UKACC International Conference on Control 2012
Cardiff, UK, 3-5 September 2012

Stabilisation of Multi-Input Nonlinear Systems Using Associated Angular Approach

Zahra Sangelaji
Department of Engineering Systems
University of Greenwich
UK
Email: z.sangelaji@greenwich.ac.uk

***Abstract*—In this paper the stabilisation of multi-input nonlinear systems is studied using the associated angular approach. In this method, a nonlinear system is converted into two associated subsystems, the so-called radial and spherical subsystems. For a single input nonlinear system, the control is designed using the one dimensional radial system to stabilise the radial and consequently the original nonlinear system. For multi-input systems, the control is also designed based on the radial subsystem, however, the method is not straightforward in comparison with single input systems. The control law includes a weighting function which is determined based on the system performance and control action. Some examples are presented to illustrate the effect of various scenarios of using the proposed method.**

I. INTRODUCTION

In recent years, various methods have been developed to design a control for many classes of nonlinear systems including linearisation [1], optimal control [2], [3], H_∞ control [4], [5], sliding mode control (SMC) [6] using quantised feedback [7] and adaptive control [8]-[9]. Output feedback control design is also a method for stabilisation of a broad class of nonlinear systems which has been studied in the last two decades. This method is mainly used when the system output is measurable and some states are not available or they are very difficult to measure [1]. Full state-feedback control design methods are utilized to stabilise a nonlinear system globally, particularly, when the states are measurable. These methods include sliding mode control [10], backstepping [11], zero dynamics based on high gain [12] and neural network [13]. However many established methods only guarantee the local stability [1] or ultimate boundedness of the states [14]. Sangelaji and Banks [15]-[16] have proposed associated angular approach for the global stabilisation of a general class of single-input nonlinear systems by using the angular form. In this method, the system is converted into two nonlinear subsystems. The trajectories of a subsystem which move on a sphere is termed the spherical subsystem and the other, a scalar nonlinear system is called the radial subsystem. The method straightforwardly yields a controller when there is no singularity except the origin in the input map function. For single-input systems, the control law is generally simple for many cases and the method is applicable to a large class of nonlinear systems. Whenever the input map of the radial subsystem is zero, the radial control is not definable. In this case, some mild conditions are proposed to guarantee the system stability [17] or the radial control or control design method should be modified such that the designed control is definable everywhere within the operating region and also stabilises the system. The radial control can be continuous or discontinuous depending on the structure of the input map. The method was originally established for single input nonlinear systems [15], [16]. In this paper the method is extended to multi-input nonlinear systems which are not straightforward, because the control can not be driven using the associated angular subsystems as proposed for single input systems. The proposed control stablises the multi-input nonlinear system. The presented method can be applied to any nonlinear system while most of the existing methods are applicable to particular classes of nonlinear systems. The drawback of the angular method is the singularity points in which the control is not definable. Similar methods as established for single input nonlinear systems are required to remove the singularities.

This paper is organised as follows: In Section II the associated angular method is studied. In Section III the multi-input system is presented. In Section IV, the special cases of the design weighting matrix, are considered. Examples illustrating the control design process are presented in Section V. Finally conclusions are presented in Section VI.

II. ASSOCIATED ANGULAR METHOD FOR SINGLE INPUT NONLINEAR SYSTEMS

Consider the nonlinear system:

$$\dot{x} = A(x) + B(x)u \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, u is the scalar control, $A(x) \in \mathbb{R}^n$ and $B(x) \in \mathbb{R}^n$.

Let $\mathbb{S}^n \subseteq \mathbb{R}^n$ be the unit n-ball, i.e. $S^n = \{z \in \mathbb{R}^n : \|z\| = 1\}$ and $\mathbb{R}^+$ be the set of positive real numbers. The map

$$\varphi : \mathbb{R}^n - \{0\} \to \mathbb{R}^+ \times \mathbb{S}^n$$
$$x \to \big(\|x\|, \frac{x}{\|x\|}\big)$$

is a diffeomorphism from $\mathbb{R}^n - \{0\}$ onto $\mathbb{R}^+ \times \mathbb{S}^n$. Note that even as x tends to zero, $\dfrac{x}{\|x\|}$ $(= z)$ is on the unit ball. The origin is removed from the domain of the function φ; otherwise the origin corresponds to infinity pair $(0, z)$ where z is any point in $\mathbb{S}^n$. This obstacle can be removed if a unique pair say $(0, z_0)$ with $z_0 = (1, \ldots, 0)$ corresponds to the origin. Using diffeomorphism φ the system (1) is converted into the associated radial and spherical subsystems as presented in the following Lemma.

Lemma 1. *The system (1) can be written in the form*

$$\begin{aligned} \dot{r} &= \lambda_A + \lambda_B u & (2) \\ \dot{z} &= \frac{1}{r}\big(\bar{A}(r,z) + \bar{B}(r,z)u\big) & (3) \end{aligned}$$

from which the following control is obtained

$$u = -\frac{\lambda_A + \alpha r}{\lambda_B} \tag{4}$$

where

$$\bar{A}(r,z) = A(r,z) - z^T A(r,z)z$$
$$\bar{B}(r,z) = B(r,z) - z^T B(r,z)z$$

Also $\lambda_A = z^T A(r,z)$, $\lambda_B = z^T B(r,z)$, $r = \|x\|$, $z = \dfrac{x}{\|x\|}$ *and* $\alpha > 0$ *is a constant real number. Moreover, the control (4) stabilises the system (1).*

Proof: Since $r = \|x\|$ and $r^2 = x^T x$,

$$2r\dot{r} = 2x^T \dot{x} \tag{5}$$

Therefore

$$r\dot{r} = x^T\big(A(x) + B(x)u\big) \tag{6}$$

and

$$\dot{r} = \frac{x^T(A(x) + B(x)u)}{r} \tag{7}$$

Substituting $x = rz$ into (7) yields

$$\begin{aligned} \dot{r} &= z^T(A(r,z) + B(r,z)u) \\ &= \lambda_A + \lambda_B u & (8) \end{aligned}$$

On the other hand, using $z = \dfrac{x}{r}$ and (6) one can obtain

$$\begin{aligned} \dot{z} &= \frac{1}{r}\dot{x} - \frac{\dot{r}}{r^2}x \\ &= \frac{1}{r}\Big(A(r,z) + B(r,z)u\Big) - \frac{1}{r^3}\Big(x^T A(r,z) + x^T B(r,z)u\Big)x \\ &= \frac{1}{r}\big(\bar{A}(r,z) + \bar{B}(r,z)u\big) & (9) \end{aligned}$$

with

$$\bar{A}(r,z) = A(r,z) - z^T A(r,z)z$$
$$\bar{B}(r,z) = B(r,z) - z^T B(r,z)z$$

Select the control

$$u = -\frac{\lambda_A + \alpha r}{\lambda_B} \tag{10}$$

where $\alpha > 0$ is a real number. Then from (6)

$$r\dot{r} = -\alpha r^2 \tag{11}$$

So $\dot{r} = -\alpha r$ which guarantees the stability of the subsystem (2) and therefore, the system (1).

Note that the z-subsystem operates on the unit ball and r-subsystem is scalar. The real positive number α is a design parameter and only affects the degree of the stability of the system. In other words, for large values of α the state settling time is shorter in comparison with small values of α. One way to ensure the accessibility of the control (10) is to consider some specific constrains on α.

III. Multi-input systems

The control design and stabilisation problem using the angular approach, which has been studied in section II is only applicable to single-input nonlinear systems and its extension to a general class of nonlinear system is not straightforward. The degree of nonlinearity in the system is not an important issue for using this method, while the most existing methods are applicable for specific nonlinear classes of nonlinear system in which the structure and the nature of nonlinearities affect the process of the control design. In this section, the angular method is extended to design an appropriate control for multi-input nonlinear systems and the stabilisation criteria are also presented.

Consider the multi-input nonlinear affine system

$$\dot{x} = A(x) + B(x)u \tag{12}$$

where $A(x) \in \mathbb{R}^{n\times n}$, $B(x) \in \mathbb{R}^{n\times m}$, $u \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$. Let $r = \|x\|$ and $z = \frac{x}{\|x\|}$ then

$$\begin{aligned} r\dot{r} &= x^T \dot{x} \\ &= x^T A(x) + x^T B(x)u \end{aligned}$$

Therefore

$$\dot{r} = \frac{1}{r}(\lambda_A(x) + (\lambda_B(x)u) \tag{13}$$

$$\dot{z} = \frac{1}{r}\Big[\big(A - z^T Az\big) + \big(B - z^T Bz\big)u\Big] \tag{14}$$

where $\lambda_A = x^T A(x)$, $\lambda_B = x^T B(x) \in \mathbb{R}^{1\times m}$ and $u \in \mathbb{R}^{m\times 1}$. Suppose that there is an $\alpha > 0$ such that $\dot{r} = -\alpha r$. The condition $\dot{r} = -\alpha r$ is a sufficient condition for stability of the system. Therefore the equation (13) implies

$$\lambda_B u = -\lambda_A(x) - \alpha r^2 \tag{15}$$

Since the r-subsystem is a one-dimensional system, the vector control input u should be selected such that (15) is satisfied. Select the control

$$u = -\frac{(x^T A(x) + \alpha r^2) R B^T(x) x}{x^T B(x) R B^T(x) x} \tag{16}$$

where the weighting matrix R is nonsingular. Substituting control (16) in (13) yields

$$\begin{aligned}\dot{r} &= \frac{1}{r}(x^T A(x) + x^T B(x)u) \\ &= \frac{1}{r}[x^T A(x) + x^T B(x)\frac{(-x^T A(x) - \alpha r^2)RB^T(x)x}{x^T B(x)RB^T(x)x}] \\ &= \frac{1}{r}(x^T A(x) + \frac{(-x^T A(x) - \alpha r^2)x^T B(x)RB^T(x)x}{x^T B(x)RB^T(x)x}) \\ &= \frac{1}{r}(x^T A(x) - x^T A(x) - \alpha r^2) \\ &= -\alpha r\end{aligned}$$

Therefore, if $x^T BRB^T x \neq 0$ the control (16) stabilises the system (12). When $x^T BRB^T x = 0$ the control (16) should be modified. The methods in [16] for removing the singularities, i.e. for points belong to the set

$$\Pi = \{x \in \mathbb{R}^n : x^T BRB^T x = 0\}$$

may straightforwardly be extended to the nonlinear system (12).

Remark 1. *When $m = 1$, the system (12) is a single input system. In this case, $B^T(x)x \in \mathbb{R}$ and $R \in \mathbb{R}$. If $B^T(x)x \neq 0$ the control (16) coincides with the control 10*

$$u = -\frac{x^T A(x) + \alpha r^2}{x^T B(x)}$$

IV. Selection of the weighting matrix R

The control (16) depends on the weighting matrix R and can be selected based on desired system performance and control action. Various selection of the weighting matrix R in (16) yields various alternative controls. However, for any selection of R, the control (16) stabilises the system. Consider the following cases:

(*i*) Let $R = \beta I_m \in \mathbb{R}^{m\times m}$ where $\beta > 0$. Then

$$u = -\frac{(\lambda_A(x) + \alpha r^2)B^T(x)x}{x^T B(x)B^T(x)x} \quad (17)$$

or

$$u = -\frac{x^T(A(x) + \alpha x)B^T(x)x}{x^T B(x)B^T(x)x}$$

Therefore, for any $\beta > 0$ the selection of $R = \beta I$ does not yield different control law. In fact, any β result in the same control as is given by (17).

(*ii*) Assume that B is full rank. Select $R = (B^T B)^{-1}$. The control (16) is now in the following form

$$u = -\frac{(\lambda_A(x) + \alpha r^2)(B^T B)^{-1}B^T x}{x^T B(B^T B)^{-1}B^T x} \quad (18)$$

or

$$u = -\frac{x^T(A + \alpha x)(B^T B)^{-1}B^T x}{x^T B(B^T B)^{-1}B^T x}$$

Note that usually R is considered a symmetric positive-definite matrix. However, the weighting matrix R may be considered only as a nonsingular matrix. Control (18) when R is nonsingular matrix guarantees the stability of the multi-input system (12).

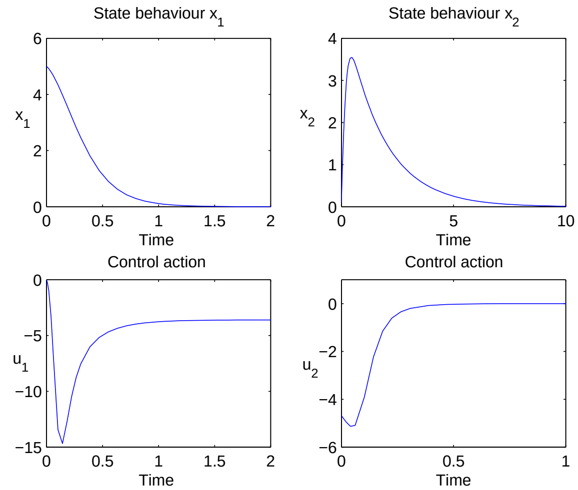

Fig. 1. Responses of the multi-input system using the radial control (16)

V. Examples

In this section two examples are presented to show the various scenarios of design of an angular controller. In the first example, the control is straightforwardly obtained as there is no singularities (except the origin), if the parameters are appropriately selected. The second example indicates the case when there are singularities. In this case a suitable condition is required.

A. Example 1

Consider the system

$$\begin{aligned}\dot{x}_1 &= (-1 + x_1)x_1 + x_1 u_2 \\ \dot{x}_2 &= 4x_1 + 3x_2 + x_2 u_1\end{aligned}$$

The state space representation of the nonlinear system is

$$\dot{x} = \begin{pmatrix} -1 + x_1 & 0 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 & x_1 \\ x_2 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad (19)$$

For this system, $x^T Ax$ and $B^T x$ are given by

$$\begin{aligned}x^T Ax &= x_1^2(-1 + x_1) + 4x_1 x_2 + 3x_2^2 \\ B^T x &= \begin{pmatrix} x_2^2 \\ x_1^2 \end{pmatrix}\end{aligned}$$

The control (17) with $R = I_2$ is

$$\begin{aligned}u &= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= -\frac{x_1^2(-1 + x_1) + 4x_1 x_2 + 3x_2^2 + \alpha(x_1^2 + x_2^2)}{x_2^4 + x_1^4}\begin{bmatrix} x_2^2 \\ x_1^2 \end{bmatrix}\end{aligned}$$

The simulation results are shown in Figure 2 for $\alpha = 0.3$.

Now consider

$$R = \begin{pmatrix} \gamma & 0 \\ 0 & \beta \end{pmatrix}$$

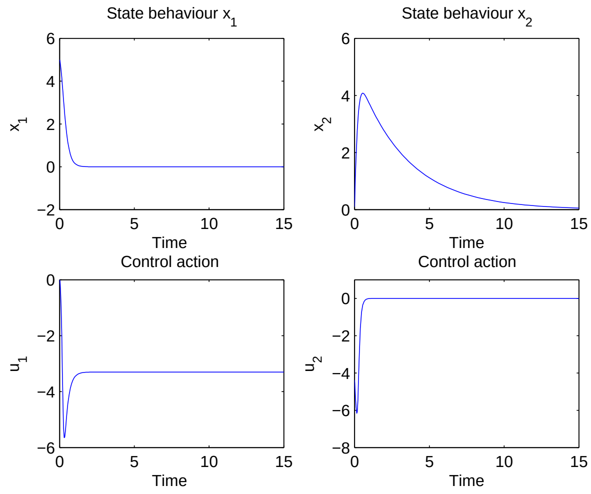

Fig. 2. Responses of the multi-input system using the radial control (17).

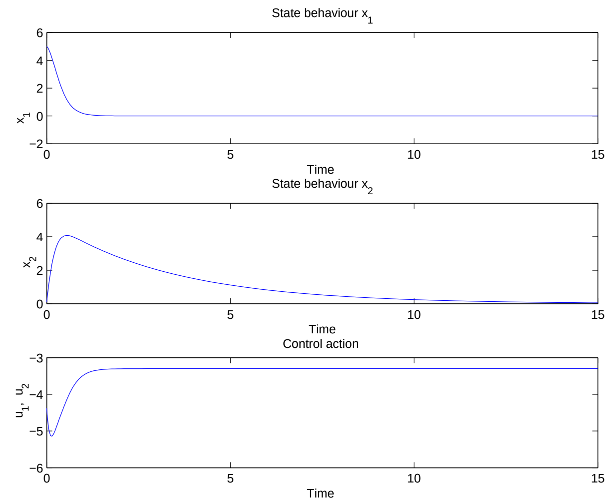

Fig. 3. Responses of the multi-input system using the radial control (18).

where $\gamma \neq \beta$ are positive numbers. Then

$$RB^T x = \begin{pmatrix} \gamma x_2^2 \\ \beta x_1^2 \end{pmatrix}$$

The control (16) is now in the following form

$$\begin{aligned} u &= \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ &= -\frac{x_1^2(-1+x_1)+4x_1x_2+3x_2^2+\alpha(x_1^2+x_2^2)}{\gamma x_2^4+\beta x_1^4}\begin{pmatrix} \gamma x_2^2 \\ \beta x_1^2 \end{pmatrix} \end{aligned}$$

If the weighting matrix R is selected such that $\alpha\beta < 0$. Then the control (20) is not defined for all x satisfying $\gamma x_2^4 + \beta x_1^4 = 0$ and therefore the number of singular points are infinite, whilst if $\alpha\beta > 0$ the control (20) is definable for all $x \in \mathbb{R}^n - \{0\}$. Thus, the selection of R is significantly important for designing multi-input nonlinear systems. This example shows that a suitable selection of R is a way for removing the singularities. The simulation results are depicted in Figure 1. The values for $\alpha = 0.3$, $\gamma = 6$, $\beta = 0.3$ and initial conditions $x_0 = [5, 0.1]$ are considered for simulation.

In this example, B is full rank for all $x_1 \neq 0$ and $x_2 \neq 0$. Therefore, R can be selected as $R = (B^T B)^{-1}$. Therefore,

$$B^T B = \begin{pmatrix} x_2^2 & 0 \\ 0 & x_1^2 \end{pmatrix}, \quad (B^T B)^{-1} = \begin{pmatrix} \frac{1}{x_2^2} & 0 \\ 0 & \frac{1}{x_1^2} \end{pmatrix}$$

and

$$x^T A x = x_1^2(-1+x_1) + 4x_1x_2 + 3x_2^2$$
$$x^T B(B^T B)^{-1} B^T x = x_2^2 + x_1^2$$

Since for $(x_1, x_2) \neq (0, 0)$, $x^T B(B^T B)^{-1} B^T x \neq 0$, the control is defined for all $(x_1, x_2) \neq 0$ and in this case both control laws u_1 and u_2 are the same

$$u_1 = u_2 = -\frac{x_1^2(-1+x_1)+4x_1x_2+3x_2^2+\alpha(x_1^2+x_2^2)}{x_1^2+x_2^2}$$

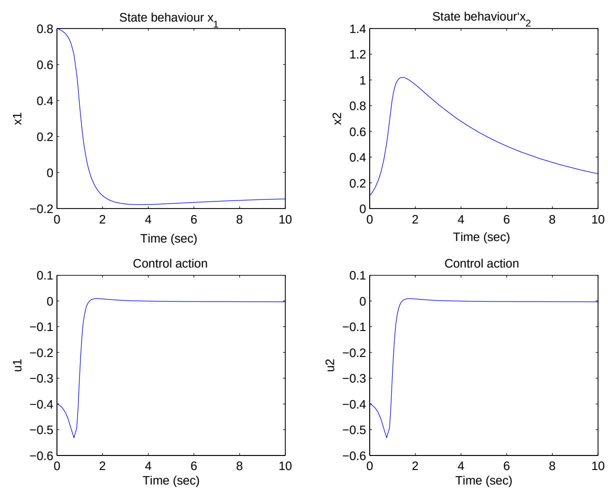

Fig. 4. Responses of the multi-input system using the radial control (20) and $\alpha = 0.4$.

Figure 3 shows the simulation results for $\alpha = 0.3$. This example illustrates that different selections of R yield various controls. When $R = (B^T B)^{-1}$ the two control inputs u_1 and u_2 are the same, whilst other choice of R presents a control vector with different components. Therefore, based upon the desired system performance and response, the weighting function may be selected.

B. Example 2

Consider the system

$$\dot{x}_1 = x_2 + x_1u_1 - x_2u_2$$
$$\dot{x}_2 = x_1x_2 - x_2u_1 + x_1u_2$$

The system can be written as

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ x_2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} x_1 & -x_2 \\ -x_2 & x_1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

For this example

$$B^T x = \begin{pmatrix} x_1^2 - x_2^2 \\ 0 \end{pmatrix}$$

$$B^T B = \begin{pmatrix} x_1^2 + x_2^2 & -2x_1x_2 \\ -2x_1x_2 & x_1^2 + x_2^2 \end{pmatrix}$$

and

$$(B^T B)^{-1} = \frac{1}{(x_1^2 - x_2^2)^2} \begin{pmatrix} x_1^2 + x_2^2 & 2x_1x_2 \\ 2x_1x_2 & x_1^2 + x_2^2 \end{pmatrix}$$

Therefore,

$$x^T A x = x_1 x_2^2 + x_1 x_2$$

$$(B^T B)^{-1} B^T x = \frac{1}{x_1^2 - x_2^2} \begin{pmatrix} x_1^2 + x_2^2 \\ 2x_1x_2 \end{pmatrix}$$

$$x^T B (B^T B)^{-1} B^T x = x_1^2 + x_2^2$$

Assume that for all nonzero x_1 and x_2, $x_1 \neq x_2$. Then the control is

$$u = -\frac{x_1 x_2^2 + x_1 x_2 + \alpha(x_1^2 + x_2^2)}{x_1^2 - x_2^2} \begin{pmatrix} 1 \\ \frac{2x_1x_2}{x_1^2 + x_2^2} \end{pmatrix} \tag{20}$$

The simulation results are shown in Figure 4.

Note that all angular methods proposed for single input systems can straightforwardly be extended to multi-input systems.

VI. Conclusions

In this paper the control design using the angular approach for multi-input nonlinear systems have been addressed. The radial control law includes a nonsingular weighting matrix which yield various control laws whenever it is not selected as a multiplication of an identity matrix. The weighting matrix does not necessarily have to be a positive-definite, however it only needs to be a nonsingular matrix. A suitable selection of weighting matrix can prevent any singularities in the radial control law. The control (16) has been designed such that when the system is single-input, this control coincides with the control (10). In single-input case, the weighing matrix is not required. In fact, in this case the weighting matrix is only a number.

All methods for single input systems for removing the singularities which have been presented in [15] and [16] are required to be extended to the multi-input nonlinear systems. In particular, similar methods as that presented in [16] should be established for removing the singularities including modifying the control, imposing a sufficient condition on design parameters, using the weighting norm and dynamical radial method.

References

[1] Isidori, A., Nonlinear control systems II, Springer, London, 1999.

[2] Banks, S. P., On the optimal control of nonlinear systems, Systems and Control Letters, 1986, 6, pp. 337–343

[3] Naidu, D. S., Optimal Control Systems, CRC Press, 2002.

[4] Knobloch, H. W., Isidori, A., and Flokerzi, D., Topics in Control Theory, DMV seminar band 22, Birkhauser, Berlin, 1993.

[5] Green M. and Limebeer, D. J. N., *Linear Robust Control*, Prentice Hall, New Jersey 1995.

[6] Lukyanov, A. G., and Utkin, V. I., Method of reducing equations for dynamic systems to a regular form, Automation and Remote Control, 42, 1981 pp. 413–420.

[7] Liu, J., and Elia, N., Quantized feedback stabilization of non-linear affine systems, Int. J. Control, 77, pp. 239ï£¡249.

[8] Ge,S.S., Wang, Z. and Lee, T.H., Adaptive stabilization of uncertain nonholonomic systems by state and output feedback, Automatica 39, 2003, pp. 1451-1460.

[9] Astolfi, A. and Ortega, R., Invariant manifolds, asymptotic immersion and the (adaptive) stabilization of nonlinear systems. In: Zinober, A. and Owens, D., Editors, 2002. *Nonlinear and adaptive control*, Springer, Berlin.

[10] Utkin, V. I., *Sliding Modes in Control and Optimisation*, Springer-Verlag, Berlin, 1992.

[11] Freeman, R., and Kokotovicï£¡, P., Robust Nonlinear Control Design: State-Space and Lyapunov Techniques. Boston, MA: Birkhï£¡user, 1996.

[12] Byrnes, C. I., and Isidori, I., Asymptotic stabilization of minimum-phase nonlinear systems, IEEE Trans. Automat. Control, vol. AC-36, No. 10, pp. 1122-1137, 1991.

[13] Meyer-Baese, A., Koshkouei, A. J., Emmett, M. R. and Goodall D., Global stability analysis and robust design of multi-time-scale biological networks under parametric uncertainties. Neural Networks 22, pp 658-663, 2009.

[14] Song, B., Hedrick, J. K., and Howell, a., robust stabilization and ultimate boundedness of dynamic surface control systems via convex optimization, int. J. Control, , Vol. 75, No. 12, 870 -881, 2002.

[15] Sangelaji, Z. and Banks, S. P., Radial Control Design for a Class of Nonlinear Systems. Journal of Control and Intelligent Systems, 37(3), 144-151, 2009.

[16] Sangelaji, Z. and Banks, S. P., Stabilisation of Nonlinear Systems Using Weighted Angular Method, IET Control Theory and Applications, 3(4), 445-451, 2009.

[17] Sangelaji, Z. and Banks, S. P., Control design and stabilization analysis of non-linear systems using angular representations. Proc. IMechE, Part I: J. Systems and Control Engineering, 222(I6), 457-463, 2008.