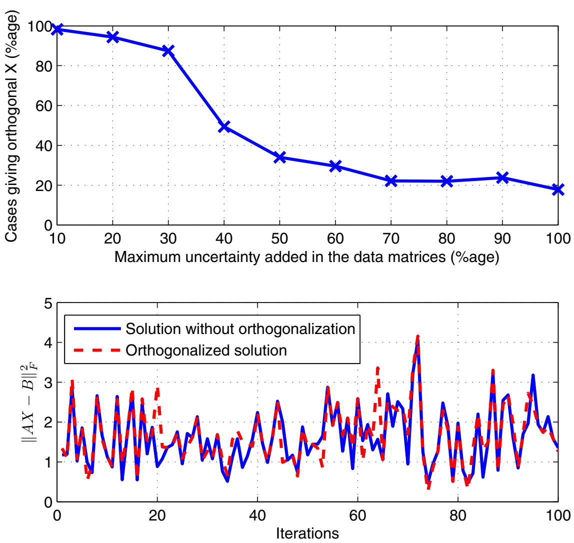

Fig. 4. First subplot shows number of orthogonal solutions (%age) verses uncertainty bounds, and second subplot compares cost using the SDR solution before and after orthogonalization.

D. Orthogonalization step

Lastly, we analyze the orthogonalization step and its effects on the cost value. For the first point, we performed simulations by varying maximum uncertainty level in the the nominal data and plotted the number of cases (in percentage), which give an orthogonal X. Results are shown in the first subplot of Figure 4. The analysis shows that the number of tests having orthogonal X increase significantly, when the size of the uncertainty is small, supporting Remark 2. Further, we compare the cost variation using the optimal solution of the approximate robust problem (21) and its orthogonalized solution for 200 runs of random data with within bounded uncertainty, showing that the cost variation due to the orthogonalization is not much.

VI. CONCLUSIONS AND FUTURE WORK

We presented an approach based on semidefinite relaxations to solve different formulations of the orthogonal Procrustes problem. It was demonstrated that the relaxation of the nominal problem results in no gap between the actual and the relaxed problems. The SDR framework also allows to handle uncertainties in the data matrices. It was further demonstrated that while considering uncertainties, the gap is not necessarily zero, which results in a solution not satisfying the orthogonality constraint of the original problem. In such cases, orthogonalization was proposed for the obtained solution. However, there are some issues, which need further analysis. In this regard, our future work may consider further study of the relaxation gap with uncertainties. Moreover, some work may be done on casting these problems in a standard SDP framework and analyzing its computational complexity using some efficient SDP algorithm.

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