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Polynomial Accelerated Algorithm Base on Minimizing TV for Computerized Tomographic Image Reconstruction

Hui Kang,Hongxia Gao
Engineering Research Center for
Precision Electronic Manufacturing Equipments
of Ministry of Education
College of Automation Science and Engineering
South China University of Technology
GuangZhou, China
Email: hxgao@scut.edu.cn ,spiritcherry@126.com

Yueming Hu
Engineering Research Center for
Precision Electronic Manufacturing Equipments
of Ministry of Education
College of Automation Science and Engineering
South China University of Technology
GuangZhou, China
Email: auymhu@scut.edu.cn

***Abstract*—Based on total variation minimization (TV) algorithm and the theory of polynomial acceleration, we prompted a modified algorithm, which is called P-TV in this paper. This new algorithm has a better noise immunity and fewer iteration numbers than the traditional TV algorithm. P-TV algorithm can be divided into four steps. Compared to the traditional TV algorithm, the P-TV algorithm applies the theory of polynomials to ART iteration step to enforce data consistency with the projection data and applies the positivity constraint after the GRAD-step too, which accelerate convergence of image reconstruction. Simulation results prove that when the iteration number is the same, the quality of reconstructed image from P-TV algorithm is better than traditional TV algorithm. When the quality of image reconstructed by both algorithms is nearly the same, the iteration number of P-TV algorithm is much less than the number of traditional TV algorithm. At last we applied the two algorithms to CT image reconstruction from the noisy data. All results show that the P-TV algorithm is effective.**

***Index Terms*—CT; Few-views; Image reconstruction; TV algorithm; Polynomial acceleration algorithm**

I. INTRODUCTION

Computed Tomography (CT) image reconstruction is widely used in medicine, industrial inspection and so on. It is an imaging technique to get the cross-sectional information of the image according to projection measurement data which are in different angles. In various forms of computerized tomographic image reconstruction, one of the main issues centers on how to estimate an accurate image from few views data. In two dimensional (2D) CT, the most common problems are the non-sufficient data reconstruction problems[1][2]. Here, we focus on the few-views problem, which obtains the projection data only in some specific angles. Two widely used iterative algorithms for tomographic reconstruction are the algebraic reconstruction technique (ART) [3]and the expectation-maximization (EM) algorithm[4]. For the case where the data are not sufficient to determine a unique solution to the imaging model, the ART algorithm will find the imagethat is consistent with the data and minimizes the sum-of-squares of the image pixel values. But this algorithm will lead to conspicuous artifacts in reconstructed images. Although there are many improvements based on ART, such as SIRT and MART, the effect on elimination of artifacts is not so obvious. In the last few years, total variation minimization (TV) methods, which originates in the field of compressing sensing (CS)[5][6], for CT reconstruction from sparse and noisy data were developed by Sidky et al [7]. This method is effective on artifacts elimination; we can get a high quality reconstruction image with it. Li Yi of North University of China and Liu Baodong of Chongqing University of China both make some modifications about TV algorithm [8][9]. But traditional TV algorithm is time consuming. In this paper, we develop an iterative image reconstruction algorithm, which is combined TV algorithm with theory of polynomial acceleration[10]. We call it P-TV algorithm which accelerates convergence of image reconstruction. Each iteration of P-TV algorithm can be divided into four steps: the POCS-step (Projection onto Convex Sets), which applies the theory of polynomials to ART iteration to enforce data consistency with the projection data; the positivity constraint step, which constrains the values of all reconstructed points within [0, 1]; the GRAD-step, which reduces the TV of the image estimate; the positivity constraint step again, this step is added to the conditional TV algorithm additionally, which can accelerate convergence of image reconstruction.

This paper is organized as follows. In Section2, we describe the CT reconstruction problem and the TV algorithm. In Section 3, we introduce the theory of polynomial acceleration and promote the modified TV algorithm. In Section4, we give the numerical simulation results. In Section5, we conclude this paper.

II. TV ALGORITHM

A. Mathematic Model of CT reconstruction

The image is described by a vector $\vec{f}$ of length N_{image} with elements$f_j, j = 1, 2N_{image}$, $N_{image} = n \times n$ where the integer n is the width and height of the 2D image array. The projection-data is described by a vector $\vec{p}$ of length $N_{data}$with elements p_i

$$p_i = \sum_{j=1}^{N_{image}} w_{ij} f_j, i = 1, 2, \ldots, N_{data} \tag{1}$$

N_{data} is the total number of projection-data. w_{ij} is the element of the system matrixrepresent the impact of pixel f_j to projection-data p_i and the value of w_{ij} is equal to the length of ith projection through the jth pixel[11]. The system matrix W is composed of N_{data} row vectors. The general expression for the algorithm discussed here involves inversion of a discrete-to-discrete linear transform

$$\vec{p} = W\vec{f} \tag{2}$$

B. TV algorithm

CT reconstruction always is an ill-posed problem, especially when it is few-views condition. The typical method to solve this problem is the regularization. Recently, regularization based on minimizing TV of the image becomes a hot topic[7][12-15].In fact, minimizing TV of the image is a progress to minimize the l_1-norm of the gradient image. So, the most elements of gradient image will be zeros and the original image will tend to constant in most pixels [16]. Each iteration of TV algorithm consists of two phases: POCS and gradient descent. The POCS phase is further broken down into two steps that enforce data consistency and positivity. As a result, the steps comprising each loop are: DATA-step, which enforces data consistency with the projection data; the POS-step, which ensures a non-negative image; and the GRAD-step, which reduces the TV of the image estimate. We can refer to references[7] for concrete steps of TV algorithm. During the process of DATA-step, TV algorithm employs the traditional ART iteration formula to enforce data consistency. And in the process of GRAD-step, parameter is a constant, which is used to control the convergence speed. So, in the beginning of iteration, $f^{(TV\_GRAD)}$ will fall off rapidly to certain value. But with the increase of iteration, $f^{(TV\_GRAD)}$ will fall off slowly. Therefore, we prompted a modified TV algorithm, called it P-TV based on TV algorithm and the theory of polynomial acceleration, which will introduce below. P-TV algorithm has a better performance in convergence speed than TV algorithm. This performance will be discussed in Section 4.

III. THEORY OF POLYNOMIAL ACCELERATION

The traditional iteration method just used the information of image in last step, such as ART. The theory of polynomial acceleration will use all the information in each iteration[10]. The general formula is as follows:

$$F^k = u_1 F^{k-1} + u_2 F^{k-2} + \ldots + u_k F^0 + w_k r^{k-1} \tag{3}$$

$$k = 1, 2, \ldots, u_1 + u_2 + \ldots + u_k = 1, w_k \neq 0$$

r^{k-1} is remainder term. We can obtain the following formula from (3)

$$F^k - F^0 = \sum_{j=0}^{k-1} \varepsilon_j (W^* W)^j r_0 = q^{k-1}(W^* W) r_0 \tag{4}$$

$$q^{k-1}(\gamma) = \varepsilon_0 + \varepsilon_1 \gamma + \ldots + \varepsilon_{k-1} \gamma^{k-1} \tag{5}$$

$q^{k-1}(\gamma)$ is a polynomial of K-1 order and

$$q^{k-1}(\gamma) = \sum_{j=0}^{k-1} (1-\gamma)^j = \frac{1-(1-\gamma)^k}{\gamma} \tag{6}$$

In order to use polynomial iteration effectively, promoting an algorithm just used the information of F^{k-1} andF^{k-2}.

$$F^k = F^{k-1} + u_k(F^{k-1} - F^{k-2}) + w_k W^*(P - WF^{k-1}) \tag{7}$$

$$u_k = \frac{(k-1)(2k-3)(2k+2v+1)}{(k+2v-1)(2k+4v-1)(2k+2v-3)} \tag{8}$$

$$w_k = \frac{(k+v-1)(2k+2v-1)}{(k+2v-1)(2k+4v-1)} \tag{9}$$

We combined this theory with TV algorithm, the improved algorithm of TV are as follows:

1) Initialization:

$$f^{(0)} = 0, \alpha = 0.2, N_{grad} = 20, v = 0.8$$

2) Data projection iteration, for $k = 1, 2, \ldots, N_{data}$:

$$f^{(k)} = f^{(k-1)} + u_k(f^{(k-1)} - f^{(k-2)}) + w_k \frac{p_i - w_{ij} f^{(k-1)}}{\sum_{j=1}^{N_{image}} w_{ij}^2} w_{ij} \tag{10}$$

$$i = 1, 2 \ldots, N_{data} \; j = 1, 2 \ldots, N_{image}$$

$$u_k = \frac{(k-1)(2k-3)(2k+2v+1)}{(k+2v-1)(2k+4v-1)(2k+2v-3)} \tag{11}$$

$$w_k = \frac{(k+v-1)(2k+2v-1)}{(k+2v-1)(2k+4v-1)} \tag{12}$$

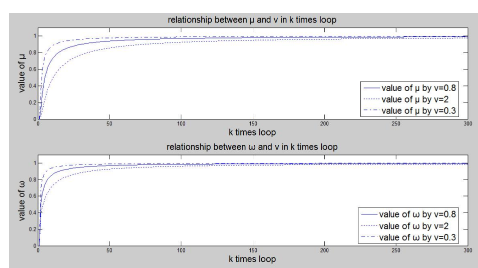

Fig. 1. tendency of u_k and w_k with the addition of iteration number

Fig.1. show the tendency of u_k and w_k with the addition of iteration number. v is a constant which controls the convergence rate of iteration. We can see from Figure, different value of v will lead to different convergence rate about u_k and w_k .Furthermore, different u_k and w_k will lead to different convergence rate in reconstruction iteration. Here, we set v=0.8 according to lots of experiments.

3) Positivity constraint:

$$f_{pose}^{(k)} = \begin{cases} 1, & f_j^{(TV\_GRAD)} > 1 \\ f_j^{(k)}, & 0 < f_j^{(TV\_GRAD)} < 1 \\ 0, & else \end{cases} \tag{13}$$

$$j = 1, 2 \ldots, N_{image}$$

4) TV gradient descent initialization:

$$d_A(n) = \| f^{(k-1)} - f_{pose}^{(k)} \| \tag{14}$$

$$f^{(TV\_GRAD)} = f_{pose}^{(k)} \tag{15}$$

5) TV gradient descent for $m = 1, 2, \ldots N_{grad}$:

$$v_j^{(m)} = \frac{\partial \| f^{(TV\_GRAD)} \|_{TV}}{\partial f_j^{(TV\_GRAD)}} \tag{16}$$

$$f^{(TV\_GRAD)} = f^{(TV\_GRAD)} - \alpha d_A \frac{v_j^{(m)}}{\| v_j^{(m)} \|} \tag{17}$$

6) Positivity constraint:

$$f_{pose}^{(TV\_GRAD)} = \begin{cases} 1, & f_j^{(TV\_GRAD)} > 1 \\ f_j^{(TV\_GRAD)}, & 0 < f_j^{(TV\_GRAD)} < 1 \\ 0, & else \end{cases} \tag{18}$$

7) Stop condition: The iteration will stop when there is no appreciable change in following formula, else return to 2).

$$| f_{pose}^{(k)} - f_{pose}^{(k-1)} | < \varepsilon \tag{19}$$

IV. NUMERICAL SIMULATION RESULTS

The true image solution is taken to be the Shepp-Logan image shown in below discredited on a 128*128 pixel grid. This phantom is often used in evaluating tomographic reconstruction algorithms. We take parallel beam in experiment and scanning parameters are given in table 1.

TABLE I
SCANNING PARAMETERS

Parameters	Values
Image size(pixel)	128*128
Projection group number	18(every 10^o)
Detector number	185
Angular range	$[0, \pi]$

For comparison, the images are reconstructed by use of TV algorithm and P-TV algorithm. Fig.2. Show the reconstructions results from noiseless data. It is shown clearly when the iteration number is same (both are 39), the reconstruct image quality of P-TV is better than TV.

For further comparison, we compare the image profiles [8] of line 50 and line 103 as fig.3. We can see image reconstructed by P-TV algorithm is closer to the original image than image reconstructed by TV algorithmespecially in some peak value. In order to show the P-TV algorithm's good performance on convergence speed, we show a group of

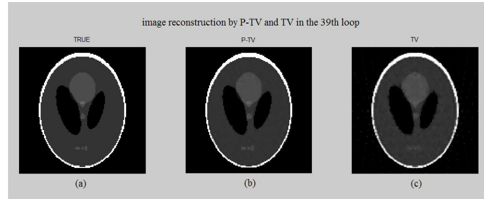

Fig. 2. Reconstruct Image (a) original image; (b) reconstructed image by P-TV algorithm; (c) reconstructed image by TV algorithm

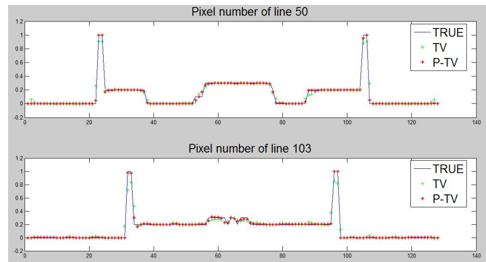

Fig. 3. Profiles (a) original image; (b) reconstructed image by P-TV algorithm

pictures as below. Picture (b) is obtained by P-TV algorithm, picture (c) is reconstructed by traditional TV algorithm. We can see that the quality of picture (b) and picture (c) is nearly the same, but the iteration number of picture (c) is 305 while the iteration number of picture (b) is only 39. That is to say, P-TV algorithm's convergence rate is nearly ten times as much as TV algorithm.

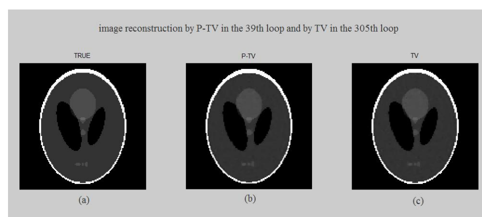

Fig. 4. Reconstruct Image (a) original image; (b) reconstructed image by P-TV algorithm; (c) reconstructed image by TV algorithm

To verify the algorithm's stability for noise, we have also applied the two algorithms to noisy data generated by adding Gaussian noise to the projection data. The standard deviation of the Gaussian noise is 0.1% of the maximum value of the projection data. The results are shown in Fig.5. It indicates that P-TV algorithm has better noise immunity than TV algorithm.

We also show the mean square error (MSE) of the reconstructed images used two algorithms and give them in Fig.6. We can see P-TV algorithm converge faster than TV algorithm and MSE of P-TV is smaller.

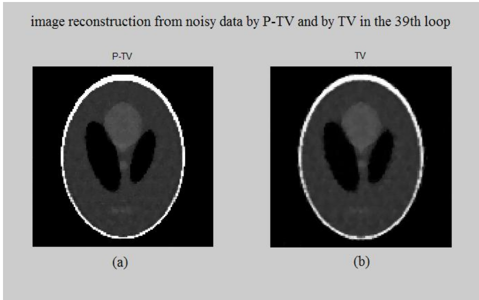

Fig. 5. Reconstruct Image with Gaussian noise: reconstructed image by P-TV algorithm; (b) reconstructed image by TV algorithm

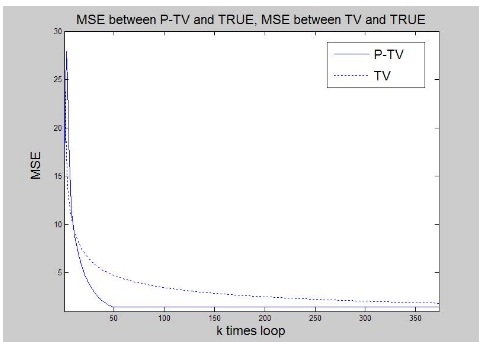

Fig. 6. Reconstruct Image with Gaussian noise: reconstructed image by P-TV algorithm; (b) reconstructed image by TV algorithm

V. Conclusion

In this paper, a P-TV algorithm based on TV algorithm and the theory of polynomial acceleration is prompted. The experimental results showed evidence that the algorithm is effective and stable. And the most prominent advantages of this algorithm are the rapid convergence speed and the less time consuming. This algorithm can be applied to 3D CT configurations without or with small modifications.

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References

[1] L.Li, K. Kang, Z. Chen, L. Zhang and Y. Xing, A general region-of-interest image reconstruction approach with truncated Hilbert transform, Journal of X-Ray Science and Technology 17 (2009), 135-152.

[2] H.Yu, Y. Ye and G. Wang, Interior reconstruction using the truncated Hilbert transform via singular value decomposition, Journal of X-Ray Science and Technology 16 (2008), 243-251.

[3] G.T. Herman, Image Reconstruction from Projection: the Fundamentals of Computerized Tomography, Academic Press, New York, 1980.

[4] H.H. Barrett and K.J. Myers, Foundations of Image Science, John Wiley Sons, Inc. Hoboken, New Jersey, 2004.

[5] E. Candes, J. Romberg and T. Tao, Robust uncertainty principles: exact signal reconstruction from highly incomplete Frequency Information, IEEE Trans Inf Theory 52(2)(2006),489-509.

[6] E. Candes, J. Romberg and T. Tao, Stable signal recovery from incomplete and inaccurate measurements, Commun Pure Appl Math 59(7) (2006), 1207-1223.

[7] E.Y. Sidky, C. Kao, and X. Pan. Accurate image reconstruction from few-views and limited-angle data in divergent-beam CT[J].Journal of X-Ray Science and Technology,(2006),14:119-139.

[8] Baodong Liu. A Thesis Submitted to Chongqing University in Partial Fulfillment of the Requirement for the Degree of Doctor of Engineering. [D](2010). Chongqing University of China.

[9] Li Yi. The study of limited angle three-dimensional CT image reconstruction algorithm. (2011), North University of China.

[10] Yu Chen. Research on Inverse Problems Solving and Image Reconstruction Algorithm For Electrical Capacitance Tomography System. (2010), Harbin University of Science and Technology of China.

[11] TianGe Zhuang. Principle and Algorithm of CT. Shanghai Jiaotong University press, 1992.

[12] Sidky E Y and Pan X. Image reconstruction in circular cone-beam computed tomography by constrained, total-variation minimization [J].Phys.Med.Biol. 2008, 53:4777-4807.

[13] Sidky E Y and Pan X. Accurate image reconstruction in circular cone-beam computed tomography by total variation minimization: a preliminary investigation [J].IEEE Nucl. Sci. Conf.Rec.2006, 5:2904-7.

[14] Sidky E Y,Chartrand R and Pan X. Image reconstruction from few views by non-convex optimization [J].IEEE Nucl. Sci. Conf. Rec., 2007,5:3526-30.

[15] LaRoque S J,Sidky E Y and Pan X. Accurate image reconstruction from few-view and limited-angle data in diffraction tomography [J].Opt.Soc.Am.A,2008,25:1772-82.

[16] Gengsheng Zeng. Medical Image Reconstruction [M].Beijing: Higher Education Press, 2010