UKACC International Conference on Control 2012
Cardiff, UK, 3-5 September 2012

Optimal Output Regulation of Minimum Phase Nonlinear Systems

Attaullah Y. Memon[†]

Abstract— **This paper studies the design of an optimal stabilizing controller for output regulation of minimum phase nonlinear systems in the Lyapunov redesign framework of our earlier work [6], and investigates the asymptotic robustness properties of the overall feedback design, given that the optimal stabilizing controller itself possesses strong robustness properties by construction. The motivation comes from the flexibility of incorporating any stabilizing controller within the proposed framework, and we seek for control design methods that yield stabilizing controllers with some additional desirable properties like optimality, disturbance rejection and robustness in the presence of matched uncertainties e.g. static nonlinearities, uncertain parameters and the unmodeled fast dynamics. We exploit the optimal control design methods developed by Kokotovic and his co-researchers [3], [4], [7] for nonlinear systems, where it is shown that in addition to achieving the asymptotic stability of the system and minimizing a cost functional, the optimal feedback control guarantees stability margins which characterize the robustness properties.**

Index Terms— **Nonlinear Systems, Inverse Optimal Control, Lyapunov Redesign, Output Regulation**

I. INTRODUCTION

In this paper, the problem of optimal output regulation of minimum-phase nonlinear systems is considered. The output regulation problem deals with the design of a controller to make the output of a plant asymptotically track reference signals and reject disturbance signals, both produced by an autonomous external system called the *exosystem*. In our earlier work [6], we used the Lyapunov redesign and saturated high-gain feedback approach to design the stabilizing compensator, and included a conditional servocompensator by modifying the original controller that yields asymptotic error regulation without degrading the transient performance. One special feature of the Lyapunov redesign framework of [6] is that it allows us to start with *any* stabilizing controller and then include a conditional servocompensator by modifying the original controller to achieve the desired control objectives. This flexibility of incorporating any stabilizing controller within our framework motivates us to seek for control design methods that yield stabilizing controllers with some additional desirable properties like optimality, disturbance rejection and robustness in the presence of matched uncertainties e.g. static nonlinearities, uncertain parameters and the unmodeled fast dynamics. Herein, we take into consideration the optimal control design methods developed by Kokotovic and his co-researchers [3], [4], [7] for the stabilization of nonlinear systems, where it is shown that in addition to achieving the asymptotic stability of the system and minimizing a cost functional, the optimal feedback control guarantees stability margins which characterize the robustness properties. A major handicap in designing such controller is that it requires the solution of the complicated Hamilton-Jacobi-Bellman (HJB) partial differential equations. Kokotovic and co-researchers introduced an inverse approach [7] to the optimal control design for nonlinear systems, which abrogates the requirement of solving the HJB equations in order to design optimal feedback controllers. We incorporate an optimal stabilizing controller in the Lyapunov redesign framework of [6], to investigate the problem of output regulation of nonlinear systems using conditional servocompensators. We concentrate on the asymptotic robustness properties of the overall Lyapunov-redesign + conditional servocompensator framework, given that the optimal stabilizing controller itself possesses strong robustness properties (e.g. robustness to matched uncertainties and unknown disturbances) by design.

The rest of the paper is organized as follows. We present a brief review of Lyapunov redesign framework of [6] in the next section. Section III introduces the definitions of stability margins for nonlinear systems which are due to [7], and can be considered as the starting point of control design in this paper. Section IV states the problem formulation and assumptions, and is followed by the closed-loop analysis in Section V. A simple example is worked out in Section VI. Finally, Section VII draws the conclusions.

II. OUTPUT REGULATION USING CONDITIONAL SERVOCOMPENSATORS

In this section we briefly review the Lyapunov redesign approach to output regulation problem using conditional servocompensators [6]. Consider the SISO nonlinear system

$$\begin{aligned} \dot{\xi} &= \tilde{f}(\xi, w) + \tilde{g}(\xi, w)u \\ e &= \tilde{h}(\xi, w) \end{aligned} \tag{1}$$

where $\xi \in R^n$ is the state, u is the control input, e is the regulation error and the functions $\tilde{f}$, $\tilde{g}$ and $\tilde{h}$ are sufficiently smooth. The plant is subjected to a vector of *exogenous* input variables, which are generated by the known exosystem

$$\dot{w} = S_0 w \tag{2}$$

where S_0 has distinct eigenvalues on the imaginary axis and $w(t)$ belongs to a compact set $\mathcal{W}$. Suppose that for all $w \in \mathcal{W}$, there exist a continuously differentiable mapping $\xi = \pi(w)$, with $\pi(0) = 0$, and a continuous mapping $\chi(w)$, generated by the internal model

$$\frac{\partial \tau(w)}{\partial w} S_0 w = S\tau(w), \qquad \chi(w) = \Gamma\tau(w)$$

[†] The author is with the Faculty of Electronics & Power Engineering, PN Engineering College, National University of Sciences & Technology, Karachi, Pakistan. (E-mail: attaullah@pnec.edu.pk)

978-1-4673-1558-6/12/$31.00 ©2012 IEEE

where S has distinct eigenvalues on the imaginary axis, such that

$$\begin{aligned} \frac{\partial\pi(w)}{\partial w}S_0 w &= \tilde{f}(\pi,w)+\tilde{g}(\pi,w)\chi(w) \\ 0 &= h(\pi,w) \end{aligned} \quad (3)$$

With the change of variables $x=\xi-\pi$, the system (1) can be represented by

$$\dot{x}=f(x,w)+g(x,w)[u-\chi(w)] \quad (4)$$

The system (4) is in the form where the state feedback regulation problem can be formulated as a state feedback stabilization problem by treating $\chi(w)$ as a matched uncertainty. Suppose there is a locally Lipschitz function $\psi(x,w)$, with $\psi(0,w)=0$, and a continuously differentiable Lyapunov function $V(x,w)$, possibly unknown, such that

$$\alpha_1(\|x\|)\leq V(x,w)\leq \alpha_2(\|x\|) \quad (5)$$

$$\frac{\partial V}{\partial w}S_0 w+\frac{\partial V}{\partial x}[f(x,w)+g(x,w)\psi(x,w)]\leq -W(x) \quad (6)$$

$\forall x\in\mathcal{X}\subset R^n$, $w\in\mathcal{W}$, where $W(x)$ is a continuous positive definite function and α_1 and α_2 are class $\mathcal{K}$ functions. The system (4) can be re-written as

$$\begin{aligned} \dot{x} &= f(x,w)+g(x,w)\psi(x,w) \\ &\quad +g(x,w)u-g(x,w)[\chi(w)+\psi(x,w)] \end{aligned} \quad (7)$$

Let $\Omega=\{V(w,x)\leq c_1\}\subset\mathcal{X}$ be a compact set for some $c_1>0$ and $\delta(x)$ be a function such that

$$\|\chi(w)+\psi(x,w)\|\leq\delta(x) \qquad \forall x\in\Omega, \quad \forall w\in\mathcal{W} \quad (8)$$

Suppose $(\partial V/\partial x)g(x,w)$ can be expressed as

$$(\partial V/\partial x)g(x,w)=\upsilon(x)H(x,w) \quad (9)$$

where $\upsilon(x)$ is a known, locally Lipschitz function, with $\upsilon(0)=0$, and $H(x,w)$ is a, possibly unknown, function such that $0<\theta\leq|H(x,w)|\leq k$, $\forall x\in\Omega$, $\forall w\in\mathcal{W}$.

A *conditional servocompensator* [6] is introduced via the saturated high-gain feedback controller

$$u=-\alpha(x)sat\,(s/\mu) \quad (10)$$

where $s=\upsilon(x)+K_1\sigma$, the saturation function is defined as

$$sat\,(s/\mu)=\begin{cases} \frac{s}{|s|} & \text{if} \quad |s|\geq\mu \\ \frac{s}{\mu} & \text{if} \quad |s|\leq\mu \end{cases} \quad (11)$$

and σ is the output of the conditional servocompensator

$$\dot{\sigma}=(S-JK_1)\sigma+\mu J sat\left(\frac{s}{\mu}\right) \quad (12)$$

with $\mu>0$ being the width of the boundary layer. The pair (S,J) is controllable and K_1 is chosen such that $S-JK_1$ is Hurwitz. The function $\alpha(x)$ is chosen to satisfy

$$\alpha(x)\geq\frac{k}{\theta}\delta(x)+\alpha_0, \quad \alpha_0>0 \quad (13)$$

It is shown in [6] that if $\sigma(0)$ is $O(\mu)$, the state $\sigma(t)$ of the conditional servocompensator (12) will always be $O(\mu)$.

The analysis in [6] shows that, for sufficiently small μ, every trajectory of the closed-loop system (2), (4), (10) and (12) asymptotically approaches a disturbance-dependant manifold of the form $\{x=0,\sigma=\bar{\sigma}\}$, on which the regulation error is zero. The state feedback design is extended to output feedback for a class of minimum-phase, input-output linearizable systems. For this class of systems, the state feedback control can be designed as a partial state feedback law that does not use the states of the internal dynamics. A reduced-order high-gain observer is used to estimate the states of the linearizable part of the system, which are derivatives of the output. The output feedback controller, obtained by replacing the states by their estimates, recovers the transient and asymptotic properties of the state feedback controller. The performance recovery is shown using the separation principle of [1] and [2].

III. Inverse optimal control design [7]

It is well known that the optimal control as a design tool guarantees robustness and stability margins. This design approach deals with the problem of finding a feedback control $u(x)$ for the nonlinear system

$$\dot{x} \;=\; f(x)+g(x)u \quad (14)$$

with the objective that the $u(x)$ achieves asymptotic stability of the equilibrium $x=0$ and minimizes the cost functional

$$J=\int_0^\infty (l(x)+u^T R(x)u)dt \quad (15)$$

where $l(x)\geq 0$ and $R(x)>0$ for all x. A direct determination of the optimal feedback law $u(x)$ for nonlinear optimal control problems requires us to solve the Hamilton-Jacobi-Bellman (HJB) partial differential equations. On the other hand, the robustness properties achieved as a result of the optimality do not depend on a particular choice of functions $l(x)\geq 0$ and $R(x)>0$. This motivated Freeman and Kokotovic' [3], [4] to pursue the development of the design methods that solve the inverse problem of optimal stabilization. In the inverse approach, a stabilizing feedback is designed first and then shown to be optimal for the cost functional (15). The problem is *inverse* since the functions $l(x)\geq 0$ and $R(x)>0$ are determined through the stabilizing feedback design process rather than being chosen by the designer.

A. Design of the Stabilizing Inverse Optimal Control

A stabilizing control law $u(x)$ solves an inverse optimal control problem for the system (14) if it can be expressed as

$$u=-k(x)=-\frac{1}{2}R^{-1}(x)(L_gV(x))^T, \; R(x)>0, \quad (16)$$

where $V(x)$ is a positive semidefinite function (to be called a Control Lyapunov Function (CLF), hereafter) and satisfies the following condition, with $u=-\frac{1}{2}k(x)$,

$$\dot{V}=L_fV(x)-\frac{1}{2}L_gV(x)k(x)\leq 0 \quad (17)$$

With the choice of $l(x) \triangleq -L_fV(x) + \frac{1}{2}L_gV(x)k(x) \geq 0$, $V(x)$ is a solution of the HJB equation

$$l(x) + L_fV(x) - \frac{1}{4}(L_gV(x))R^{-1}(x)(L_gV(x))^T = 0 \quad (18)$$

Therefore, the control law $u(x)$ is an inverse optimal stabilizing control law for the system (14) if it achieves the asymptotic stability of $x = 0$ for the system (14) and is of the form (16) with $V(x)$ that satisfies the condition (17).

The importance of the existence of a CLF in the framework of inverse optimal control design is that, when a CLF is known, an inverse optimal stabilizing control law can be given by Sontag's formula [8]

$$u_s(x) = \begin{cases} -\Big(c_0 + \frac{a_x+\sqrt{{a_x}^2+({b_x}^T b_x)^2}}{{b_x}^T b_x}\Big)b_x \ , & b_x \neq 0 \\ 0 \ , & b_x = 0 \end{cases} \quad (19)$$

where $a_x = L_fV(x)$, $b_x = (L_gV(x))^T$, and c_0 is a positive constant. It is shown in [7] that the control law (19) is Lipschitz continuous at $x = 0$, if $V(x)$ is a CLF that satisfies the *small control property*[1] for the nonlinear system (14), and is optimal stabilizing for the cost functional

$$J = \int_0^\infty (\frac{1}{2}p(x)b^T(x)b(x) + \frac{1}{2p(x)}u^TR(x)u)dt \quad (20)$$

where

$$p(x) = \begin{cases} c_0 + \frac{a_x+\sqrt{{a_x}^2+({b_x}^T b_x)^2}}{{b_x}^T b_x} \ , & b_x \neq 0 \\ c_0 \ , & b_x = 0 \end{cases} \quad (21)$$

An important consequence of the optimality of the control law (19) is that it has a sector stability margin $(\frac{1}{2}, \infty)$, and, under certain assumptions, it achieves a disk stability margin $D(\frac{1}{2})$. These stability margins, which are defined below, provide guaranteed robustness in the presence of matched uncertainties e.g. static nonlinearities, uncertain parameters and the unmodeled fast dynamics.

B. Stability Margins for Nonlinear Systems

The basic robustness properties of nonlinear feedback systems can be characterized in terms of stability margins, e.g. *gain*, *sector* and *disk stability margins*. Consider the nonlinear feedback system shown in Figure.1 where u and y are of the same dimension and Δ represents modeling uncertainty at the input side. Under the nominal conditions, the feedback loop consists of the (nominal) nonlinear plant H with the nominal control $u = -k(x) = -y$, and Δ is identity. The nominal system is denoted by (H, k) and the perturbed system by (H, k, Δ). The input uncertainties can be static or dynamic. The two most common static uncertainties are *unknown static nonlinearity* and *unknown parameters*. The dynamic uncertainty arises due to unmodeled fast dynamics of the system. The following definitions are due to [7].

[1]A nonlinear system $\dot{x} = f(x, u)$, with a known Lyapunov function $V(x)$, is said to satisfy the *small control property* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $\|x\| < \delta$ there exists u with $\|u\| < \epsilon$ so that $\dot{V}(x)$ is negative definite.

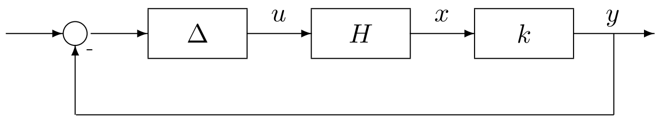

Fig. 1. Nonlinear Feedback Loop with the Control Law $u = k(x)$ and an Input Uncertainty Δ

Definition 1. *(Gain Margin) The nonlinear system (H, k) is said to have a gain margin (α, β) if the perturbed closed-loop system (H, k, Δ) is globally asymptotically stable for any Δ which is of the form $diag\{\kappa_1, \cdots, \kappa_m\}$ with constants $\kappa_i \in (\alpha, \beta), i = 1, \cdots, m$.*

Definition 2. *(Sector Margin) The nonlinear system (H, k) is said to have a sector margin (α, β) if the perturbed closed-loop system (H, k, Δ) is globally asymptotically stable for any Δ which is of the form $diag\{\varphi_1(\cdot), \cdots, \varphi_m(\cdot)\}$ where $\varphi_i(\cdot)$'s are locally Lipschitz static nonlinearities which belong to the sector (α, β).*

Definition 3. *(Disc Margin) The nonlinear system (H, k) is said to have a disc margin $D(\alpha)$ if the perturbed closed-loop system (H, k, Δ) is globally asymptotically stable for any Δ which is globally asymptotically stable and input feedforward passive [7], with a radially unbounded storage function.*

It follows from the definition of disc margin that a nonlinear system having a disc margin $D(\alpha)$ also has gain and sector margins (α, ∞). A disk margin guards against two types of input uncertainties: static nonlinearities and dynamic uncertainties arising from unmodeled fast dynamics of the system.

IV. PROBLEM FORMULATION AND CONTROL DESIGN

Consider the single-input single-output minimum-phase nonlinear system

$$\begin{aligned} \dot{\zeta} &= \tilde{f}(\zeta, w) + \tilde{g}(\zeta, w)\varphi(u) \\ e &= \tilde{h}(\zeta, w) \end{aligned} \quad (22)$$

where $\zeta \in R^n$ is the state, u is the control input, and e denotes the regulation error. The nonlinear function $\varphi(u)$ belongs to a sector $[\theta_1, \theta_2]$ and satisfies the inequality

$$\theta_1 u^2 \leq u\varphi(u) \leq \theta_2 u^2, \quad 0 \leq \theta_1 \leq \theta_2 \quad (23)$$

The plant is subjected to a set of *exogenous* input variables w that belong to a compact set $\mathcal{W} \in R^w$, which include unknown disturbances to be rejected and references to be tracked. The functions $\tilde{f}$, $\tilde{g}$ and $\tilde{h}$ are sufficiently smooth in ζ on a domain $\Xi \subset R^n$ and are continuous in w for $w \in \mathcal{W}$. Our goal is to design a controller to asymptotically regulate e to zero.

We now cast the given output regulation problem in the Lyapunov redesign framework of Section II. As described there, $\zeta = \pi(w)$ is a zero-error invariant manifold and $\chi(w)$ is the steady-state control that maintains the motion on this manifold, in the presence of any exogenous input w, which is generated by the exosystem (2). With the change of variables $x = \zeta - \pi(w)$, the system (22) can be represented by

$$\dot{x} = f(x, w) + g(x, w)[\varphi(u) - \chi(w)] \tag{24}$$

where $f(x, w) = \tilde{f}(x + \pi, w) - \tilde{f}(\pi, w) + [\tilde{g}(x + \pi, w) - \tilde{g}(\pi, w)]\chi(w)$ and $g(x, w) = \tilde{g}(x + \pi, w)$.

In what follows, we assume that an optimal stabilizing state feedback controller $\psi(x)$ is available for the nominal system

$$\dot{x} = f(x, w) + g(x, w)u \tag{25}$$

such that the origin of the nominal closed-loop system

$$\dot{x} = f(x, w) + g(x, w)\psi(x) \tag{26}$$

is uniformly asymptotically stable. It is shown in [7] that if $\psi(x)$ is an optimal stabilizing state feedback controller for the system (25) for a cost functional

$$J = \int_0^\infty (l(x) + u^T R(x)u)dt \tag{27}$$

with $l(x) \geq 0$ and $R(x) > 0$ for all x, then it achieves a sector margin $(\frac{1}{2}, \infty)$. The optimal stabilizing feedback control $\psi(x)$ takes the form

$$\psi(x) = -\frac{1}{2}R^{-1}(x)(L_g V(x))^T, \; R(x) > 0, \tag{28}$$

where the *optimal value function* $V(x)$ is radially unbounded, and is such that the time-derivative of V along the solutions of the closed-loop system (26) is

$$\dot{V} = L_f V(x) - \frac{1}{4}L_g V(x)R^{-1}(x)(L_g V(x))^T \leq 0 \tag{29}$$

As reviewed in Section III, we use the inverse design approach [7], in which a stabilizing feedback controller is designed first and then shown to be optimal for the cost functional (27). When $V(x)$ (called *Control Lyapunov Function*, hereafter) is known, an inverse optimal stabilizing feedback control $\psi(x)$ for the nominal system (25) can be given by Sontag's formula (19), that yields in a sector stability margin $(\frac{1}{2}, \infty)$, and, if $R(x) = I$, it achieves a disk stability margin $D(\frac{1}{2})$.

Assumption 1. *There exists a smooth positive-definite, and radially unbounded function $V(x, w)$ for the system (25) that satisfies*

$$L_g V(x, w) = 0 \quad \Rightarrow L_f V(x, w) < 0, \quad \forall x \neq 0 \tag{30}$$

Remark 1. *Any Lyapunov function whose time-derivative can be rendered negative definite is a CLF. The importance of CLF concept in the framework of inverse optimality is that, when a CLF is known, an inverse optimal stabilizing control such as (19) can be designed, and the CLF becomes an optimal value function.*

It follows that, with a known V, the optimal feedback control $\psi(x)$ can robustly stabilize the system (25) in the presence of any sector-bounded nonlinearity φ that belongs to a sector $[\theta_1, \theta_2]$. Our goal is to show that with the optimal feedback control $\psi(x)$ we can solve the problem of robust output regulation in the presence of sector nonlinearity φ. The results of [6] can not be directly applied since the nature of problem differs in that, the Equation (4) depends linearly on control, whereas in the current problem the control depends on a sector-bounded nonlinear function φ. Furthermore, with Assumption 1, Equations (5)-(6) can be written as

$$\alpha_1(\|x\|) \leq V(x, w) \leq \alpha_2(\|x\|) \tag{31}$$

$$\frac{\partial V}{\partial w}S_0 w + \frac{\partial V}{\partial x}[f(x, w) + g(x, w)\varphi(\psi(x))] \leq -W(x) \tag{32}$$

$\forall x \in X \subset R^n$ and $\forall w \in \mathcal{W}$, where α_1 and α_2 are some class $\mathcal{K}$ functions, $W(x)$ is a continuous positive definite function, and X is a given domain that contains the origin.

The system (24) can also be written as

$$\begin{aligned} \dot{x} \;\; = \;\; & f(x, w) + g(x, w)\varphi(\psi(x)) + g(x, w)\varphi(u) \\ & - g(x, w)[\chi(w) + \varphi(\psi(x))] \end{aligned} \tag{33}$$

We use Lyapunov redesign to construct the saturated high-gain feedback controller to deal with the uncertain term $[\chi(w) + \varphi(\psi(x))]$. Let $\Omega = \{sup_{w \in \mathcal{W}} V(x, w) \leq c_1\} \subset X$, for some $c_1 > 0$, and $\delta(x)$ be a continuous function, independent of the sector-bounded nonlinearity $\varphi(\cdot)$, such that

$$\|\chi(w) + \varphi(\psi(x))\| \leq \delta(x), \quad \forall x \in \Omega, \; \forall w \in \mathcal{W} \tag{34}$$

For simplicity, with $H = 1$, $(\partial V/\partial x)g(x, w)$ as given in (9) can be expressed as

$$(\partial V/\partial x)g(x, w) = \upsilon(x), \quad \forall x \in \Omega, \; \forall w \in \mathcal{W} \tag{35}$$

where $\upsilon(x)$ is a known, locally Lipschitz function, with $\upsilon(0) = 0$. We introduce the conditional servocompensator via the saturated high-gain feedback controller

$$u = -\alpha(x) sat\,(s/\mu) \tag{36}$$

where $s = \upsilon(x) + K_1\sigma$, the continuous function $\alpha(x)$ is chosen such that

$$\alpha(x) \geq \delta(x) + \alpha_0, \quad \alpha_0 > 0 \tag{37}$$

the saturation function is defined as in (11) and σ is output of the conditional servocompensator (12).

V. CLOSED-LOOP ANALYSIS

We will now show that, for sufficiently small μ, the set $\Phi = \Omega \times \left\{V_0(\sigma) \leq \mu^2 c_2\right\}$ is a subset of the region of attraction, and for all initial conditions in Φ, every trajectory of the closed-loop system

$$\begin{aligned} \dot{w} \;\; &= \;\; S_0 w \\ \dot{x} \;\; &= \;\; f(x, w) + g(x, w)\varphi(\psi(x)) \\ & \quad + g(x, w)\varphi\left(-\alpha(x) sat\,(s/\mu)\right) \\ & \quad - g(x, w)[\chi(w) + \varphi(\psi(x))] \\ \dot{\sigma} \;\; &= \;\; (S - JK_1)\sigma + \mu J sat\,(s/\mu) \end{aligned} \tag{38}$$

asymptotically approaches an invariant manifold on which the error is zero. The forthcoming analysis follows the outline of the analysis in [6], with various technical differences due to the nature of the problem under consideration. We start by showing that the set Φ is positively invariant and there is a class $\mathcal{K}$ function ρ such that every trajectory in Φ enters the set $\Phi_\mu = \{\|x\| \leq \rho(\mu)\} \times \left\{V_0(\sigma) \leq \mu^2 c_2\right\}$ in finite time and stays thereafter. The derivative of $V(x,w)$ along the trajectories of the closed-loop system (38) satisfies

$$\begin{aligned}
\dot{V} &= \frac{\partial V}{\partial w}S_0 w + \frac{\partial V}{\partial x}[f(x,w) + g(x,w)\varphi(\psi(x))] \\
&\quad + \frac{\partial V}{\partial x}g(x,w)\varphi\left(-\alpha(x)sat\left(s/\mu\right)\right) \\
&\quad - \frac{\partial V}{\partial x}g(x,w)[\chi(w) + \varphi(\psi(x))] \\
&= \frac{\partial V}{\partial w}S_0 w + \frac{\partial V}{\partial x}[f(x,w) + g(x,w)\varphi(\psi(x))] \\
&\quad + \upsilon(x)\varphi\left(-\alpha(x)sat\left(s/\mu\right)\right) \\
&\quad - \upsilon(x)[\chi(w) + \varphi(\psi(x))] \\
&\leq -W(x) + [s - K_1\sigma]\varphi\left(-\alpha(x)sat\left(s/\mu\right)\right) \\
&\quad - [s - K_1\sigma][\chi(w) + \varphi(\psi(x))] \\
&= -W(x) + s\varphi\left(-\alpha(x)sat\left(s/\mu\right)\right) \\
&\quad - (K_1\sigma)\varphi\left(-\alpha(x)sat\left(s/\mu\right)\right) \\
&\quad - s[\chi(w) + \varphi(\psi(x))] + (K_1\sigma)[\chi(w) + \varphi(\psi(x))]
\end{aligned}$$

Inside Φ, $\|\sigma\| \leq \mu\sqrt{c_2/\lambda_{min}(P_0)}$. Using this along with (23), (37) and (11), it can be shown that when $|s| \geq \mu$, we have [2]

$$\begin{aligned}
\dot{V} &\leq -W(x) - \theta_1\alpha(x)\left|s\right| + \theta_1\alpha(x)\left\|K_1\right\|\left\|\sigma\right\| \\
&\quad + \delta(x)\left|s\right| + \delta(x)\left\|K_1\right\|\left\|\sigma\right\|
\end{aligned}$$

$$\begin{aligned}
\dot{V} &\leq -W(x) - [\theta_1\alpha(x) - \delta(x)]\left|s\right| + [\theta_1\alpha(x) \\
&\quad + \delta(x)]\left\|K_1\right\|\left\|\sigma\right\| \\
&\leq -W(x) + \mu\gamma_1 \qquad (39)
\end{aligned}$$

where $\gamma_1 = \max_{x\in\Omega} k_0[\theta_1\alpha(x) + \delta(x)]$, in which the constant $k_0 = \|K_1\|\sqrt{c_2/\lambda_{min}(P_0)}$.

Similarly, when $|s| \leq \mu$, we have

$$\begin{aligned}
\dot{V} &\leq -W(x) - \theta_1\alpha(x)\frac{s^2}{\mu} + \theta_1\alpha(x)\left\|K_1\right\|\left\|\sigma\right\|\frac{s}{\mu} \\
&\quad + \delta(x)\left|s\right| + \delta(x)\left\|K_1\right\|\left\|\sigma\right\| \\
&\leq -W(x) + \mu\gamma_2 \qquad (40)
\end{aligned}$$

[2]When $|s| \geq \mu$, from (23) and (11), we have:

$$\begin{aligned}
\theta_1 u^2 &\leq u\varphi(u) \\
\theta_1\left(-\alpha(x)\frac{s}{|s|}\right)^2 &\leq -\alpha(x)sat\left(s/\mu\right)\varphi\left(-\alpha(x)sat\left(s/\mu\right)\right) \\
-\alpha(x)\theta_1\frac{s^2}{|s|} &\geq s\varphi\left(-\alpha(x)sat\left(s/\mu\right)\right) \\
\Rightarrow -\alpha(x)\theta_1\left|s\right| &\geq s\varphi\left(-\alpha(x)sat\left(s/\mu\right)\right)
\end{aligned}$$

Similarly, when $|s| \leq \mu$, we have:

$$-\alpha(x)\theta_1(s^2/\mu) \geq s\varphi\left(-\alpha(x)sat\left(s/\mu\right)\right)$$

where $\gamma_2 = \max_{x\in\Omega} k_0\left[\theta_1\alpha(x) + \delta(x)(1 + 1/k_0)\right] \geq \gamma_1$. From (39) and (40),

$$\dot{V} \leq -W(x) + \mu\gamma_2, \quad \forall(x,\sigma) \in \Phi$$

Hence, from [5, Theorem 4.18], for sufficiently small μ, Φ is positively invariant and all trajectories starting in Φ enter the positively invariant Φ_μ in finite time and stay thereafter.

In the next step, we use $V_s = \frac{1}{2}s^2$, and Assumption 2, below, to show that the trajectories reach the boundary layer $\{|s| \leq \mu\}$ in finite time.

Assumption 2. $(\partial\upsilon/\partial x)g(x,w)$ *can be expressed as*

$$(\partial\upsilon/\partial x)g(x,w) = \beta(x), \quad k_p \leq |\beta(x)| \leq k_q, \quad k_q > k_p > 0$$

for all $x \in \{\|x\| \leq \rho(\mu)\}$ *and for all* $w \in \mathcal{W}$*. Furthermore,* $\alpha(0) \geq \left(\frac{k_q}{\theta_1 k_p}\right)\delta(0) + \alpha_0$, $\alpha_0 > 0$.

For $(x,\sigma) \in \Phi_\mu$

$$\begin{aligned}
s\dot{s} &= s\frac{\partial\upsilon}{\partial x}[f(x,w) + g(x,w)\varphi(\psi(x))] \\
&\quad + s\beta(x)\varphi\left(-\alpha(x)sat\left(s/\mu\right)\right) \\
&\quad - s\beta(x)[\chi(w) + \varphi(\psi(x))] + sK_1(S - JK_1)\sigma \\
&\quad + \mu sK_1 Jsat\left(s/\mu\right)
\end{aligned}$$

When $|s| \geq \mu$, we have

$$\begin{aligned}
s\dot{s} &\leq -k_p\theta_1\alpha(x)\left|s\right| + k_q\left\|[\chi(w) + \varphi(\psi(x))]\right\|\left|s\right| \\
&\quad + \left\|\frac{\partial\upsilon}{\partial x}[f(x,w) + g(x,w)\varphi(\psi(x))]\right\|\left|s\right| \\
&\quad + \left\|\sigma\right\|\left\|K_1\right\|\left\|(S - JK_1)\right\| \\
&\quad + \mu\left\|K_1\right\|\left\|J\right\|\left|s\right|
\end{aligned}$$

Inside Φ_μ, $\|\sigma\| \leq \mu\sqrt{c_2/\lambda_{min}(P_0)}$. Also, the function $\frac{\partial\upsilon}{\partial x}[f(x,w) + g(x,w)\varphi(\psi(x))]$ is continuous such that $\frac{\partial\upsilon}{\partial x}[f(0,w) + g(0,w)\varphi(\psi(x)(0,w))] = 0$. Therefore, the norm $\left\|\frac{\partial\upsilon}{\partial x}[f(x,w) + g(x,w)\varphi(\psi(x))]\right\|$ together with the norms $\|\sigma\|\,\|K_1\|\,\|(S_J K_1)\|$, $\mu\,\|K_1\|\,\|J\|$, $\|\alpha(x) - \alpha(0)\|$, and $\|\delta(x) - \delta(0)\|$ can be bounded by a class $\mathcal{K}$ function $\rho_1(\mu)$. Hence,

$$\begin{aligned}
s\dot{s} &\leq -\theta_1 k_p\alpha(0)\left|s\right| + k_q\delta(0)\left|s\right| + \rho_1(\mu)\left|s\right| \\
\Rightarrow \quad \dot{V}_s &\leq -k_p\left[\theta_1\alpha_0 - \frac{\rho_1(\mu)}{k_p}\right]\left|s\right| \qquad (41)
\end{aligned}$$

Thus, for sufficiently small μ, all trajectories inside Φ_μ would reach the boundary layer $\{|s| \leq \mu\}$ in finite time. Inside the boundary layer, the closed-loop system (38) is given by

$$\begin{aligned}
\dot{w} &= S_0 w \\
\dot{x} &= f(x,w) + g(x,w)\varphi(\psi(x)) \\
&\quad + g(x,w)\varphi\left(-\alpha(x)s/\mu\right) \\
&\quad - g(x,w)[\chi(w) + \varphi(\psi(x))] \qquad (42) \\
\dot{\sigma} &= S\sigma + J\upsilon(x)
\end{aligned}$$

By following the analysis in [6], it can be shown that inside the boundary layer, the trajectories of the closed-loop system (42) will asymptotically approach an invariant manifold on which the regulation error is zero. These conclusions are formally summarized in the following theorem.

Theorem 1. *Under stated assumptions, consider the closed-loop system (38). Suppose* $w(0) \in \mathcal{W}$. *Then, there exists* $\mu^* > 0$ *such that* $\forall \mu \in (0, \mu^*]$, *the set* $\Psi = \Omega \times \{V_0(\sigma) \leq \mu^2 c_2\}$ *is a subset of the region of attraction, and for all initial conditions in* Ψ, *the state variables are bounded and* $\lim_{t\to\infty} e(t) = 0$.

VI. ILLUSTRATIVE EXAMPLE

Consider the nonlinear system

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 \\ \dot{\zeta}_2 &= 2\zeta_1\zeta_2 + u + d(t) \\ y &= \zeta_1 \end{aligned} \tag{43}$$

It is desired to achieve optimal regulation of the system's output y to a constant reference signal r_0 in the presence of a disturbance signal, $d(t) = d_0 sin(\omega t)$. Both these signals are generated by the exosystem

$$\dot{w} = \begin{bmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} w, w(0) = \begin{bmatrix} d_0 \\ 0 \\ r_0 \end{bmatrix}, \begin{bmatrix} d(t) \\ r_0 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_3 \end{bmatrix}$$

With change of variables $x_1 = \zeta_1 - w_3, x_2 = \zeta_2$, we have

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 2(x_1 + w_3)x_2 + u + w_1 \\ e &= x_1 \end{aligned} \tag{44}$$

To achieve a sector margin for the nominal system, we use a CLF to design an optimal stabilizing control [7]. With

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, P = \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix},$$

the Ricatti inequality $A^T P + PA - PBB^T P < 0$ is satisfied for any $c \in (0, 1)$. Then, $V = x^T Px$ is a CLF for the nominal nonlinear system. Assumption 1 is satisfied with this CLF which is also an optimal value function. Furthermore, using Sontag's Formula (19), we get the optimal stabilizing control law for the nominal system as

$$\begin{aligned} \psi(x) &= -2x_1x_2 \\ &\quad - \frac{\alpha_1 x_1 + \sqrt{(2x_1x_2\alpha_2 + x_2\alpha_1)^2 + {\alpha_2}^4}}{\alpha_2} \end{aligned} \tag{45}$$

in which $\alpha_1 = x_1 + cx_2$, and $\alpha_2 = cx_1 + x_2$. This optimal stabilizing control law has two desirable properties, namely, it has a sector margin $(\frac{1}{2}, \infty)$, and it can achieve a disk margin $(\frac{1}{2})$ for the nominal system. The system (44) can also be written as

$$\begin{aligned} \dot{x} &= Ax + B[f(x, w) + g(x, w)(u - \chi(w))] \\ e &= Cx \end{aligned} \tag{46}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

$$f(x, w) = 2(x_1 + w_3)x_2 + w_1 + g(x, w)\chi(w)$$

and $\chi(w) = -\left(\frac{w_1 + 2w_3}{g(w)}\right)$. With this formulation, the problem fits into the Lyapunov redesign framework as given by (33) with an optimal stabilizing controller (45), and by using the control u of the form (36), it can be shown that the system yields zero steady-state regulation error in the presence of disturbance signal generated by the exosystem.

VII. CONCLUSIONS

This paper incorporates an optimal stabilizing controller in the Lyapunov redesign framework of [6], and investigates the asymptotic robustness properties of the overall feedback design, given that the optimal stabilizing controller itself possesses strong robustness properties by design. The motivation comes from the flexibility of incorporating any stabilizing controller within the framework as proposed in [6], and we seek for control design methods that yield stabilizing controllers with some additional desirable properties like optimality, disturbance rejection and robustness in the presence of matched uncertainties e.g. static nonlinearities, uncertain parameters and the unmodeled fast dynamics. The synthesis of such controller requires the solution of the complicated Hamilton-Jacobi-Bellman (HJB) partial differential equations. Here, we exploit the inverse optimal control design methods presented in [7] for nonlinear systems, where it is shown that in addition to achieving the asymptotic stability of the system and minimizing a cost functional, the optimal feedback control law which is designed based on the existence of a Control Lyapunov Function (CLF), guarantees stability margins which characterize the system's robustness properties. A simple illustrative example is also presented to delineate the overall design process.

VIII. ACKNOWLEDGEMENTS

The author gratefully acknowledges the valuable guidance provided by Professor Hassan K. Khalil.

REFERENCES

[1] Atassi A. N., Khalil H. K., *A Separation Principle for the Control of a Class of Nonlinear Systems*, IEEE Transactions on Automatic Control; **46**: 742 − 746; 2001.

[2] Atassi A. N., *A Separation Principle for the Control of a Class of Nonlinear Systems*, PhD Thesis; College of Engineering, Michigan State University, East Lansing, MI; 1999.

[3] Freeman, R. A., and Kokotovic', P., *Optimality of robust nonlinear feedback controls*, SIAM Journal on Control and Optimization; Volume 34 , Issue 4, pp. 1365 − 1391, 1996.

[4] Freeman, R. A., and Kokotovic', P., *Robust control of nonlinear systems*, Birkhauser, Boston, 1996.

[5] Khalil, H. K., *Nonlinear Control Systems, 3rd edition*, Prentice-Hall: Upper Saddle River, New Jersey, 2002.

[6] Memon A. Y., Khalil H. K., *Output Regulation of Nonlinear Systems Using Conditional Servocompensators*, Automatica, 46 (7), pp. 1119-1128, 2010.

[7] Sepulchre, R., Jankovic', M., and Kokotovic', P., *Constructive nonlinear control*, Springer, London, 1997.

[8] Sontag, E. D., *Remarks on stabilization and input-to-state stability*, In proceedings of 28th Conference on Decision & Control, Tampa, FL, pp. 1376 − 1378, 1989.