UKACC International Conference on Control 2012
Cardiff, UK, 3-5 September 2012

A variable structure observer for unknown input estimation in sampled systems

L. Orihuela, S. K. Spurgeon, X. G. Yan and F. R. Rubio

***Abstract*— This paper considers the design of a variable structure observer for unknown input estimation and/or fault reconstruction in systems where the process measurements are sampled. It is well known that the principle of the equivalent injection signal from the sliding mode domain can be used for reconstruction of unknown inputs but much of the associated theory is predicated on output sampling of infinite frequency. Sample rate may be a physical constraint of the process and the reconstruction properties of such continuous time sliding mode observers degrade under this constraint. This paper explores how a recently developed ultimately bounded stable variable structure discrete time observer can be used for unknown input estimation. The main novelty of the approach is that the design of the observer is written as an optimization problem with linear constraints with the output sampling incorporated explicitly in the model used for observer design. The design methodology is shown to have advantages in terms of reconstruction accuracy when the performance is compared to that of a classical sliding mode observer on a case study.**

I. INTRODUCTION

Variable structure systems were perhaps originally best known for their potential as a robust control method [1], [2], [3]. They are characterised by a suite of feedback control laws and a decision rule. The decision rule, termed the switching function, has as its input some measure of the current system behaviour and produces as an output the particular feedback controller which should be used at that instant in time. In sliding mode control, variable structure control systems are designed to drive and then constrain the system state to lie within a neighbourhood of the switching function. The paradigm has several advantages: the dynamic behaviour of the system may be specified by the choice of switching function and the system is completely insensitive to an important class of uncertainties. A disadvantage of the methodology has been the fundamentally discontinuous control signal which, in theoretical terms, must switch with infinite frequency to provide total rejection of uncertainty. Control implementation via approximate, smooth strategies is widely reported, but in such cases total invariance is routinely lost.

In contrast, the application of sliding mode methods to the observer problem is less mature and has some fundamentally different properties [4]. The discontinuous injection signals which were perceived as problematic for many control applications have no disadvantages for software based observer frameworks. The ability to generate a sliding motion on the error between the measured plant output and the output of the observer ensures that a sliding mode observer produces a set of state estimates that are precisely commensurate with the actual output of the plant. Further, analysis of the average value of the applied observer injection signal, the so-called equivalent injection signal, contains useful information about the mismatch between the model used to define the observer and the actual plant [5]. This property has been employed for general unknown input estimation as well as for fault reconstruction [6], [7], [8]. The results obtained to date most frequently require that an ideal sliding motion is attained in finite time and the effects of sampling on the physical measurements used to drive the observer are typically not considered within the observer design frameworks. However, in the presence of a sampled output, the ideal sliding mode cannot be achieved. Indeed, the error dynamics in the observer may become unstable if the sampling frequency is reduced significantly. The effect of output sampling on the performance of a sliding mode observer designed using classical continuous variable structure control theory has been discussed by several authors, see for example [9], where the fast sampling required for fault reconstruction via such a sliding mode observer on a motor experiment is reported.

In practice, the sample rate is not always a parameter that can be selected by the designer and in this case consideration must be given to developing design methods that incorporate the sampling characteristics if good estimates are to be obtained for the unknown inputs. Recent work has considered the development of a sliding mode observer in the presence of sampled output information and its application to fault reconstruction by using the delayed continuous-time representation of the sampled-data system, for which a set of Linear Matrix Inequalities (LMIs) provide conditions for ultimate boundedness of the solution [10]. An alternative approach is to consider a discrete time observer design methodology. Compared to the continuous time case, the literature in this area is sparse. Several contributions develop sliding mode observers for systems with a single output [11], [12], [13]. More recently, [14] studied sliding mode observers for a class of discrete-time multi-output systems, but the design of the observer is largely heuristic, and it is not possible to ensure *a priori* the stability of the observer. A variable structure observer design framework for discrete-time multi-output systems which uses Linear Matrix Inequalities to constructively exploit the degrees of freedom

L. Orihuela and F. R. Rubio would like to acknowledge CICYT (Grant DPI2010-19154) for funding this work.

L. Orihuela and F. R. Rubio are with with Dpto. de Ingeniería de Sistemas y Automática. Escuela Superior de Ingenieros. University of Seville, Spain orihuela,rubio@us.es

S. K. Spurgeon and X. G. Yan are with the School of Engineering and Digital Arts. University of Kent, United Kingdom S.K.Spurgeon, X.Yan@kent.ac.uk

978-1-4673-1558-6/12/$31.00 ©2012 IEEE

within the design has been recently developed [15].

This paper extends this later framework to incorporate the estimation of faults and/or unknown inputs as a design requirement and assesses the degree to which the methodology overcomes problems of sampled data implementation frameworks for variable structure based signal reconstruction.

The paper is structured as follows. The problem is stated in Section II. The variable structure observer is proposed in Section III. The estimation of unknown inputs using the discrete-time observer is studied in Section IV. Some examples are presented in Section V. The paper ends with conclusions and a discussion of future research directions.

II. PROBLEM STATEMENT

Consider the multi-output, discrete-time linear system described by

$$x(k+1) = Ax(k) + Dw(k), \quad (1)$$
$$y(k) = Cx(k), \quad (2)$$

where $x \in \mathcal{R}^n$ is the state, $y \in \mathcal{R}^p$ is the output and $w \in \mathcal{R}^m$ is the unknown input. Matrices A, D, C are known with appropriate dimensions. It is assumed that the pair (A, C) is observable. Note that no control input affects the system, as the observer problem is the focus of this paper. The inclusion of the control signal can be trivially dealt with, as both the observer and the system are subject to the same input and thus the control signal has no affect on the dynamics of the error between the system and observer.

Assume that $rank(CD) = m$ and the invariant zeros of the triple (A, D, C) lie inside the unit circle. Then, there exists a linear change of coordinates T_o (see [3]) such that the system can be written as:

$$x_1(k+1) = A_{11}x_1(k) + A_{12}z(k) + D_1w(k), \quad (3)$$
$$z(k+1) = A_{21}x_1(k) + A_{22}z(k) + D_2w(k), \quad (4)$$
$$y(k) = z(k), \quad (5)$$

where $z(k) \in \mathcal{R}^p$, $x_1 \in \mathcal{R}^{n-p}$ and $D_2 \in \mathcal{R}^{p\times m}$. Matrix A_{11} is stable and D_2 has full column rank.

Unlike [10] and similar approaches, the disturbance process or unknown input $w(k)$ affects both dynamics. The classical nomenclature used in sliding mode theory, namely, unmatched and matched disturbances has been adopted (see [3]). It is assumed to be bounded by

$$\|w(k)\| \leq \xi, \ \forall k,$$

where ξ is a known positive scalar.

The following section is devoted to the design of an ultimately bounded variable structure observer for the system (1), by driving the observation error $e(k)$ to the vicinity of the equilibrium point $e(k) = 0$ in finite time and maintaining it in the neighbourhood thereafter. Due to the presence of the disturbances, asymptotic stability is not possible. However, the proposed observer reduces the ultimate bound on the response when compared to classical observation strategies. In Section IV, the properties of this observer will be used in order to estimate the unknown input $w(k)$.

III. VARIABLE STRUCTURE OBSERVER

The proposed variable structure observer for the multi-output system (3)-(5) is defined as

$$\hat{x}_1(k+1) = A_{11}\hat{x}_1(k) + A_{12}y(k) \quad (6)$$
$$\hat{z}(k+1) = A_{21}\hat{x}_1(k) + A_{22}\hat{z}(k) - (A_{22} - A_{22}^s)e_z(k) + \nu(k) \quad (7)$$

where $e_z(k) = \hat{z}(k) - y(k)$ and $A_{22}^s \in \mathcal{R}^{p\times p}$ is a design matrix. The variable structure term $\nu(k)$ is defined by:

$$\nu(k) = Bf_{sat}(e_z(k), \Delta) = B \begin{bmatrix} sat\left(\frac{e_{z1}(k)}{\Delta}\right) \\ sat\left(\frac{e_{z2}(k)}{\Delta}\right) \\ \vdots \\ sat\left(\frac{e_{zp}(k)}{\Delta}\right) \end{bmatrix} \quad (8)$$

where Δ is a positive scalar and $B \in \mathcal{R}^{p\times p}$ is also a design matrix. The function $sat(.)$ is defined as:

$$sat\left(\frac{e_{zi}(k)}{\Delta}\right) = \begin{cases} sgn(e_{zi}(k)), & |e_{zi}(k)| > \Delta \\ \frac{e_{zi}(k)}{\Delta}, & |e_{zi}(k)| \leq \Delta \end{cases}$$

It can be viewed as a set of p unidimensional switching functions. Each one switches whenever the associated component of the output observation error e_{zi}, $i = 1..., p$ crosses the boundary of the region.

Let the state estimation errors be $e_1(k) = \hat{x}_1(k) - x_1(k)$ and $e_z(k) = \hat{z}(k) - z(k)$. It follows that the error dynamics are

$$e_1(k+1) = A_{11}e_1(k) - D_1w(k), \quad (9)$$
$$e_z(k+1) = A_{21}e_1(k) + A_{22}^s e_z(k) - D_2w(k) + \nu(k). \quad (10)$$

where $y(k) = z(k)$ in (5) is used to obtain the equations above.

In order to study the stability of the observer, a Lyapunov framework is used. Defining Δ as the ultimate boundedness, the objective of this section is to design the variable structure observer in such a way that Δ is minimized ensuring that the forward increment of the Lyapunov function is negative for all k such that $|e_z(k)| \geq \Delta$. Consider the following Lyapunov function:

$$V(k) = e_1^T(k)P_1e_1(k) + e_z^T(k)P_2e_z(k), \quad (11)$$

where P_1, P_2 are positive definite matrices of appropriate dimensions.

Due to the presence of the saturation, $\nu(k)$ is a nonlinear function. A linear representation of the saturation is introduced which will be useful when designing the observer via linear matrix inequalities. This idea was presented in [15].

Denote by $\mathfrak{E}_\beta$ the set of states such that

$$\mathfrak{E}_\beta = \{(e_1(k), e_z(k)) \mid e_1(k) \in \mathcal{R}^{n-p}, e_z(k) \in \mathcal{R}^p, \\ V(k) = e_1^T(k)P_1e_1(k) + e_z^T(k)P_2e_z(k) \leq \beta^{-1}\},$$

for a positive scalar β. The following lemma gives the linear representation of the nonlinear dynamics.

Lemma 1. [15] Given $\beta > 0$, assume that there exists a matrix $H_z \in \mathcal{R}^{p\times p}$, such that $|h_{zi}e_z| \le 1$, for all $e_z \in \mathfrak{E}_\beta$, where h_{zi} denotes the i-th row of H_z. Then, for $(e_1, e_z) \in \mathfrak{E}_\beta$, the observation error dynamic system (9)-(10) with switching function (8) admits the following representation:

$$\begin{aligned} e_1(k+1) &= A_{11}e_1(k) - D_1w(k), \\ e_z(k+1) &= A_{21}e_1(k) + A_{22}^s e_z(k) - D_2w(k) \\ &+ B\sum_{j=1}^{2^p}\lambda_j(k)A_je_z(k), \end{aligned}$$

where:

$$\begin{aligned} A_j &= F_jK + F_j^- H_z, \quad j = 1,..,2^p, \\ \sum_{j=1}^{2^p}\lambda_j(k) &= 1, \quad \lambda_j(k) \ge 0, \forall k > 0, \\ K &= diag\{1/\Delta, ..., 1/\Delta\}, \end{aligned}$$

with F_j a diagonal matrix with diagonal elements that are either 1 or 0, and $F_j^- \triangleq I_m - F_j, \forall j$.

From Lemma 1, it can be concluded that the linear representation for the saturation is valid only if the region $\mathfrak{E}_\beta$ (defined by the parameter β) and a corresponding matrix H_z can be found such that the error remains in $\mathfrak{E}_\beta$ for all k. Hence, this must be an additional constraint in the design of the observer.

The following theorem presents the main result of this section.

Theorem 1. Given the disturbance bound ξ and the size β of the set $\mathfrak{E}_\beta$, if positive definite matrices P_1, P_2, matrices A_{22}^s, B, H_z of appropriate dimensions and scalar $\tau > 0$ solve the following optimization problem

$$\min_{A_{22}^s, B, P_1, P_2, H_z, \tau} \Delta \tag{12}$$

subject to (13)-(14), then the observation error dynamic system (9)-(10) is ultimately bounded stable. The minimum boundedness is Δ.

Proof. It is first proved that conditions (13)-(14) imply that, given a positive Δ, the forward increment of the Lyapunov function (11) decreases for all $|e_z(k)| \ge \Delta$.

The inequalities (14) guarantee that $|h_{zi}e_z| \le 1$, ($i = 1, ..., p$) for all $(e_1, e_z) \in \mathfrak{E}_\beta$. This results from the fact that any error belonging to $\mathfrak{E}_\beta$ satisfies

$$\begin{aligned} &\beta e_1^T(k)P_1e_1(k) + \beta e_z^T(k)P_2e_z(k) \le 1 \\ &\qquad \Rightarrow \beta e_z^T(k)P_2e_z(k) \le 1 \end{aligned}$$

Then, for $(e_1, e_z) \in \mathfrak{E}_\beta$, the following inequalities

$$2 \ge 1 + \beta e_z^T(k)P_2e_z(k) \ge 2|h_{zi}e_z|$$

imply that $|h_{zi}e_z| \le 1$ for $i = 1, ..., p$. The latter inequality, which can be written as

$$\begin{bmatrix} 1 & \pm e_z^T \end{bmatrix} \begin{bmatrix} 1 & h_{zi} \\ * & \beta P_2 \end{bmatrix} \begin{bmatrix} 1 \\ \pm e_z \end{bmatrix} \ge 0,$$

is satisfied by (14).

As the assumption of Lemma 1 is verified, the polytopic description of the system given in that lemma holds. Using this linear representation, the Lyapunov function (11) at $k+1$ for vertex j of the polytope is[1]

$$\begin{aligned} &V_j(k+1) = \\ &= (A_{11}e_1 - D_1w)^T P_1 (A_{11}e_1 - D_1w) \\ &+ \big(A_{21}e_1 + M_{2j}e_z - D_2w\big)^T P_2 \big(A_{21}e_1 + M_{2j}e_z - D_2w\big), \end{aligned}$$

where

$$M_{2j} = A_{22}^s + BF_jK + BF_j^- H_z.$$

The forward increment of the Lyapunov function will be

$$\begin{aligned} \Delta V_j(k) =\ & V_j(k+1) - V_j(k) \\ &= -e_1^TQ_1e_1 - 2e_1^TA_{11}^TP_1D_1w + w^TD_1^TP_1D_1w \\ &+ e_1^TA_{21}^TP_2A_{21}e_1 + 2e_1^TA_{21}^TP_2M_{2j}e_z \\ &- 2e_1^TA_{21}^TP_2D_2w + e_z^T(M_{2j}^TP_2M_{2j} - P_2)e_z \\ &- 2e_z^TM_{2j}^TP_2D_2w + w^TD_2^TP_2D_2w \end{aligned} \tag{15}$$

where $-Q_1 \triangleq A_{11}^TP_1A_{11} - P_1$. The positive term $\tau w^T(k)w(k)$ can be bounded by:

$$\tau w^T(k)w(k) \le \tau\xi^2 \le \frac{\tau\xi^2}{\Delta^2}e_z^T(k)e_z(k),$$

taking into account that $\|e_z(k)\| \ge \Delta$. Therefore,

$$\frac{\tau\xi^2}{\Delta^2}e_z^T(k)e_z(k) - \tau w^T(k)w(k) \ge 0 \tag{16}$$

From (15) and (16), it follows that

$$\begin{aligned} \Delta V_j(k) &= V_j(k+1) - V_j(k) \\ &= -e_1^T\left(A_{11}^TP_1A_{11} - P_1 + A_{21}^TP_2A_{21}\right)e_1 + \\ &2e_1^TA_{21}^TP_2M_{2j}e_z - 2e_1^T\left(A_{11}^TP_1D_1 + A_{21}^TP_2D_2\right)w \\ &+ e_z^T\left(M_{2j}^TP_2M_{2j} - P_2\right)e_z - 2e_z^TM_{2j}^TP_2D_2w \\ &+ w^T\left(D_1^TP_1D_1 + D_2^TP_2D_2\right)w \\ &+ \frac{\tau\xi^2}{\Delta^2}e_z^T(k)e_z(k) - \tau w^T(k)w(k) \end{aligned}$$

Then, the increment of the Lyapunov function can be written in the following quadratic manner:

$$\Delta V_j(k) \le \zeta^T(k)\Xi_j\zeta(k),$$

where the stacked state vector is

$$\zeta(k) = \begin{bmatrix} e_1(k) \\ e_z(k) \\ w(k) \end{bmatrix}$$

and the symmetric matrix Ξ_j is given in equation (13).

The inequalities (13) imply that matrices Ξ_j are negative definite for all the vertices of the polytope, and then, the forward increment of the Lyapunov function will be negative for all $\zeta(k) \ne 0$ (see [16]), which ensures the asymptotic stability of the system.

[1]The time script k has been removed for ease of exposition.

$$\begin{bmatrix} A_{11}^T P_1 A_{11} - P_1 + A_{21}^T P_2 A_{21} & A_{21}^T P_2 M_{2j} & -A_{11}^T P_1 D_1 - A_{21}^T P_2 D_2 \\ * & M_{2j}^T P_2 M_{2j} - P_2 + \frac{\tau\xi^2}{\Delta^2} I & -M_{2j}^T P_2 D_2 \\ * & * & -\tau I + D_1^T P_1 D_1 + D_2^T P_2 D_2 \end{bmatrix} < 0, j = 1, ..., 2^p, \quad (13)$$

$$\begin{bmatrix} 1 & h_{zi} \\ * & \beta P_2 \end{bmatrix} \geq 0, i = 1, ..., p. \quad (14)$$

where

$$M_{2j} = A_{22}^s + BF_j K + BF_j^- H_z.$$

Finally, it must be ensured that the state of the system remains in $\mathfrak{E}_\beta$. To demonstrate this, the fact that the set $\mathfrak{E}_\beta$ is an invariant set is utilised so that from any initial condition $e_1(0), e_z(0)$ in $\mathfrak{E}_\beta$, any error $e_1(k), e_z(k)$ will belong to $\mathfrak{E}_\beta$, for all $k \geq 0$. The reason is clear as $\Delta V(k)$ is negative definite, then $V(k) \leq V(0) \leq \beta^{-1}$.

Finally, the optimization problem is introduced to minimize the size of the ultimate boundedness Δ. This concludes the proof. □

Theorem 1 does not give any insights into the design of the observer. There are many unknown matrices that must be designed: some related to the observer dynamics A_{22}^s, B and some are needed for stability considerations such as P_1, P_2, H_z. There are also constants Δ, β, τ to be selected. In the following subsection some modifications on the conditions of Theorem 1 are introduced in such a way that some Linear Matrix Inequalities (LMIs) are obtained, which can be efficiently solved using appropriate software tools.

A. OBSERVER DESIGN VIA LMI

Assume that the matrices A_{22}^s and H_z have been well designed. Imposing a particular choice of A_{22}^s, define the dynamics of the observation error when there are no disturbances (see eq. (10)). The following lemma can be used to design the observer.

Lemma 2. Given matrices A_{22}^s, H_z and scalars β, ξ, if positive definite matrices P_1, P_2, matrix W of appropriate dimensions and scalar $\tau > 0$ solve the following optimization problem

$$\min_{P_1, P_2, W, \tau} \Delta$$

subject to

$$\begin{bmatrix} -P_1 & 0 & 0 & A_{11}^T P_1 & A_{21}^T P_2 \\ * & -P_2 + \tau\frac{\xi^2}{\Delta^2} I & 0 & 0 & \Theta_j \\ * & * & -\tau I & -D_1^T P_1 & -D_2^T P_2 \\ * & * & * & -P_1 & 0 \\ * & * & * & * & -P_2 \end{bmatrix} < 0, \quad j = 1, ..., 2^p, \quad (17)$$

$$\begin{bmatrix} 1 & h_{zi} \\ * & \beta P_2 \end{bmatrix} \geq 0, \quad i = 1, ..., p. \quad (18)$$

with

$$\Theta_j = A_{22}^{sT} P_2 + K^T F_j W^T + H_z^T F_j^- W^T,$$

then the observation error dynamic system (9)–(10) is ultimately bounded stable by taking $B = P_2^{-1} W$. Matrix h_{zi} denotes the i-th row of H_z.

Proof. See Appendix.

The optimization of scalar Δ can be easily carried out by means of a bisection algorithm or similar. The conditions are linear matrix inequalities with design parameters P_1, P_2, W, τ, so the problem can be easily solved using appropriate software.

IV. UNKNOWN INPUT ESTIMATION

This section is devoted to the unknown input estimation properties of the variable structure observer designed previously. Specifically, the switching function $\nu(k)$ contains useful information about the mismatch between the model used to define the observer and the actual plant.

Let $\bar{w}(k)$ denote the estimate of the unknown input. From the dynamics of the observation error (9), the actual error can be estimated as

$$e_1(k) = A_{11}^k e_1(0) + \sum_{i=0}^{k-1} (A_{11}^{k-1-i} D_1 \bar{w}(i)).$$

As A_{11} is stable, the first term vanishes in some steps, so:

$$e_1(k) \approx \sum_{i=0}^{k-1} (A_{11}^{k-1-i} D_1 \bar{w}(i)). \quad (19)$$

Note that actual $e_1(k)$ depends on past values of $\bar{w}(k)$. On the other hand, using the output error dynamics (10), and assuming that slow disturbances will imply slow e_z, then:

$$\begin{aligned} e_z(k) &\approx e_z(k+1), \\ &\approx A_{21} e_1(k) + A_{22}^s e_z(k) - D_2 \bar{w}(k) + \nu(k). \end{aligned}$$

As the observer is asymptotically stable when $|e_z(k)| > \Delta$, it can be assumed that it is evolving inside the ball for sufficiently large k. In that case,

$$\nu(k) = \frac{B}{\Delta} e_z(k) \quad \Rightarrow \quad e_z(k) = \Delta B^{-1} \nu(k),$$

assuming nonsingular B. Substituting in the previous equation:

$$D_2 \bar{w}(k) \approx A_{21} e_1(k) + [\Delta(A_{22}^s - I) B^{-1} + I] \nu(k).$$

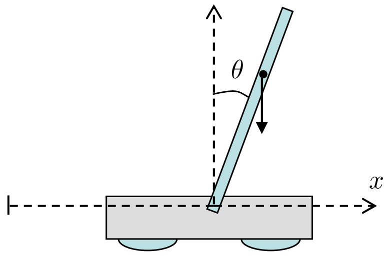

Fig. 1. Scheme of the inverted pendulum with cart

By least squares, the unknown input can be estimated using the $e_1(k)$ given in (19) and the actual value of $\nu(k)$:

$$\bar{w}(k) \approx M\left(A_{21}e_1(k) + [\Delta(A_{22}^s - I)B^{-1} + I]\nu(k)\right),$$

where $M = (D_2^T D_2)^{-1} D_2^T$. Here the matrix $D_2^T D_2$ is nonsingular because D_2 has full column rank.

V. EXAMPLE: INVERTED PENDULUM

The problem of the inverted pendulum with a cart constitutes a benchmark study for the application of nonlinear design methods, [17]. The problem also lends itself to assessment of linear frameworks, as linearization errors are a motivation for control engineers to employ robust control techniques and observers. Consider the inverted pendulum with a cart shown in Figure 1. Using the same model given in [3], the equations of motion are

$$\begin{aligned} (M+m)\ddot{x} + F_x\dot{x} + ml(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) &= u, \quad (20) \\ J\ddot{\theta} + F_\theta\dot{\theta} - mlg\sin\theta + ml\ddot{x}\cos\theta &= 0, \quad (21) \end{aligned}$$

where the values of the physical parameters used are given in Table I.

TABLE I
MODEL PARAMETERS FOR THE INVERTED PENDULUM WITH CART

M	(kg)	3.2	F_x	(kg/s)	6.2
m	(kg)	0.535	F_θ	$(kg\ m^2)$	0.009
J	$(kg\ m^2)$	0.062	g	(m/s^2)	9.8
l	(m)	0.365			

To evaluate the performance of the proposed unknown input observer, it will be compared with the continuous sliding mode observer proposed in [3]. However, to make an appropriate comparison, additional sampling will be introduced between the plant and observer in this continuous version to reflect the practical situation whereby signals from the plant are sampled. In both cases, the system must be linearised around the equilibrium point at the origin. Using $x, \theta, \dot{x}$ and $\dot{\theta}$ as system states, and assuming that only θ, x and $\dot{x}$ are available as measured outputs, the discrete-time model with sampling time $T_m = 0.1s$ is given by the triple

$$A = \left[\begin{array}{c|ccc} 0.1051 & 0 & -7.4143 & 2.5489 \\ \hline -0.0003 & 1 & -0.0119 & 0.0915 \\ 0.1053 & 0 & 2.2255 & -0.3016 \\ -0.0084 & 0 & -0.2681 & 0.8447 \end{array}\right], B = \left[\begin{array}{c} 0.0435 \\ \hline 0.0015 \\ -0.0049 \\ 0.0293 \end{array}\right]$$

$$C = \left[\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

It is assumed that in both cases, the same control signal is used. Specifically, a sliding mode controller is implemented. Furthermore, it is assumed that the unknown inputs enter the system through the same channel as the control input, that is, $D = B$. Using Lemma 2 with $\beta = 0.01$ and $\xi = 0.15$, the optimization problem leads a minimum bound of $\Delta = 0.02$. The variable structure observer is defined by

$$\begin{aligned} A_{22}^s &= \begin{bmatrix} 0.3000 & 0 & 0 \\ 0 & 0.2000 & 0 \\ 0 & 0 & 0.4000 \end{bmatrix}, \\ B &= \begin{bmatrix} -0.0062 & 0.0000 & -0.0000 \\ 0.0000 & -0.0042 & 0.0000 \\ -0.0000 & 0.0000 & -0.0088 \end{bmatrix} \end{aligned}$$

Figures 2 and 3 compare the results of both observers. The simulations have been performed using the nonlinear model for the pendulum given in equations (20)-(21). The sum of two sinusoidal functions has been applied as the unknown input.

Figure 2 shows that the proposed observer exhibits an initial transient time after which a good estimate of the unknown input is obtained. This is because some of the assumptions made in Section IV are correct only if the observation error is close to zero. Moreover, linearization errors are more apparent at the transient, when the pendulum is far from the equilibrium. However, after this transient time, Figure 3 reveals that the discrete-time observer produces a better estimate than its continuous counterpart. Comparing the error in the steady state, the continuous observer has a maximum absolute error of 0.0295 units whereas the discrete observer achieves a maximum error of 0.0183. There is thus a circa 60% difference in the error bound, with the proposed observer providing the greater accuracy of reconstruction.

VI. CONCLUSIONS

Following the same framework as in [15], an unknown input observer has been presented for sampled systems. The proposed method has several advantages. On the one hand, the design exploits all the degrees of freedom available in the observer framework proposed in [15] in a positive way. On the other hand, the quality of the estimate of the unknown inputs is higher when compared with a classical continuous sliding mode observer. In the nonlinear bench mark inverted pendulum with cart example, a circa 60% improvement in the error bound is achieved with the proposed synthesis. It is clear that the current methodology relies upon knowledge of the sample rate used for implementation and development of methods which incorporate sampling of uncertain or variable rate must now be considered.

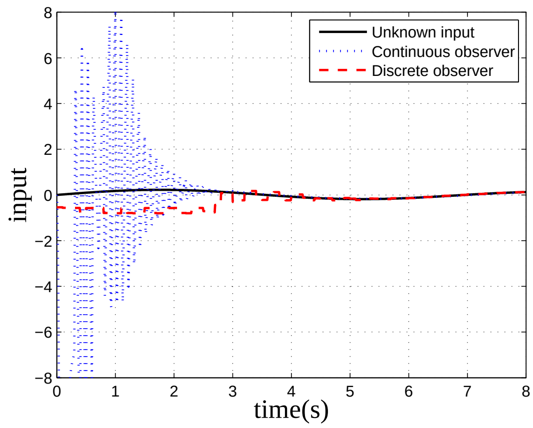

Fig. 2. Unknown input estimation with the continuous SMO (dotted line) and the discrete VSO (dashed line)

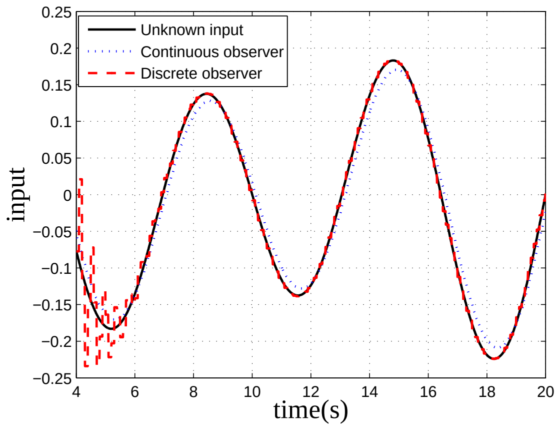

Fig. 3. Unknown input estimation with the continuous SMO (dotted line) and the discrete VSO (dashed line)

APPENDIX

It will be shown that the conditions given in Lemma 2 and in Theorem 1 are equivalent. The first set of inequalities (13), $\Xi_j < 0$ can be rewritten as:

$$\begin{bmatrix} -P_1 & 0 & 0 \\ * & -P_2+\tau\frac{\xi^2}{\Delta^2}I & 0 \\ * & * & -\tau I \end{bmatrix} + \begin{bmatrix} A_{21}^T \\ M_{2j}^T \\ -D_2^T \end{bmatrix} P_2 \begin{bmatrix} A_{21} & M_{2j} & -D_2 \end{bmatrix} + \begin{bmatrix} A_{11}^T \\ 0 \\ -D_1^T \end{bmatrix} P_1 \begin{bmatrix} A_{11} & 0 & -D_1 \end{bmatrix} < 0,$$

for $j = 1, ..., 2^p$. Using the Schur complement, the previous inequalities are equivalent to

$$\begin{bmatrix} -P_1 & 0 & 0 & A_{11}^T & A_{21}^T \\ * & -P_2+\tau\frac{\xi^2}{\Delta^2}I & 0 & 0 & M_{2j}^T \\ * & * & -\tau I & -D_1^T & -D_2^T \\ * & * & * & -P_1^{-1} & 0 \\ * & * & * & * & -P_2^{-1} \end{bmatrix} < 0,$$

for $j = 1, ..., 2^p$. Pre- and post-multiplying the previous inequalities by $diag\{I, I, I, P_1, P_2\}$ and its transpose, conditions (17) are obtained. Then, the proof is finished by direct application of Theorem 1. □

REFERENCES

[1] U. Itkis, *Control systems of variable structure*. New York: Wiley, 1976.

[2] V. I. Utkin, *Sliding modes in Control Optimisation*. Berlin: Springer-Verlag, 1992.

[3] C. Edwards and S. K. Spurgeon, *Sliding Mode Control: Theory and applications*. Taylor and Francis, 1998.

[4] S. K. Spurgeon, “Sliding mode observers - a survey,” *International Journal of Systems Science*, vol. 39, no. 8, pp. 751–764, 2008.

[5] C. Edwards, S. K. Spurgeon, and R. J. Patton, “Sliding mode observers for fault detection and isolation,” *Automatica*, vol. 36, no. 4, pp. 541–553, 2000.

[6] T. Floquet and J. P. Barbot, “Super twisting algorithm based step-by-step sliding mode observers for nonlinear systems with unknown inputs,” *International Journal of Systems Science*, vol. 38, no. 10, pp. 803–815, 2007.

[7] T. Floquet, C. Edwards, and S. K. Spurgeon, “On sliding mode observers for systems with unknown inputs,” *International Journal of Adaptive Control and Signal Processing*, vol. 21, no. 8-9, pp. 803–815, 2007.

[8] C. Edwards and C. P. Tan, “Sensor fault tolerant control using sliding mode observers,” *Control Engineering Practice*, vol. 14, no. 8, pp. 897–908, 2006.

[9] ——, “Fault tolerant control using sliding mode observers,” in *43rd IEEE Conference on Decision and Control*, Nassau, Bahamas, December 2004, pp. 5254–5259.

[10] X. Han, E. Fridman, and S. K. Spurgeon, “A sliding mode observer for fault reconstruction under output sampling: A time-delay approach,” in *50th IEEE Conference on Decision and Control and European Control Conference*, Orlando, FL, USA, December 2011, pp. 77–82.

[11] A. J. Koshkouei and A. S. I. Zinober, “Sliding mode state observers for siso linear discrete-time systems,” in *UKACC International Conference on Control*, Swansea , UK, September 1998, pp. 837–842.

[12] ——, “Sliding mode state observers for discrete-time linear systems,” *International Journal of Systems Science*, vol. 33, no. 9, pp. 751–758, 2002.

[13] S. M. Lee and B. H. Lee, “A discrete-time sliding mode controller and observer with computation time delay,” *Control Engineering Practice*, vol. 7, no. 8, pp. 943–955, 2005.

[14] K. C. Veluvolu, S. Pavuluri, Y. C. Soh, W. Cao, and Z. Y. Liu, “Observers with multiple sliding modes for uncertain linear MIMO systems,” in *IEEE Conference on Industrial Electronics and Applications*, Singapore, May 2006, pp. 1–6.

[15] L. Orihuela, X. Yan, S. K. Spurgeon, and F. R. Rubio, “Variable structure observer for discrete-time multi-output systems,” in *12th International Workshop on Variable Structure Systems*, Mumbai, India, January 2012.

[16] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in system and control theory*. Philadelphia: Society for Industrial and Applied Mathematics, 1994.

[17] J. Aracil and F. Gordillo, “El péndulo invertido: un desafío para el control no lineal,” *Revista Iberoamericana de Automática e Informática Industrial*, vol. 2, no. 2, pp. 8–19, 2005.