UKACC International Conference on Control 2012
Cardiff, UK, 3-5 September 2012

Isochronal synchronization in complex networks

The Lyapunov-Krasovskii theorem and stability in the network parameter space

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***Abstract*—Isochronal synchronization is a unique phenomenon in which physically distant oscillators wired together relax into zero-lag synchronous behavior over time. Such behavior is observed in natural processes and, recently, has been considered for promising applications in communication. Towards technological development of devices that explore isochronal sync, stability issues of the phenomenon need to be considered, both in the context of a pair or a network of coupled oscillators. This study concerns such stability issues by using the Lyapunov-Krasovskii theorem to propose a framework to study synchronization stability by using accessible parameters of the network coupling setup. As a result, relations between stability and network parameters are unveiled and the comprehension of roads leading to stability is enhanced.**

Keywords: complex networks, isochronal synchronization, Lyapunov-Krasovskii.

I. INTRODUCTION

The communication among physically distant entities is subject to time delays. This is a result of the finite time a signal requires to travel through a physical media the distance from an emitter to a receiver. Curiously, despite of such condition, chaotic oscillators were revealed to overcome time delay and synchronize with zero-lag [1-10]. Under bidirectional coupling and adequate conditions, coupling setups of chaotic oscillators are somehow capable of absorbing coupling delays and synchronize as if no delays were present in the communication process at all [1, 8].

The knowledge of such fact brought promising applications for chaos in communication, such as those presented in [1, 8]. In this context, a fundamental requirement for communication is the stability of isochronal synchronization under small disturbances [6, 7]. This is so because information itself is introduced in the communication process as a disturbance to the synchronous dynamics [1]. As such, the communication process per se is a sequence of desync and resync episodes which can be understood as the transmission of bits through the coupling link between chaotic oscillators [1]. Resync, specifically, is possible if sync is stable, and the need for stability is justified.

In the scenario featuring pairs of delay-coupled oscillators, analytical results may establish frameworks that allow the systematic study of stability of isochronal sync in rather straightforward manners. This paper presents analytical results and their use towards the comprehension of the relations between network sync stability and the network parameters. The phenomenon of isochronal sync is considered, which implies the presence of time delays in the coupling among the oscillators. On this basis, the network parameter space is swept in a search process to map regions of sync stability. The relation between the number of nodes of the network and the number of links among its nodes is shown to be critical to the stability of isochronal sync, as it was observed to be for other types of synchronization as well [12].

The paper is organized as follows: section 2 presents the theoretical basis upon which the developments and explorations are based; section 3 explores the theoretical results towards the understanding of the underlying mechanisms of sync stability; section 4 presents some final remarks.

II. ISOCHRONAL SYNC STABILITY OVER THE NETWORK PARAMETER SPACE

A. Isochronal sync and the network error equations

Consider the systems

$$\dot{x}_i(t) = Ax_i(t) + g(x_i(t)) + u_i(t) \qquad (1)$$

where $x(t) \in R^n$ is a state vector, $A \in R^n \times R^n$ is a constant matrix and $g(.) \in R^n \to R^n$ is a continuous vector field, such that the nodes are coupled exclusively by the control function $u_i(t)$ which is a feedback control function given by

$$u_i(t) = \tfrac{1}{G_{ii}} K \left(G_{ii} x_i(t) - \sum_{\substack{j=1 \\ j \neq i}}^{N} G_{ij} x_j(t-\tau) \right) \qquad (2)$$

and G_{ij} are entries of the Laplacian matrix G and $K = K^T$ is a feedback matrix to be designed. The symmetry property of the feedback matrix is desirable as it allows some simplifications later on the development. Note that equation (2) can be rewritten as

$$u_i(t) = KC_{ii} \sum_{j=1}^{N} G_{ij} x_j(t-\tau_{ij}) \qquad (3)$$

and the equations for the *ith* node of the network ($i = 1, 2, ..., N$) as

$$\dot{x}_i(t) = Ax_i(t) + g(x_i(t)) + KC_{ii}\sum_{j=1}^{N} G_{ij}x_j(t - \tau_{ij}) \quad (4)$$

where $\tau_{ij} = \tau$, for $i \neq j$, $\tau_{ij} = 0$, otherwise, and

$$C_{ii} = -\frac{1}{G_{ii}} \quad (5)$$

for node balance. Alternatively, collecting the *ith* term of the summation (3), the self-coupling, for which $\tau = 0$, and reorganizing the terms of equation (4), it now reads

$$\dot{x}_i(t) = (A - K)x_i(t) + g(x_i(t)) + KC_{ii}\sum_{\substack{j=1 \\ j\neq i}}^{N} G_{ij}x_j(t - \tau) \quad (6)$$

In the following, the state vectors of the network nodes, $x_1(t), x_2(t), ..., x_N(t)$ are collected into the network state vector $X(t) \in \mathbb{R}^{Nn}$, given by

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{pmatrix} \quad (7)$$

Thus, the equations of the delay-coupled network can be written in the compact form

$$\dot{X}(t) = I_N \otimes (A - K)X(t) + \bar{G}(X(t)) + CA_d \otimes KX(t - \tau) \quad (8)$$

where I_N is an *N-dimensional* identity matrix, $\otimes$ is the Kronecker product,

$$\bar{G}(X(t)) = \begin{pmatrix} g(x_1(t)) \\ \vdots \\ g(x_N(t)) \end{pmatrix} \quad (9)$$

is a nonlinear vector field and A_d is the network adjacency matrix that assigns $A_{d_{ij}} = A_{d_{ji}} = -1$ if nodes i and j are connected, $A_{d_{ij}} = A_{d_{ji}} = 0$ otherwise and, $A_{d_{ii}} = 0$.

At this point, the dynamical equations of the network are available and the formulation of the synchronization problem requires the definition of an error vector function $e(t)$, such that, if $\|e(t)\| \to 0$ as $t \to \infty$, then the N systems of the network are asymptotically synchronized. Towards that end, considering the definition of isochronal synchronization, one can define

$$e(t) = X(t) - X_s(t) \quad (10)$$

where

$$X_s(t) = SX(t) \quad (11)$$

being $S \in \mathbb{R}^{Nn} \times \mathbb{R}^{Nn}$ an appropriate vector coordinate transformation defined as $S = T \otimes I_n$ and being T defined as

$$T = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1N} \\ T_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ T_{N1} & \cdots & \cdots & T_{NN} \end{pmatrix} \quad (12)$$

where either $T_{ij} = 1$ or $T_{ij} = 0$ and $\sum_i T_{ij} = \sum_j T_{ij} = 1$, such that $rank(T) = N$ and

$$rank(I_N - T) = N - 1 \quad (13)$$

It can be verified that (10) yields

$$\begin{aligned} \dot{e}(t) &= \left(A - K + M_{(X(t),X_s(t))}\right)e(t) + \left(CA_d \otimes K - S[CA_d \otimes K]\right)X(t - \tau) \\ &\left(A - K + M_{(X(t),X_s(t))}\right)e(t) + \left(I_{Nn} - S\right)\left(CA_d \otimes K\right)X(t - \tau) \end{aligned} \quad (14)$$

where $M_{(X(t),X_s(t))}e(t) = \bar{G}(X(t)) - S\bar{G}(X(t))$. Note that $S\bar{G}(X(t)) = \bar{G}(X_s(t))$.

At this point, note that system (14) depends both on the error states $e(t)$ and the network states $X(t)$, which is undesirable, since the synchronization stability evaluation must be performed in the error equations alone. However, considering equations (10), (11), one can rewrite systems (14) in terms of the error variable $e(t)$ by designing an adequate matrix of coefficients E such that

$$\begin{aligned} \left(I_{Nn} - S\right)\left(CA_d \otimes K\right)X(t - \tau) &= E\left(X(t - \tau) - X_s(t - \tau)\right) \\ &= E\left(I_{Nn} - S\right)X(t - \tau) \\ &= Ee(t - \tau) \end{aligned} \quad (15)$$

and, finally,

$$\left(I_{Nn} - S\right)\left(CA_d \otimes K\right) = E\left(I_{Nn} - S\right) \quad (16)$$

Since the matrix $\left(I_{Nn} - S\right)$ is singular due to the definition of S, the equation cannot be solved for E analytically. However, it is possible to prove that a solution exists, such that equation (14) can be rewritten in the form

$$\dot{e}(t) = I_N \otimes \left(A - K + M_{(X(t),X_s(t))}\right)e(t) + Ee(t - \tau) \quad (17)$$

which depends only on the error variables $e(t)$. Note that $\|e(t)\| = 0$ implies $x_1(t) = x_2(t) = ... = x_N(t)$, which imply that the network is asymptotically synchronized. In other words, the stability of network synchronization requires that the error system (17) asymptotically establishes at the trivial fixed point.

Towards the determination of conditions for synchronization stability, consider the identity $e(t - \tau) = e(t) - \int_{t-\tau}^{t} \dot{e}(\theta)d\theta$, such that the error equations (17) can be rewritten as

$$\dot{e}(t) = I_N \otimes \left(A - K + M_{(X(t),X_s(t))}\right)e(t) + E\left(e(t) - \int_{t-\tau}^{t} \dot{e}(\theta)d\theta\right) \quad (18)$$

and, one step ahead,

$$\begin{aligned} \dot{e}(t) &= \left[I_N \otimes \left(A - K + M_{(X(t),X_s(t))}\right) + E\right]e(t) + \\ &- E\int_{t-\tau}^{t} \left[I_N \otimes \left(A - K + M_{(X(t),X_s(t))}\right)e(\theta) + Ee(\theta - \tau)\right]d\theta \end{aligned} \quad (19)$$

such that the error system is obtained.

Notice that according to our formulation, the definition of the error system dismisses the use of reference signals. It follows that the asymptotic stability of the error system (19) means that of the synchronization of the network of oscillators

(8). In the next section, a stability criterion for the trivial fixed point of the error system (19) is derived, by means of the Lyapunov-Krasovskii stability theorem [11].

B. Criterion for Isochronal sync stability

Consider the upper bound of the Lipschitz constant of the nonlinear vector field $g(.)$ over the invariant set Ω constituted by the trajectory of $x(t)$ to be given by ℓ, such that $L = \ell I$ and I is the n-dimensional identity matrix. The following result can be established [7]:

Theorem [Isochronal synchronization of delay-coupled complex networks]. *If there exists a constant matrix $P = P^T > 0$ a positive constant $\varepsilon > 0$ and a feedback gain matrix K such that*

$$W = Q - 2\varepsilon I_N \otimes A^T A - 2\varepsilon E^T E - \frac{\tau}{\varepsilon} E^T P^2 E > 0 \qquad (20)$$

holds for a matrix $Q = Q^T > 0$, where Q is given by equation

$$-Q = \left[I_N \otimes (A - K + L) + E\right]^T P + P\left[I_N \otimes (A - K + L) + E\right] \qquad (21)$$

then the delay-coupled network whose error system given by equation (19) achieves isochronal synchronization for coupling delay τ.

III. INVESTIGATING ISOCHRONAL SYNC STABILITY

This section explores the relations between (i) synchronization stability and network topology, (ii) synchronization stability and coupling delays, based on the analytical results presented in [7]. The objective is to trace the influence of such network parameters in synchronization stability.

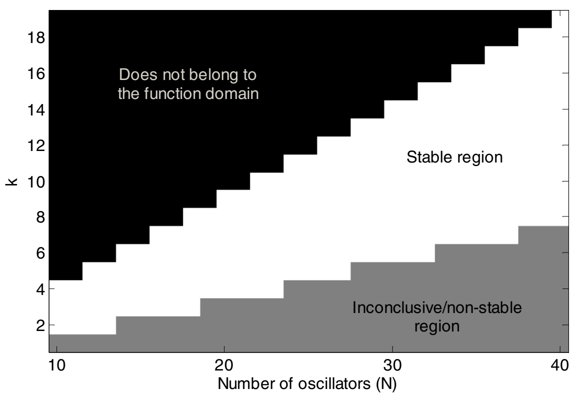

Figure 1 - Example of application of the network isochronal sync stability criterion as a stability function over the parameter domain: stability regions can be mapped from accessible network parameters. Since the stability criterion is sufficient (but not necessary), a the gray region is traced for the parameter values for which (20) is not satisfied.

A. Sync stability versus network topology

In this example, the regions of stable isochronal synchronization are established for *k-cycle* networks on the basis of the inequality (20). Such regions are subsets of the cartesian product $N \times k$. A similar systematic approach can be used to trace such map for other kinds of network, as other parameters are chosen. To make the visual analysis simple, a $2-$ *dimensional* parameter meshgrid is generated. From Figure 1, Figure 2, Figure 3 and Figure 4, the loss of stability can be traced towards the identification of the parameters that are most influential to its occurrence. It can be recognized that smaller proportions of links among nodes relatively to the number of nodes in the network has negative effect on stability of isochronal sync. This effect is illustrated by arrows which show the direction of the loss of stability. Further analysis and insights into the nature of network stability are possible as other parameters are considered in the generation of the domain for evaluation of other similar stability functions resulting from the analytical results presented in this paper. In the following, similar studies considering stability under different values of coupling delay are considered.

B. Time-delays versus number of links in k-cycle networks

At first, the form that the delay term appears in the inequality (20) suggests that stability is degraded for larger values of delay. Although this is not a general rule, the qualitative behavior of DDEs is shown to change as delays increase, due to the occurrence of Hopf bifurcations [13], which induce oscillatory and unstable behavior. The stability map in the parameter space $\tau \times k$ shows this relation, and it is illustrated in Figure 4 and Figure 5.

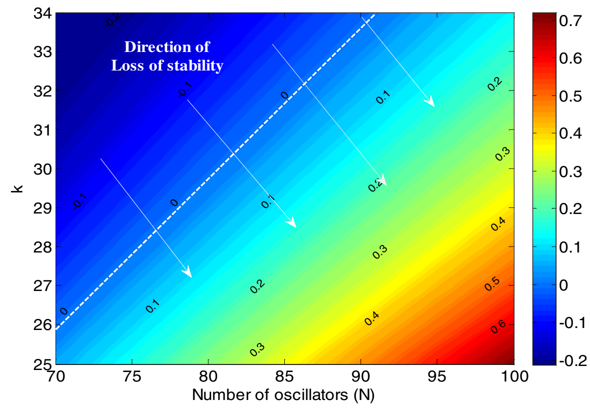

Figure 2 – Stability function over the set $N \times k$ of parameter space of the of the k-cycle network of Lorenz oscillators: the stability region has negative maximum eigenvalue of $-W$ and sync stability is lost as the number of oscillators increases and/or the number of links decreases.

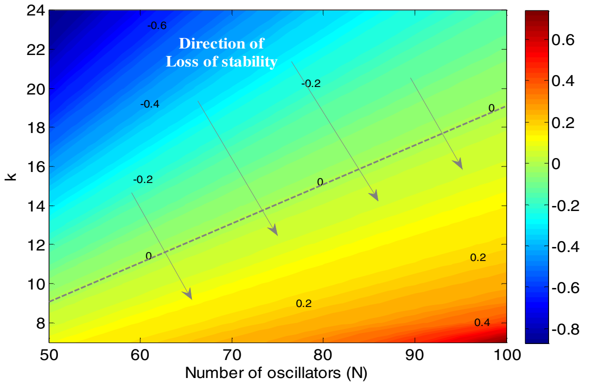

Figure 3 - Stability function over the set $N \times k$ of the parameter space of the k-cycle network of Rössler oscillators: the stability region has negative maximum eigenvalue of $-W$ and sync stability is lost as the number of oscillators increases and/or the number of links decreases.

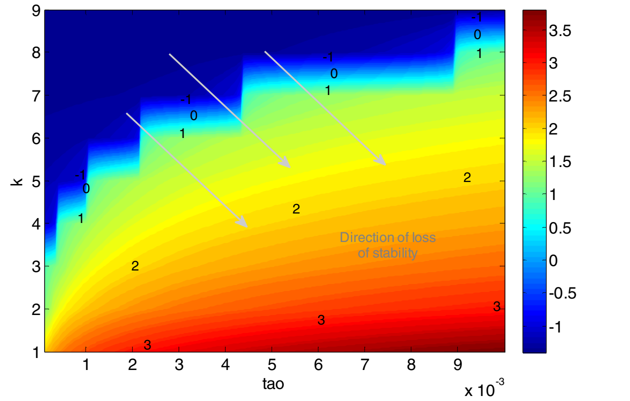

Figure 4 – Stability function over the set $\tau \times k$ of the parameter space of a k-cycle network with N = 20 Rössler oscillators: the region of stable sync has negative maximum eigenvalue of $-W$. In this case, sync stability is lost as the number of links decreases and the time-delay increases.

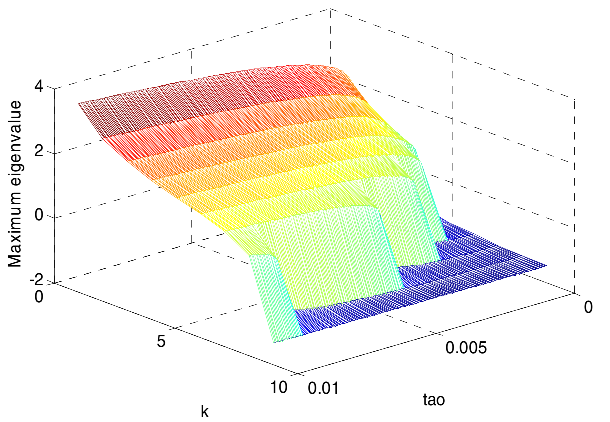

Figure 5 – Mesh of the stability function plotted in Figure 4: few links among nodes and larger values of time-delays render network synchronization less prone to synchronization.

C. Sync stability versus individual node dynamics

A careful observation of the stability criterion (20) reveals that larger values of the Lipschitz ℓ constant also contribute negatively to stability. Moreover, it can be concluded that the substitution of the nonlinear terms of the nodes' dynamics with their upper bound, performed in the development of the stability criterion, is a source of restrictiveness, since it considers a worst-case scenario.

From a different perspective, this also illustrates the fact that some chaotic networks are more likely to synchronize than others, depending on the intrinsic dynamics of its nodes. Moreover, larger upper bounds of the nonlinear vector field g over $x(t)$ reflect on the entries of the feedback matrix K , which are supposed to lead the network nodes to isochronal synchronization, and thus counteract the effect of perturbations. Note that the value of K also affects the delay term and, as such, may as well act as a perturbation to stability. Thus, not necessarily larger values of the entries of K imply enhancement of stability, as observed also in non-delayed networks [12].

IV. Final Remarks

The underlying mechanisms of network synchronization are topics of great interest within the context of complex networks. This paper presented analytical results that allow the determination of stability of isochronal synchronization in complex networks of delay-coupled chaotic oscillators. The relations between stability and the parameters of the network, as developed in this paper, help in the understanding of such mechanisms.

Acknowledgment

The authors thank FAPESP and CPNq for the support on this research.

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