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Stability Analysis and Control of Linear Periodic Time-Delay Systems with State-Space Models Based on Semi-Discretization

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***Abstract*—Stability analysis and control for linear periodic time-delay systems described by state space models are investigated in this paper. Semi-discretization method is used to develop a mapping of the system response in a finite-dimensional state space. The stability region and stability boundary can be found by comparing the maximum absolute value of the mapping's eigenvalues with 1. More importantly, an efficient stability criterion is presented for linear periodic neutral systems. Besides, minimization of the maximum absolute value of the mapping's eigenvalues leads to optimal control gains. Two numerical examples are given to illustrate the proposed method's effectiveness.**

Keywords-periodic time-delay systems; periodic neutral systems; stability analysis; feedback control; semi-discretization.

I. INTRODUCTION

Generally speaking, linear periodic time-delay systems can be divided into three categories according to the system complexity: (i) systems with a single time-delay, (ii) systems with multiple time-delays and (iii) neutral systems. Linear periodic systems with a single or multiple time-delays are actually special cases of linear periodic neutral systems.

In recent years, the stability analysis of linear time-delay systems, especially neutral systems, has attracted considerable attention. Among different derivation methods for stability criteria, one main method is based on Lyapunov-Krasovskii functional (LKF) and linear matrix inequality (LMI) [1-5]. In [1] delay-dependent stability conditions were obtained for neutral systems with time-varying delays in terms of LMIs and descriptor model transformation. Reference [2] derived a stability criterion which was formulated in an LMI for uncertain neutral systems with norm-bounded or time-varying uncertainty. A discretized LKF approach was developed to analyze the stability of linear neutral systems with mixed neutral and discrete delays in [3]. Combining the parameterized model transformation method with a method taking the relationships between the terms in the Leibniz-Newton formula into account and using LMI, [4] put forward delay-dependent robust stability criteria and a stabilizing method for neutral systems. Reference [5] studied the stability of neutral type systems with uncertain time-varying delays and norm-bounded uncertainties by utilizing the input-output approach and LKF. Besides, some other methods have also been introduced, e.g. [6] presented necessary and sufficient conditions for delay-dependent stability by principle of the argument.

Meanwhile, control for linear time-delay systems has also been investigated intensively using LMI, LKF and sliding-mode control (SMC) [7-12]. Reference [7] proposed a sufficient condition for the solvability of the design of memoryless state feedback controllers in terms of LMI and gave an explicit expression for the desired controller. In [8], a criterion for the existence of dynamic output feedback controllers was derived based on the LMI and LKF, and a parameterized characterization of the controllers was also given. In [9], LMI optimization approach is used to design the robust output dynamic observer-based controls. Reference [10] raised a robust control design method with LMI and discretized parameter-dependent LKF. In [11], the design of SMC for a class of neutral delay systems with uncertainties in both the state matrices and the input matrix was studied. Reference [12] put forward a robust adaptive SMC design scheme for discrete-time state-delay systems with mismatched uncertainties and external disturbances.

Although there are a large number of references discussing linear systems with time-delay, one type of time-delay system, i.e. linear periodic time-delay systems, has not received much attention. Semi-discretization has been introduced to study this type of time-delay systems. This method proposes to discretize some spatial or temporal variables and treat the rest of them as continuous variables, and thus the exact solution of linear systems can be obtained to construct a very accurate mapping of the state vector over a mapping time step. Therefore semi-discretization is able to handle periodic time-delay systems. Semi-discretization was utilized for the stability analysis of second order linear periodic time-delay system with a single delay represented by ordinary differential equations (ODEs) in [13]. In [14] the stability analysis of higher order systems with both discrete and continuous time-delays was studied based on semi-discretization. Reference [15] reported an application of semi-discretization to the feedback controls and optimal gain design of linear periodic systems with a single time-delay.

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Reference [16] extended this idea to the stability analysis of linear neutral systems represented by ODEs.

In this paper, we further extend this method to the investigation regarding stability analysis and control design of all types of linear periodic time-delay systems described by state space models.

The rest of this paper is organized as follows. In Section II, semi-discretization method is deployed to derive stability criteria for different types of linear periodic time-delay systems. Also, optimal feedback control of such systems is discussed. Section III presents two numerical examples to illustrate this method's effectiveness. The conclusions are presented in Section IV.

II. MAIN RESULTS

In this section, two topics are covered: Subsection A presents stability criteria for three types of linear periodic time-delay systems. Subsection B introduces one scheme for the optimal feedback control design.

A. Stability Criteria

Linear periodic time-delay systems have a general state space form as shown by (1).

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \sum_{j=1}^{m}[\mathbf{B}_j(t)\mathbf{x}(t-\tau_j) + \mathbf{C}_j(t)\dot{\mathbf{x}}(t-\tau_j)], \quad (1)$$

where $\mathbf{x} \in \mathbf{R}^N$, and the coefficient matrices $\mathbf{A}(t)$, $\mathbf{B}_j(t)$ and $\mathbf{C}_j(t)$ are all periodic matrices with period T. m is the number of time-delays. Without loss of generality, it is assumed that τ_j's are already arranged such that $\tau_1 < \tau_2 < \cdots < \tau_m$.

With different constraints on m and $\mathbf{C}(t)$'s, (1) is able to represent different types of linear periodic time-delay systems. The general forms of periodic time-delay systems with a single delay and periodic time-delay systems with multiple delays are given by (2) and (3).

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{x}(t-\tau) \quad (2)$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \sum_{j=1}^{m}\mathbf{B}_j(t)\mathbf{x}(t-\tau_j) \quad (3)$$

Because (2) and (3) are special cases of (1), stability criteria for linear periodic systems with a single or multiple time-delays are derivable from the criterion for linear periodic neutral systems. Hence Theorem 1 will be first introduced as a stability criterion for linear periodic neutral systems, and then the stability criteria for the other two types of systems are given in Corollaries 1 and 2, respectively.

Theorem 1: Suppose for one linear periodic time-delay system, the mapping of the state vector over one period is obtained using semi-discretization, and is shown as

$$\mathbf{y}_{p+1} = \mathbf{\Phi}\mathbf{y}_p, \quad (4)$$

where p indicates the index of one period, and $\mathbf{y}_p$ is the state vector at the beginning of the pth period. $\mathbf{\Phi}$ is the product of transition matrices over the period, namely,

$$\mathbf{\Phi} = \left(\prod_{i=0}^{k-1}\mathbf{H}_i^T\right)^T, \quad (5)$$

where $\mathbf{H}_i$ is the transition matrix over a time interval $[t_i, t_{i+1}]$, and satisfies (6). Note that the period T is equally partitioned into k intervals.

$$\mathbf{y}_{i+1} = \mathbf{H}_i\mathbf{y}_i, \quad (6)$$

where i indicates the index of one time interval, and $\mathbf{y}_i$ is the state vector at the beginning of the ith interval.

The transition matrices $\mathbf{H}_i$'s are given in (7).

$$\mathbf{H}_i = \begin{bmatrix} \mathbf{H}_{i,11} & \mathbf{H}_{i,12} \\ \mathbf{H}_{i,21} & \mathbf{H}_{i,22} \\ \mathbf{H}_{i,31} & \mathbf{H}_{i,32} \\ \mathbf{H}_{i,41} & \mathbf{H}_{i,42} \end{bmatrix}_{2(n_m+1)N\times 2(n_m+1)N} \quad (7)$$

where

$$\mathbf{H}_{i,11} = \begin{bmatrix} \mathbf{Q}_i & \mathbf{0}_{N\times(n_1-1)N} & \mathbf{P}_i\mathbf{B}_{1,i} & \cdots & \mathbf{0}_{N\times(n_m-n_{m-1}-1)N} & \mathbf{P}_i\mathbf{B}_{m,i} \end{bmatrix}_{N\times(n_m+1)N},$$

$$\mathbf{H}_{i,12} = \begin{bmatrix} \mathbf{0}_{N\times n_1 N} & \mathbf{P}_i\mathbf{C}_{1,i} & \cdots & \mathbf{0}_{N\times(n_m-n_{m-1}-1)N} & \mathbf{P}_i\mathbf{C}_{m,i} \end{bmatrix}_{N\times(n_m+1)N},$$

$$\mathbf{H}_{i,21} = \begin{bmatrix} \mathbf{I}_{n_m N\times n_m N} & \mathbf{0}_{n_m N\times N} \end{bmatrix}_{n_m N\times(n_m+1)N},$$

$$\mathbf{H}_{i,22} = \mathbf{0}_{n_m N\times(n_m+1)N},$$

$$\mathbf{H}_{i,31} = \begin{bmatrix} \mathbf{A}_i\mathbf{Q}_i & \mathbf{0}_{N\times(n_1-2)N} & \mathbf{B}_{1,i} & \mathbf{A}_i\mathbf{P}_i\mathbf{B}_{1,i} & \cdots & \mathbf{B}_{m,i} & \mathbf{A}_i\mathbf{P}_i\mathbf{B}_{m,i} \end{bmatrix}_{N\times(n_m+1)N},$$

$$\mathbf{H}_{i,32} = \begin{bmatrix} \mathbf{0}_{N\times(n_1-1)N} & \mathbf{C}_{1,i} & \mathbf{A}_i\mathbf{P}_i\mathbf{C}_{1,i} & \cdots & \mathbf{C}_{m,i} & \mathbf{A}_i\mathbf{P}_i\mathbf{C}_{m,i} \end{bmatrix}_{N\times(n_m+1)N},$$

$$\mathbf{H}_{i,41} = \mathbf{0}_{n_m N\times(n_m+1)N},$$

$$\mathbf{H}_{i,42} = \begin{bmatrix} \mathbf{0}_{n_m N\times N} & \mathbf{I}_{n_m N\times n_m N} \end{bmatrix}_{n_m N\times(n_m+1)N}.$$

In (7):

$$\begin{aligned} &\mathbf{A}_i = \mathbf{A}(t_i), \mathbf{B}_{j,i} = \mathbf{B}_j(t_i), \mathbf{C}_{j,i} = \mathbf{C}_j(t_i), \\ &\mathbf{Q}_i = e^{\mathbf{A}_i\Delta t}, \mathbf{P}_i = \int_{t_i}^{t_{i+1}} e^{\mathbf{A}_i(\Delta t-\hat{t})}d\hat{t}, \\ &n_j = \tau_j / \Delta t,\ j = 1,2,...,m. \end{aligned} \quad (8)$$

Let $|\lambda|_{\max}$ denote the maximal absolute value of eigenvalues of $\mathbf{\Phi}$. Then the stability region is determined by

$$|\lambda|_{\max} < 1, \quad (9)$$

and the stability boundary is determined by

$$|\lambda|_{\max} = 1. \quad (10)$$

Proof: First let us discretize the period T into an integer k intervals of equal length Δt such that $T = k\Delta t$.

Consider (1) in a time interval $t \in [t_i, t_{i+1}]$, where $t_i = i\Delta t, i = 0,1,2,...,k$. In each small time interval $[t_i, t_{i+1}]$, the delayed responses $\mathbf{x}(t-\tau_j)$ and time-dependent coefficient matrices are assumed to be constant. Thus we have the notations in (8). Besides we denote

$$\begin{aligned} &\mathbf{x}(t_i) = \mathbf{x}_i, \\ &\mathbf{x}(t_{i+1}) = \mathbf{x}_{i+1}, \\ &\mathbf{x}(t-\tau_j) = \mathbf{x}(i\Delta t - n_j \Delta t) = \mathbf{x}_{i-n_j}. \end{aligned} \tag{11}$$

Within one particular interval, (1) becomes:

$$\dot{\mathbf{x}}(t) - \mathbf{A}_i \mathbf{x}(t) - \sum_{j=1}^{m} [\mathbf{B}_{j,i}\mathbf{x}(t-\tau_j) + \mathbf{C}_{j,i}\dot{\mathbf{x}}(t-\tau_j)] = 0. \tag{12}$$

The general solution of (12) is given by

$$\mathbf{x}(t) = e^{\mathbf{A}_i(t-t_i)}\mathbf{x}_i + \int_{t_i}^{t} e^{\mathbf{A}_i(t-t_i-\hat{t})} \sum_{j=1}^{m} [\mathbf{B}_{j,i}\mathbf{x}(\hat{t}-\tau_j) + \mathbf{C}_{j,i}\dot{\mathbf{x}}(\hat{t}-\tau_j)] d\hat{t}, \tag{13}$$

where $t \in [t_i, t_{i+1}]$.

Let t be t_{i+1} in (13), then

$$\mathbf{x}_{i+1} = \mathbf{Q}_i \mathbf{x}_i + \mathbf{P}_i \sum_{j=1}^{m} (\mathbf{B}_{j,i}\mathbf{x}_{i-n_j} + \mathbf{C}_{j,i}\dot{\mathbf{x}}_{i-n_j}), \tag{14}$$

where the definition of $\mathbf{Q}_i$ and $\mathbf{P}_i$ is given in (8).

In order to construct a mapping over one time interval, one $2(n_m+1)\times N$ dimensional state vector is defined as:

$$\mathbf{y}_i = \begin{bmatrix} \mathbf{x}_i^T & \mathbf{x}_{i-1}^T & \cdots & \mathbf{x}_{i-n_m}^T & \dot{\mathbf{x}}_i^T & \dot{\mathbf{x}}_{i-1}^T & \cdots & \dot{\mathbf{x}}_{i-n_m}^T \end{bmatrix}^T. \tag{15}$$

To find the transition matrix $\mathbf{H}_i$ is equivalent to find a mapping from $\mathbf{y}_i$ to $\mathbf{y}_{i+1}$. $\mathbf{y}_{i+1}$ has the form in (16).

$$\mathbf{y}_{i+1} = \begin{bmatrix} \mathbf{x}_{i+1}^T & \mathbf{x}_i^T & \cdots & \mathbf{x}_{i-n_m+1}^T & \dot{\mathbf{x}}_{i+1}^T & \dot{\mathbf{x}}_i^T & \cdots & \dot{\mathbf{x}}_{i-n_m+1}^T \end{bmatrix}^T. \tag{16}$$

All entries except $\mathbf{x}_{i+1}^T$ and $\dot{\mathbf{x}}_{i+1}^T$ in $\mathbf{y}_{i+1}$ also appear in $\mathbf{y}_i$, which indicates that corresponding entries in $\mathbf{H}_i$ are identity matrices with appropriate dimensions.

Moreover, (14) actually provides a representation of $\mathbf{x}_{i+1}$ using the entries in $\mathbf{y}_i$. Hence the last problem lies in the representation of $\dot{\mathbf{x}}_{i+1}$.

Let t be t_{i+1} in (12), and it becomes:

$$\dot{\mathbf{x}}_{i+1} = \mathbf{A}_i \mathbf{x}_{i+1} + \sum_{j=1}^{m} (\mathbf{B}_{j,i}\mathbf{x}_{i-n_j+1} + \mathbf{C}_{j,i}\dot{\mathbf{x}}_{i-n_j+1}). \tag{17}$$

Plug (14) into (17) and we get (18).

$$\begin{aligned} \dot{\mathbf{x}}_{i+1} &= \mathbf{A}_i \mathbf{Q}_i \mathbf{x}_i + \\ &\sum_{j=1}^{m} [\mathbf{A}_i \mathbf{P}_i \mathbf{B}_{j,i}\mathbf{x}_{i-n_j} + \mathbf{B}_{j,i}\mathbf{x}_{i-n_j+1} + \mathbf{A}_i \mathbf{P}_i \mathbf{C}_{j,i}\dot{\mathbf{x}}_{i-n_j} + \mathbf{C}_{j,i}\dot{\mathbf{x}}_{i-n_j+1}]. \end{aligned} \tag{18}$$

Combining the results above together, and the transition matrix over $[t_i, t_{i+1}]$ can then be expressed as (17).

The mapping of the state vector over one period $T = k\Delta t$ is therefore shown by (4), and $\mathbf{\Phi}$ is given by (5).

The stability of the system is determined by the eigenvalues of $\mathbf{\Phi}$, i.e.

$$\left|\mathbf{y}_{p+1}\right| \le |\lambda|_{\max} \left|\mathbf{y}_p\right|. \tag{19}$$

When $|\lambda|_{\max} < 1$, $\mathbf{\Phi}$ is a contraction, and the system is asymptotically stable. Then the stability region and stability boundary can be found using (9) and (10), respectively.

Moreover, (19) also indicates that the smaller $|\lambda|_{\max}$ is, the faster the system response converges to zero. Hence $|\lambda|_{\max}$ also provides an indicator of the control performance.

□

Furthermore, as particular cases of the previous result, the stability criteria for periodic time-delay systems with a single delay and multiple delays are presented in the sequel.

Corollary 1: For a linear periodic time-delay system with one single time-delay, the stability region and stability boundary are determined by (9) and (10), respectively. The mapping over one period and the corresponding mapping matrix $\mathbf{\Phi}$ are the same as in Theorem 1, and transition matrices $\mathbf{H}_i$'s can be instead calculated using (20).

$$\mathbf{H}_i = \begin{bmatrix} \mathbf{H}_{i,1} \\ \mathbf{H}_{i,2} \end{bmatrix}_{(n+1)N\times(n+1)N}, \tag{20}$$

where

$$\begin{aligned} &\mathbf{H}_{i,1} = \begin{bmatrix} \mathbf{Q}_i & \mathbf{0}_{N\times(n-1)N} & \mathbf{P}_i\mathbf{B}_i \end{bmatrix}, \\ &\mathbf{H}_{i,1} = \begin{bmatrix} \mathbf{I}_{nN\times nN} & 0_{nN\times N} \end{bmatrix}, \\ &n = \tau / \Delta t, \end{aligned}$$

and $\mathbf{Q}_i$, $\mathbf{P}_i$ and $\mathbf{B}_i$ are the same as in (8).

Corollary 2: For a linear periodic time-delay system with multiple time-delays, the stability region and stability boundary are determined by (9) and (10), respectively. The mapping over one period and the corresponding mapping matrix $\mathbf{\Phi}$ are the same as in Theorem 1, and transition matrices can be found with (21).

$$\mathbf{H}_i = \begin{bmatrix} \mathbf{H}_{i,1} \\ \mathbf{H}_{i,2} \end{bmatrix}, \tag{21}$$

where

$$\mathbf{H}_{i,1} = \begin{bmatrix} \mathbf{Q}_i & \mathbf{0}_{N\times(n_1N-N)} & \mathbf{P}_i\mathbf{B}_{1,i} & \cdots & \mathbf{0}_{N\times(n_mN-n_{m-1}N)} & \mathbf{P}_i\mathbf{B}_{m,i} \end{bmatrix},$$
$$\mathbf{H}_{i,2} = \mathbf{I}_{(n_m+1)N\times(n_m+1)N},$$
$$n_j = \tau_j / \Delta t,\ j = 1,2,...,m,$$

and $\mathbf{Q}_i$, $\mathbf{P}_i$ and $\mathbf{B}_{j,i}$ are the same as in (8).

B. Optimal Feedback Control

In this section, we consider the optimal feedback control design for linear periodic time-delay systems.

Equation (22) is a general linear periodic time-delay system with feedback control, which can be either delayed or not.

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \sum_{j=1}^{m}[\mathbf{B}_j(t)\mathbf{x}(t-\tau_j) + \mathbf{C}_j(t)\dot{\mathbf{x}}(t-\tau_j)] + \mathbf{D}(t)\mathbf{u}(t), \quad (22)$$

where $\mathbf{x} \in \mathbf{R}^N$, $\mathbf{u} \in \mathbf{R}^M$. $\mathbf{A}(t)$, $\mathbf{B}_j(t)$, $\mathbf{C}_j(t) \in \mathbf{R}^{N\times N}$, and $\mathbf{D}(t) \in \mathbf{R}^{N\times M}$ are periodic matrices with period T.

According to whether delays exist in $\mathbf{u}(t)$ or not, $\mathbf{u}(t)$ have two different forms which are represented by (23).

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) \text{ or } \mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t-\tau_u). \quad (23)$$

where $\mathbf{K} \in \mathbf{R}^{M\times N}$ is the gain matrix.

Obviously, (22) is able to be transformed into the form of (1). Therefore, the stability region and stability boundary can be obtained by using stability criteria presented in Subsection A. Furthermore, optimal control gains can also be found, and the search scheme is discussed as follows.

If the problem is restricted in a finite and compact region $\mathbf{\Omega}$ in the parametric space $\mathbf{K}$, we can find the regions of stability and optimal control gains in the region to minimize $|\lambda|_{\max}$. This leads to one optimization problem:

$$\min_{\mathbf{K}\in\mathbf{\Omega}}[\max|\lambda(\mathbf{\Phi})|] \text{ subject to } |\lambda|_{\max} < 1. \quad (24)$$

This optimization formulation offers an approach to design feedback controls for linear periodic time-delay systems, and the control performance criterion is chosen as the decay rate of the mapping $\mathbf{\Phi}$ over one period. For some further discussion, please refer to Sheng and Sun [15].

III. EXAMPLES

In this section, two examples are presented to illustrate semi-discretization's validity. All computations are carried out by Matlab R2010b.

EXAMPLE 1: Equation (25) describes a linear time-invariant neutral system.

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{x}(t-\tau) + \mathbf{C}\dot{\mathbf{x}}(t-\tau), \quad (25)$$

where

$$\mathbf{A} = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},\ \mathbf{C} = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix},\ 0 \le c \le 1.$$

When $c = 0$, this system is simplified to be a linear time-delay system with a single time-delay, so Corollary 1 is applied to get the upper stability bound of τ; when $c \ne 0$, this system is a linear neutral system, and Theorem 1 is used.

The upper stability bound is calculated with different criteria, which are given in [1, 2, 4 and 6]. The comparison with the criteria presented in this paper is listed in Table I.

Table I suggests that the upper bounds obtained by the methods in this paper are much less conservative and much more accurate than those of [1, 2 and 4], and meanwhile they are comparable with those of [6].

TABLE I. COMPARISON OF TIME-DELAY'S UPPER BOUND FOR EXAMPLE 1

Criteria \ c	0	0.5	0.9
[1]	4.47	1.14	0.13
[2]	4.35	3.62	0.99
[4]	4.47	3.67	1.41
[6]	6.15-6.2	4.735-4.74	1.57-1.575
Semi-Discretization	6.17	4.67	1.52

EXAMPLE 2: A linear periodic neutral system is given by (26).

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \sum_{j=1}^{2}[\mathbf{B}_j\mathbf{x}(t-\tau_j) + \mathbf{C}_j\dot{\mathbf{x}}(t-\tau_j)], \quad (26)$$

where

$$\mathbf{A}(t) = \begin{bmatrix} -0.9 & -1 \\ 1+0.3\sin 2t & 0.9+0.4\cos 4t \end{bmatrix},$$
$$\mathbf{B}_1 = \begin{bmatrix} 0 & 0 \\ -k_1 & -k_2 \end{bmatrix},\ \mathbf{B}_2 = \begin{bmatrix} 1 & 0.4 \\ 0 & 1 \end{bmatrix},$$
$$\mathbf{C}_1 = \begin{bmatrix} 0 & -0.4 \\ -0.5 & 0 \end{bmatrix},\ \mathbf{C}_2 = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.5 \end{bmatrix},$$
$$\tau_1 = \frac{\pi}{10},\ \tau_2 = \frac{\pi}{5}.$$

In this example, the coefficient matrix $\mathbf{A}(t)$ is periodic with period π, and the feedback controller is delayed with delay $\pi/10$.

Theorem 1 is utilized for searching the stability region on the $k_1 - k_2$ plane. Besides, the optimal pair of control gains is also found by solving the optimization problem stated by (24).

The stability region and the optimal control gain pair are shown by Fig. 1, where the labels of contours are the maximum absolute values of eigenvalues of the mapping matrices, i.e. $|\lambda|_{\max}$. The stability region is within the bold black line, and the optimal gain pair is $(k_1, k_2) = (0.3556, 2.8256)$, with $|\lambda|_{\max}$ = 0.3234, which is indicated by red "+". Besides, the blue circle and black triangle indicate $(0.8, 3)$ and $(1, 2)$, respectively.

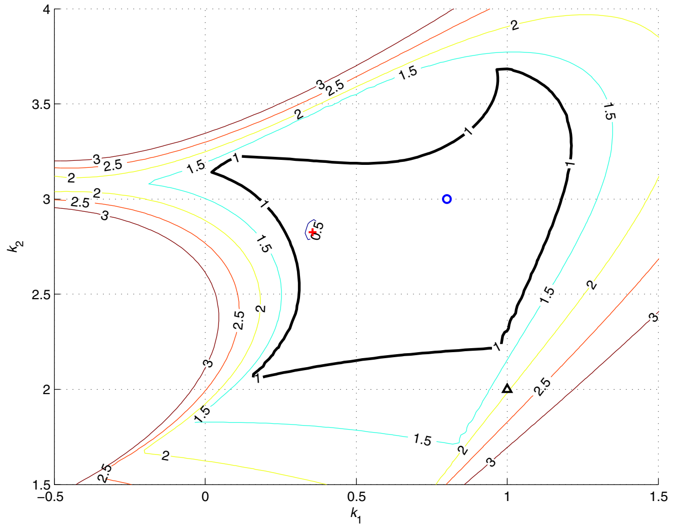

Figure 1. Stability region and the optimal pair of control gains.

Fig. 2 shows simulation results when different pairs of control gains are applied for (26), where the horizontal axis is simulation time, and the vertical axis is the norm of state, which is defined as $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2}$. The initial conditions are set to be $\mathbf{x}_0 = [0 \quad 0.1]^T$. Note that $(0.3556, 2.8256)$ is the optimal controller, and therefore the corresponding system response converges to zero fastest. $(0.8, 3)$ is within the stability region, but the corresponding $|\lambda|_{\max}$ is 0.8403, which is larger than that when the optimal controller is applied, so the system response converges to zero asymptotically yet slowly. $(1, 2)$ is one pair causing the system to be unstable, with $|\lambda|_{\max} = 1.9439 > 1$, and hence the system response increases very quickly.

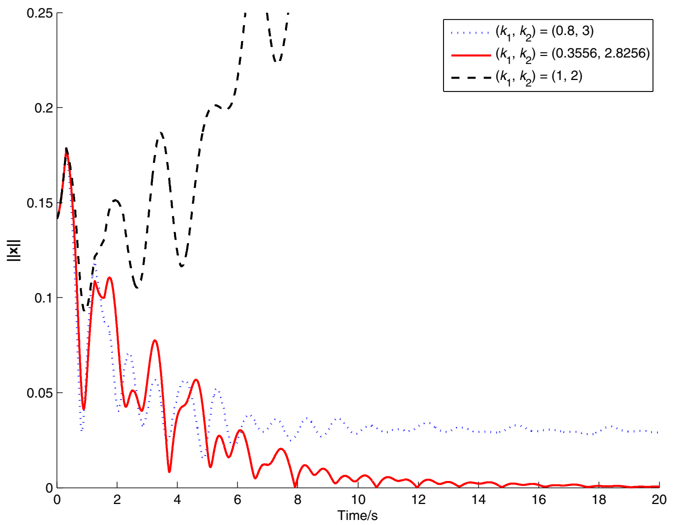

Figure 2. System responses with different control gains.

IV. CONCLUSION

In this paper, stability analysis and control design of linear periodic time-delay systems have been investigated. Based on the semi-discretization method, stability criteria have been derived for different types of linear periodic time-delay systems, and optimal feedback control design scheme has also been presented. Two illustrative examples have been given to demonstrate the merits of the obtained results.

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