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# The benefits of nonlinear cubic viscous damping on the force transmissibility of a Duffing-type vibration isolator

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*Abstract*—**Vibration isolation systems with nonlinear stiffness under sinusoidal excitation exhibit unwanted jump phenomena and superharmonics when they are lightly damped. These characteristics can be suppressed by linear viscous damping but the force transmissibility over the high frequency range increases as a result. In this study, nonlinear viscous damping will be chosen to solve this problem with the aid of a single-degree-of-freedom model with cubic stiffness. Simulation results show that nonlinear viscous damping can reduce the resonant peak as well as suppressing the adverse properties of nonlinear stiffness, jumps and harmonics, without compromising the transmissibility over the high frequency range. Nonlinear damping preserves the benefits of linear damping while removing the undesirable effects over the non-resonant regions and therefore improves the overall performance.**

**Keywords: nonlinear damping, nonlinear spring, jump phenomena, duffing, OFRF**

## I. Introduction

Passive vibration isolation systems are employed in many engineering applications where objects require protection from undesired vibration force or displacement. They are usually installed between the source of the disturbance and equipment requiring protection in order to reduce the level of vibration transmitted to the object. A single-degree-of-freedom (sdof) vibration isolator, modelled by a linear mass-spring-damper system, has been well-studied by many authors [1]–[9]. The need for nonlinear vibration isolation and the recent developments are summarised by Ibrahim [10].

The nonlinearity usually takes the form of either a nonlinear spring force or a nonlinear damping force. There are a number of works focusing on Duffing-type isolators where the restoring spring force is a cubic function of the displacement [11]–[14]. Compared to the linear case, the nonlinear stiffness term could lower the force transmissibility around the resonant frequency range. In some cases, the resonant frequency can be reduced, resulting in a larger frequency range of vibration isolation. One major drawback of nonlinear stiffness, however, is the jump phenomena, where the transmissibility suddenly jumps up or down when there is a small change in the frequency of the excitation signal. The inherent instability and the resulting high level of higher harmonics may cause some concerns. The effects of the nonlinear spring force on an sdof system was compared with that of the nonlinear damping force by Zhang et al. [15]. Their results showed that nonlinear damping is preferable as a significant reduction of the peak transmitted force can be achieved without bifurcations in the system. The study by Lang et al [16] showed that the nonlinear viscous damping force could complement the drawbacks of linear viscous damping by modifying the force transmissibility in the resonant region without causing any detrimental effects over the non-resonant regions.

The application of nonlinear stiffness to vibration isolation has a longer history than the use of nonlinear viscous damping. Because of the different pros and cons of these two nonlinear forces, this paper aims to examine the benefits brought by nonlinear viscous damping when it is incorporated into an isolator with nonlinear stiffness. The nonlinear stiffness may be an internal property of the isolator or it may be an inherit property of the system that requires protection from vibration. The isolation system with cubic stiffness is represented by an sdof mass-spring-damper model. The transmissibility performance over a range of frequencies for different levels of linear and nonlinear damping obtained by simulation will be displayed. The comparison of the transmissibility curves will reveal the benefits of nonlinear viscous damping over linear viscous damping. The jump phenomena and the harmonics caused by the spring nonlinearity can be suppressed by either linear or nonlinear damping but only nonlinear damping can maintain the performance over high frequency regions simultaneously.

An sdof mass-spring-damper model with nonlinear spring and damping forces under sinusoidal excitation is described in Section II. The influence of nonlinear damping on this model are given in Section III. Simulation results and discussions are provided in Section IV and finally a conclusion is given in Section V.

## II. SDOF VIBRATION ISOLATORS WITH SPRING AND DAMPING NONLINEARITY

Consider a single-degree-of-freedom (sdof) vibration isolation system under a sinusodial excitation force $f_{in}(t)$ as shown in Figure 1, where

$$f_{in}(t) = A\sin(\bar{\omega}t) \tag{1}$$

with magnitude $A$ and frequency $\bar{\omega}$. The force transmitted to the immobile base, $f_{out}(t)$, is related to $f_{in}(t)$ by the equations of motion of the nonlinear vibration isolation system given by

$$\begin{cases} M\ddot{x}(t) + C_1\dot{x}(t) + C_2[\dot{x}(t)]^3 \\ +K_1x(t) + K_2[x(t)]^3 = f_{in}(t) = A\sin(\bar{\omega}t) \\ \\ f_{out}(t) = C_1\dot{x}(t) + C_2[\dot{x}(t)]^3 + K_1x(t) + K_2[x(t)]^3 \end{cases} \tag{2}$$

where $x(t)$ is the displacement of the moving mass, $M$ is the mass, $C_1$ the viscous damping constant, $C_2$ the cubic viscous damping constant, $K_1$ the linear spring constant, $K_2$ the cubic spring constant. When $K_2, C_2 = 0$, System (2) is identical to a well-studied linear system. When $K_2 \neq 0$ and $C_2 = 0$, it becomes a Duffing-type vibration isolator which exhibits jump-up and jump-down phenomena as discussed in many publications [13], [17], [18].

For the purpose of general analysis, System (2) is reduced to a non-dimensional form which is non-specific to chosen values of $M$ and $K_1$. By denoting the resonant frequency $\omega_0 = \sqrt{K_1/M}$, the first equation in System (2) becomes

$$\ddot{y}(\tau) + \xi_1\dot{y}(\tau) + \xi_2[\dot{y}(\tau)]^3 + y(\tau) + \gamma[y(\tau)]^3 = \sin(\Omega\tau) \tag{3}$$

where

$$\tau = \omega_0 t, \tag{4}$$

$$\Omega = \frac{\bar{\omega}}{\omega_0}, \tag{5}$$

$$y(\tau) = \frac{K_1}{A}x(t), \tag{6}$$

$$\gamma = \frac{A^2K_2}{K_1^3}, \tag{7}$$

$$\xi_1 = \frac{C_1}{\sqrt{K_1M}}, \tag{8}$$

(9)

and

$$\xi_2 = \frac{C_2A^2}{\sqrt{(K_1M)^3}}. \tag{10}$$

System (2) can be viewed as a single-input two-output system by substituting in

$$y_1(\tau) = y(\tau) \tag{11}$$

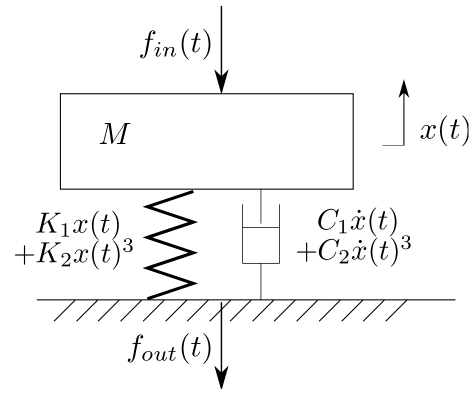


Fig. 1. Single-degree-of-freedom vibration isolation system with nonlinear cubic stiffness and nonlinear cubic viscous damping

and

$$y_2(\tau) = y_1(\tau) + \gamma[y_1(\tau)]^3 + \xi_1\dot{y}_1(\tau) + \xi_2[\dot{y}_1(\tau)]^3 \tag{12}$$

hence the model becomes

$$\begin{cases} \ddot{y}_1(\tau) + y_2(\tau) = u(\tau) = \sin(\Omega\tau) \\ y_2(\tau) = y_1(\tau) + \gamma[y_1(\tau)]^3 + \xi_1\dot{y}_1(\tau) + \xi_2[\dot{y}(\tau)]^3. \end{cases} \tag{13}$$

The force transmissibility $T_1(\Omega)$, the ratio of the magnitude of the transmitted force at the excitation frequency to that of the excitation force, can be deduced from equations (2), (4) - (10) and (13). In the time domain, this gives

$$\frac{f_{out}(t)}{A} = \frac{C_1\dot{x}(t) + C_2[\dot{x}(t)]^3 + K_1x(t) + K_2[x(t)]^3}{A} \tag{14}$$

$$= y_1(\tau) + \gamma[y_1(\tau)]^3 + \xi_1\dot{y}_1(\tau) + \xi_2[\dot{y}_1(\tau)]^3 \tag{15}$$

$$= y_2(\tau). \tag{16}$$

By applying Fourier transform to $y_2(\tau)$ and evaluating at $\omega = \Omega$ gives

$$T_1(\Omega) = \Big|Y_2(j\omega)|_{\omega=\Omega}\Big| = \Big|Y_2(j\Omega)\Big|. \tag{17}$$

Equation (17) implies that the force transmissibility of the system can be evaluated by examining the spectrum of the second output of System (13).

Because of the system's nonlinearity, the second output $y_2(\tau)$ contains the fundamental frequency of the input signal as well as harmonics. As System (13) has cubic stiffness and cubic damping, the third harmonics of the output may be significant. This harmonic effect can be described by $T_3(\Omega)$ defined as

$$T_3(\Omega) = \Big|Y_2(j\omega)|_{\omega=3\Omega}\Big|. \tag{18}$$

The numerical simulation results of System (13) will be presented in Section III. The effects of nonlinear viscous damping on the performance of vibration isolation will be shown by $T_1(\Omega)$ and $T_3(\Omega)$.

## III. The effects of nonlinear viscous damping on a Duffing-type vibration isolator

From analysis and observation, the following conclusions can be made.

**Proposition 1**

*(i)*

When $\Omega \approx 1$, there exists a $\bar{\xi_2} > 0$ such that

$$\left.\frac{d[T_1(\Omega)]^2}{d\xi_2}\right|_{\Omega\approx 1} < 0 \tag{19}$$

if $0 < \xi_2 < \bar{\xi_2}$ for $0 \leq \gamma < \bar{\gamma}$ where $\bar{\gamma}$ is the maximum value of $\gamma$ beyond which jump phenomena will be observed in System (13).

*(ii)*

When $\Omega \ll 1$ or $\Omega \gg 1$, there exists a $\bar{\xi_2} > 0$ such that

$$T_1(\Omega)|_{\xi_2>0} \approx T_1(\Omega)|_{\xi_2=0} \tag{20}$$

if $0 < \xi_2 < \bar{\xi_2}$ for $0 \leq \gamma < \bar{\gamma}$ where $\bar{\gamma}$ is the maximum value of $\gamma$ beyond which jump phenomena will be observed in System (13).

*(iii)*

For fixed values of $\gamma, \xi_1 > 0$, if System (13) exhibits jump phenomena, there exists a $\bar{\xi_2}$ such that the jumps are eliminated.

The complete proof is still under study but a brief description of the idea is provided here. System (13) is a polynomial form nonlinear differential equation model, the input and the two outputs of which can be represented by a Volterra series around the zero equilibrium point in the time domain [19]. The force transmissibility (i.e. the spectrum of the second output) can be related to the input spectrum using the output frequency response function (OFRF) concept [20]. The OFRF defines the relationship between the output spectrum (the force transmissibility) and the parameters which characterise the nonlinearity of a Volterra system ($\gamma$ and $\xi_2$) for a given input and fixed values of the linear terms. This concept is applied to a vibration isolation system with linear stiffness and nonlinear damping in a theoretical study by Lang et al. [16]. The same approach can be applied to System (13) which has an extra nonlinear stiffness term. Its force transmissibility $T_1(\Omega)$ can then be expressed as a polynomial function of $\gamma$ and $\xi_2$. The derivative of $[T_1(\Omega)]^2$ with respect to $\xi_2$ can be obtained for $\Omega \approx 1$ to prove conclusion (i) and the values of $T_1(\Omega)$ for $\xi_2 > 0$ and $\xi_2 = 0$ are compared to reach conclusion (ii).

Conclusion (iii) states that the jump phenomena can be eliminated by nonlinear damping. As the Volterra model cannot capture the jump phenomena, the OFRF concept cannot be applied here. An alternative approach, the harmonic balance method, can be used for this analysis [11], [12], [21]–[27]. The solution of the second output of System (13) is assumed to be of the form of a truncated Fourier series. Substituting this into System (13) and equating coefficients related to each harmonic components yields an expression for $T_1(\Omega)$. This well-known method has been the main tool for the analysis of Duffing systems and can now be extended to the study of a Duffing system with an additional nonlinear damping term.

The three conclusions of Proposition 1 are based on some previous analyses and observations from numerical examples given in Section IV. It is worth pointing out that for System (13), the following remark about the harmonics in the system output can also be made.

**Remark 1**

When $\Omega \approx 1$, there exists a $\bar{\xi_2} > 0$ such that

$$\left.\frac{d[T_3(\Omega)]^2}{d\xi_2}\right|_{\Omega\approx 1} < 0 \tag{21}$$

if $0 < \xi_2 < \bar{\xi_2}$ for $0 \leq \gamma < \bar{\gamma}$ where $\bar{\gamma}$ is the maximum value of $\gamma$ beyond which jump phenomena will be observed in System (13).

The present study extends the findings of Lang et al. [16] to a vibration isolation system with nonlinear stiffness. The effects of nonlinear damping are concluded by Proposition 1 and Remark 1. Equation (19), the first conclusion of Proposition 1, suggests that an increase in the value of cubic damping leads to a reduction in the force transmissibility around the resonant frequency range and the second conclusion, summarised by Equation (20), indicates that the the transmissibility over the frequency range far away from the resonant frequency remains unaffected by nonlinear damping. These findings conclude that the ideal properties of nonlinear damping found by Lang et al. [16] also hold true for systems with nonlinear stiffness for certain values of $\gamma$. The jump phenomena is addressed by the third conclusion which shows that the cubic damping, when it reaches a high enough level, can remove the jump phenomena caused by the nonlinear spring thus greatly improving the system stability. Remark 1 focuses on the superharmonics produced by the nonlinear spring. The highest level of superharmonics is produced when the excitation frequency is near the resonant frequency. The derivative of $[T_3(\Omega)]^2$ with respect to $\xi_2$ is negative so Equation (21) indicates that the third harmonic can be reduced by nonlinear damping. These are the main advantages of including nonlinear viscous damping on a Duffing-type system.

## IV. Simulation results and discussion

Numerical simulation studies were conducted using the standard MATLAB solver for ordinary differential equations (`ode45`) to demonstrate the effects of nonlinear viscous damping on a Duffing-type vibration isolation system. The results are presented in Figures 2 to 6.

The advantage of nonlinear viscous damping over conventional linear damping on an sdof vibration isolation system with linear stiffness, as described in the study by Lang et al. [16], are summarised by Figures 2 and 3. In Figure 2, the system has linear stiffness and linear damping with $\xi_1$ = 0.1, 0.2 and 0.4. As the linear damping parameter $\xi_1$ increases, $T_1(\Omega)$ around the resonant region decreases but there is an undesirable increase in $T_1(\Omega)$ in the high frequency region. In Figure 3, the value of $\xi_1$ is kept constant at 0.1 while the cubic damping parameter $\xi_2$ takes the values 0, 0.2 and 0.4. In this case, the resonance at $\Omega \approx 1$ is suppressed by the increase of $\xi_2$ while $T_1(\Omega)$ over the frequency ranges of $\Omega \ll 1$ and $\Omega \gg 1$ is unaffected.

The simulation is repeated for System (13) where the nonlinear stiffness is taken into account (i.e. $\gamma > 0$). Figure 4 shows the effects of linear damping with the presence of nonlinear stiffness. The contrast of the effects of linear and nonlinear viscous damping can be observed by comparing Figure 4 with Figures 5 and 6, where the cubic viscous damping constant $\xi_2$ takes the values 0.2, 0.4 and 0.6 for $\gamma = 0.1$ and $\gamma = 0.2$ respectively. Similar to the case with pure linear stiffness, the cubic viscous damping again modifies the resonant region without causing detrimental effects in the non-resonant regions. It is clear that the increase of $\xi_1$ leads to an increase of $T_1(\Omega)$ over the high frequency range whereas the increase of $\xi_2$ does not. Thus, nonlinear viscous damping has a significant advantage over linear viscous damping on vibration isolation even with the presence of nonlinear stiffness. These observations are summarised by the first two conclusions of Proposition 1.

Nonlinear viscous damping may also address the well-known jump-up and jump-down phenomena of systems with nonlinear stiffness. To ensure stability, jump avoidance should be an important feature of a vibration isolator. The study by Ravindra and Mallik [18] found that the jump phenomena could be eliminated by linear viscous damping. With pure linear viscous damping, the simulation results in Figure 4 shows a jump occurring at $\Omega \approx 1.4$ when $\xi_1 = 0.1$ for $\gamma = 0.2$ and $\xi_2 = 0$ but the jump no long exists when the level of linear damping increases to $\xi_1 = 0.2$. The trade-off of applying linear damping to remove the jump phenomena is the adverse effects on the transmissibility over the high frequency range. This problem can be overcome by employing nonlinear viscous damping as discussed above. In Figures 5 and 6, the jumps disappear when $\xi_2$ increases from 0 to 0.2 while the shape of the force transmissibility curve remains unchanged for $\Omega \ll 1$ and $\Omega \gg 1$. This is the conclusion (iii) of Proposition 1.

Under the excitation of a sinusoidal input force, the output of an sdof system with nonlinear stiffness exhibits some harmonics. To achieve a good level of isolation, these harmonics, which are transmitted to the base, should be minimised. The spectrum of the second output of System (13) contains a strong component at the excitation frequency $\Omega$ plus the higher harmonics at $n\Omega$, where $n = 3, 5, 7, \cdots$. The effects of the fifth and higher harmonics are neglected as their magnitudes are small. The simulation results of $T_3(\Omega)$, the third harmonics, are illustrated in Figures 7 and 8. Consider the case when $\gamma = 0.2$ and $\xi_1 = 0.1$, two peaks, one at $\Omega \approx 0.4$ and another at $\Omega \approx 1.1$, are observed in Figure 7. The magnitudes of both peaks decrease as the value of nonlinear viscous damping increases. This is the conclusion of Remark 1. While $T_1(\Omega)$ measures the force transmissibility at the fundamental frequency, $T_3(\Omega)$ indicates the level of output force at $3\Omega$. The sum of $T_1(\Omega)$ and $T_3(\Omega)$ (and the higher harmonics terms) measures the overall energy transmissibility. Nonlinear damping can reduce the peaks occurring in both $T_1(\Omega)$ and $T_3(\Omega)$ so it can also remove energy at frequencies $\Omega \approx 1$ from an sdof system with nonlinear stiffness effectively.

The effects of linear damping on the third harmonics $T_3(\Omega)$ are included in Figure 8 for completeness. The results suggest that linear damping reduces $T_3(\Omega)$ over the whole frequency range, as opposed to just the $\Omega \approx 1$ region in Figure 7. This means that linear damping is better at suppressing the higher harmonics than nonlinear damping. However, when the total energy transmitted is considered, an increase in $\xi_2$ leads to a far greater rise in $T_1(\Omega)$ than the fall in $T_3(\Omega)$ for $\Omega \gg 1$. In the nonlinear damping case, $T_3(\Omega)$ rises from 0.003 (-52 dB) to 0.008 (-42 dB) when $\xi_2$ increases from 0.2 to 0.6 at $\Omega$ = 3. This increase is very small in absolute terms. Nonlinear damping therefore remains the preferred choice over linear damping on a vibration isolation system with nonlinear stiffness.

## V. Conclusions

A vibration isolation system with nonlinear stiffness and linear viscous damping has been studied by many researchers. The force transmitted contains harmonics of the excitation frequency. The jump-up and jump-down phenomena occurs when the level of linear damping is small. This current study shows that the overall performance of vibration isolation is enhanced by the introduction of nonlinear viscous damping based on an sdof model described in Section II . The simulation results reveal important features of nonlinear viscous damping on a Duffing-type system outlined in Section III. The nonlinear damping parameter can eliminate the jump phenomena and reduce the third harmonics of the transmitted force when the excitation frequency is close to the resonant frequency. These may also be achieved by linear viscous damping but the nonlinear

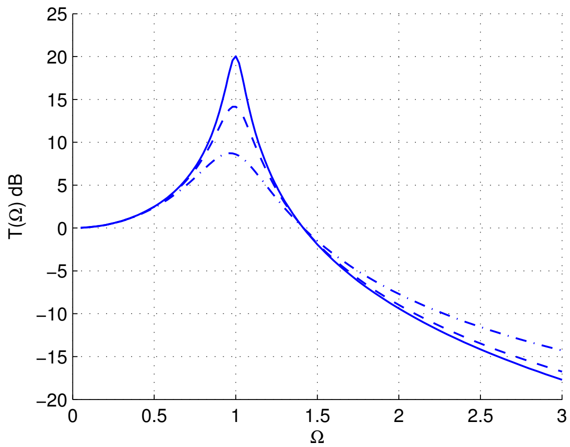


Fig. 2. The force transmissibility of System (2) with linear stiffness and linear viscous damping, where $\gamma, \xi_2 = 0$. Solid: $\xi_1 = 0.1$; Dashed: $\xi_1 = 0.2$; Dot-dashed: $\xi_1 = 0.4$.

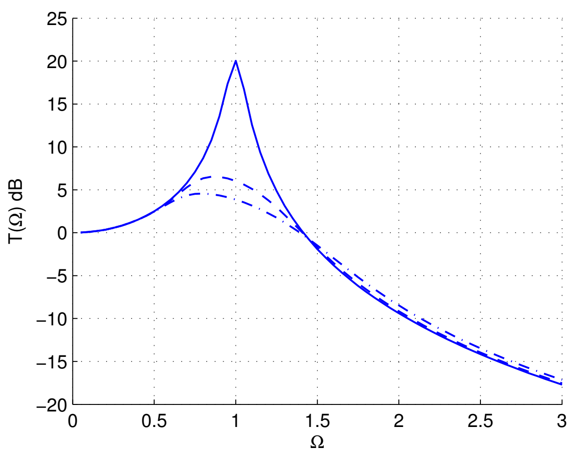


Fig. 3. The force transmissibility of System (2) with linear stiffness and nonlinear viscous damping, where $\gamma = 0$ and $\xi_1 = 0.1$. Solid: $\xi_2 = 0$; Dashed: $\xi_2 = 0.2$; Dot-dashed: $\xi_2 = 0.4$.

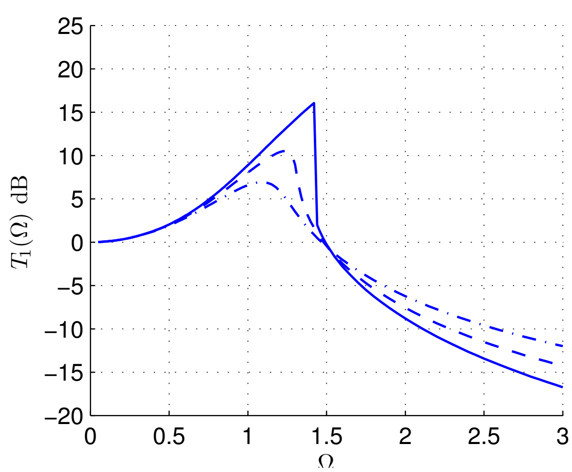


Fig. 4. The force transmissibility of System (2) with nonlinear stiffness and linear viscous damping, where $\gamma = 0.2$, $\xi_2 = 0$. Solid: $\xi_1 = 0.1$; Dashed: $\xi_1 = 0.2$; Dot-dashed: $\xi_1 = 0.4$.

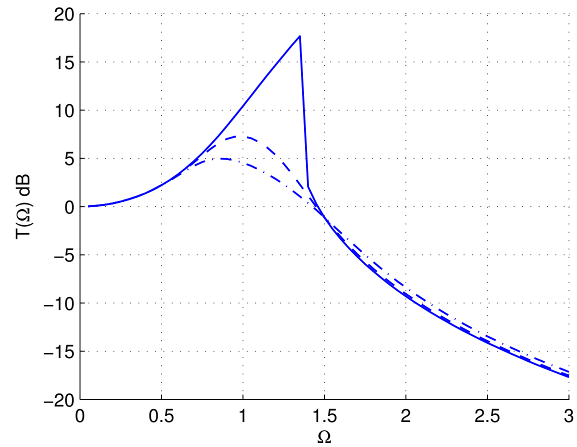


Fig. 5. The force transmissibility of System (2) with nonlinear stiffness and nonlinear viscous damping, where $\gamma, \xi_1 = 0.1$. Solid: $\xi_2 = 0$; Dashed: $\xi_2 = 0.2$; Dot-dashed: $\xi_2 = 0.4$.

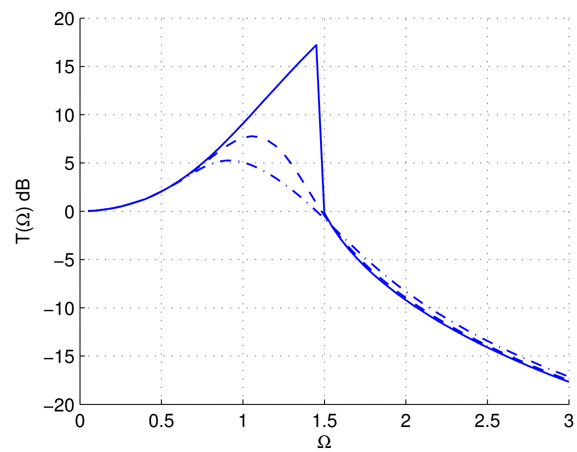


Fig. 6. The force transmissibility of System (2) with nonlinear stiffness and nonlinear viscous damping, where $\gamma = 0.2$ and $\xi_1 = 0.1$. Solid: $\xi_2 = 0$; Dashed: $\xi_2 = 0.2$; Dot-dashed: $\xi_2 = 0.4$.

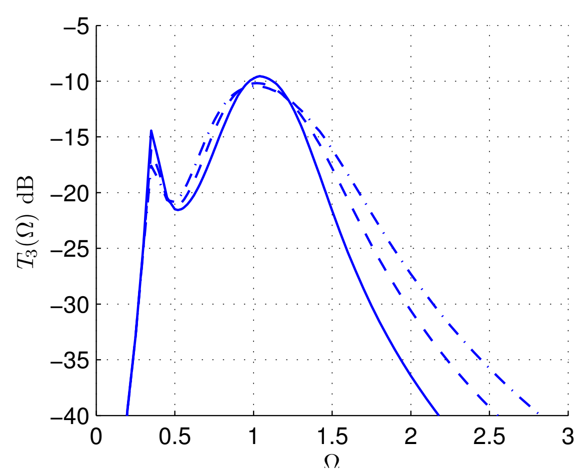


Fig. 7. The third harmonic of the force transmissibility of System (2) with nonlinear stiffness and nonlinear viscous damping, where $\gamma = 0.1$ and $\xi_1 = 0.1$. Solid: $\xi_2 = 0.2$; Dashed: $\xi_2 = 0.4$; Dot-dashed: $\xi_2 = 0.6$.

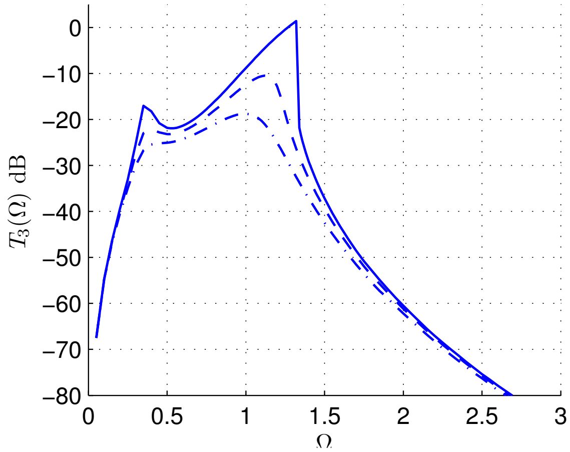


Fig. 8. The third harmonic of the force transmissibility of System (2) with nonlinear stiffness and linear viscous damping, where $\gamma = 0.1$ and $\xi_2 = 0$. Solid: $\xi_1 = 0.2$; Dashed: $\xi_1 = 0.4$; Dot-dashed: $\xi_1 = 0.6$.

damping parameter has a major additional advantage - it lowers the force transmissibility over the frequencies range around the resonant region while keeping the high frequency range unaffected. These features bring significant benefits to passive vibration isolation when nonlinear stiffness is present.

Theoretical analyses including a rigorous proof for Proposition 1 using the OFRF approach and the harmonic balance method will be provided in a later publication. Further studies will focus on the practical applications of this concept on different engineering designs. There are some recent research results on achieving nonlinear viscous damping by controlling the output of Magnetorheological dampers [28]–[30]. The authors will continue to explore this idea and apply the concept of nonlinear viscous damping to areas such as car suspension control and shock absorption for structures.

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## References


[1] C. E. Crede, *Vibration and shock isolation*. Wiley, 1951.

[2] J. C. Snowdon, *Vibration and shock in damped mechanical systems*. J. Wiley, 1968.

[3] J. B. Vernon, *Linear vibration theory*. Wiley, 1967.

[4] J. C. Snowdon, "Vibration isolation: use and characterization," *The Journal of the Acoustical Society of America*, vol. 66, p. 1245, 1979.

[5] E. Suhir, "Shock protection with a nonlinear spring," *Components, Packaging, and Manufacturing Technology, Part A, IEEE Transactions on*, vol. 18, no. 2, pp. 430–437, 1995.

[6] E. I. Rivin, *Passive vibration isolation*. Amer Society of Mechanical, 2003.

[7] P. Thureau, D. Lecler, and J. Grosjean, *An introduction to the principles of vibrations of linear systems*. Thornes, 1981.

[8] D. J. Mead, *Passive vibration control*. John Wiley & Sons Inc, 1999.

[9] M. J. Crocker, *Handbook of noise and vibration control*. Wiley, 2007.

[10] R. A. Ibrahim, "Recent advances in nonlinear passive vibration isolators," *Journal of Sound and Vibration*, vol. 314, no. 3-5, pp. 371–452, 2008.

[11] L. Liu, J. P. Thomas, E. H. Dowell, P. Attar, and K. C. Hall, "A comparison of classical and high dimensional harmonic balance approaches for a duffing oscillator," *Journal of Computational Physics*, vol. 215, no. 1, pp. 298–320, 2006.

[12] H. Hu and J. H. Tang, "Solution of a duffing-harmonic oscillator by the method of harmonic balance," *Journal of sound and vibration*, vol. 294, no. 3, pp. 637–639, 2006.

[13] M. J. Brennan, I. Kovacic, A. Carrella, and T. P. Waters, "On the jump-up and jump-down frequencies of the duffing oscillator," *Journal of Sound and Vibration*, vol. 318, no. 4-5, pp. 1250–1261, 2008.

[14] A. Carrella, M. J. Brennan, T. P. Waters, and V. Lopes, "Force and displacement transmissibility of a nonlinear isolator with high-static-low-dynamic-stiffness," *International Journal of Mechanical Sciences*, 2011.

[15] B. Zhang, S. A. Billings, Z. Q. Lang, and G. R. Tomlinson, "Suppressing resonant vibrations using nonlinear springs and dampers," *Journal of Vibration and Control*, vol. 15, no. 11, p. 1731, 2009.

[16] Z. Lang, X. Jing, S. Billings, G. Tomlinson, and Z. Peng, "Theoretical study of the effects of nonlinear viscous damping on vibration isolation of sdof systems," *Journal Of Sound And Vibration*, vol. 323, no. 1-2, pp. 352–365, 2009.

[17] T. Fang and E. H. Dowell, "Numerical simulations of jump phenomena in stable duffing systems," *International journal of non-linear mechanics*, vol. 22, no. 3, pp. 267–274, 1987.

[18] B. Ravindra and A. K. Mallik, "Hard duffing-type vibration isolator with combined coulomb and viscous damping," *International journal of non-linear mechanics*, vol. 28, no. 4, pp. 427–440, 1993.

[19] S. Boyd and L. Chua, "Fading memory and the problem of approximating nonlinear operators with volterra series," *Circuits and Systems, IEEE Transactions on*, vol. 32, no. 11, pp. 1150–1161, 1985.

[20] Z. Q. Lang, S. A. Billings, R. Yue, and J. Li, "Output frequency response function of nonlinear volterra systems," *Automatica*, vol. 43, no. 5, pp. 805–816, 2007.

[21] G. Schmidt and A. Tondl, *Non-linear vibrations*. Cambridge Univ Pr, 1986, vol. 66.

[22] M. N. Hamdan and T. D. Burton, "On the steady state response and stability of non-linear oscillators using harmonic balance," *Journal of sound and vibration*, vol. 166, no. 2, pp. 255–266, 1993.

[23] N. MacDonald, "Choices in the harmonic balance technique," *Journal of Physics A: Mathematical and General*, vol. 26, p. 6367, 1993.

[24] H. P. W. Gottlieb, "Harmonic balance approach to limit cycles for nonlinear jerk equations," *Journal of Sound and Vibration*, vol. 297, no. 1, pp. 243–250, 2006.

[25] Z. K. Peng and Z. Q. Lang, "The effects of nonlinearity on the output frequency response of a passive engine mount," *Journal of Sound and Vibration*, vol. 318, no. 1-2, pp. 313–328, 2008.

[26] R. E. Mickens, *Truly nonlinear oscillations: harmonic balance, parameter expansions, iteration, and averaging methods*. World Scientific Pub Co Inc, 2010.

[27] Z. K. Peng, G. Meng, Z. Q. Lang, W. M. Zhang, and F. L. Chu, "Study of the effects of cubic nonlinear damping on vibration isolations using harmonic balance method," *International Journal of Non-Linear Mechanics*, 2011.

[28] N. D. Sims, R. Stanway, D. J. Peel, W. A. Bullough, and A. R. Johnson, "Controllable viscous damping: an experimental study of an electrorheological long-stroke damper under proportional feedback control," *Smart materials and structures*, vol. 8, p. 601, 1999.

[29] N. D. Sims, R. Stanway, A. R. Johnson, D. J. Peel, and W. A. Bullough, "Smart fluid damping: shaping the force/velocity response through feedback control," *Journal of intelligent material systems and structures*, vol. 11, no. 12, pp. 945–958, 2000.

[30] H. Laalej and Z. Q. Lang, "Numerical investigation of the effects of mr damper characteristic parameters on vibration isolation of sdof systems under harmonic excitations," *Journal of Intelligent Material Systems and Structures*, vol. 21, no. 5, p. 483, 2010.