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Inverse Optimal Robust Control of Singularly Impulsive Dynamical Systems

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***Abstract*— In this paper for the class of nonlinear uncertain singularly impulsive dynamical systems we present optimal robust control and inverse robust optimal control results. We consider a control problem for nonlinear uncertain singularly impulsive dynamical systems involving a notion of optimality with respect to an *auxiliary cost* which guarantees a bound on the worst-case value of a nonlinear-nonquadratic hybrid cost criterion over a prescribed uncertainty set. Further we specialize result to affine uncertain systems to obtain controllers predicated on an *inverse optimal hybrid control problem*. In particular, to avoid the complexity in solving the steady-state hybrid Hamilton-Jacobi-Bellman equation we parameterize a family of stabilizing hybrid controllers that minimize some *derived* hybrid cost functional that provides flexibility in specifying the control law. The performance integrand is shown to explicitly depend on the nonlinear singularly impulsive system dynamics, the Lyapunov function of the closed-loop system, and the stabilizing hybrid feedback control law wherein the coupling is introduced via the hybrid Hamilton-Jacobi-Bellman equation. By varying the parameters in the Lyapunov function and the performance integrand, the proposed framework can be used to characterize a class of globally stabilizing hybrid controllers that can meet the closed-loop system response constraints. Obtained results for nonlinear case are further specialized to linear singularly impulsive dynamical systems with polynomial and multilinear performance functional.**

***Index Terms*— mathematical model, singularly impulsive dynamical systems, optimal robust control, inverse optimal robust control**

I. INTRODUCTION

For the class of nonlinear uncertain singularly impulsive dynamical systems presented in [2], we have developed robust stability results in [7]. In this paper we give optimal robust control and inverse robust optimal control results. For that purpose, we generalize results developed in [3]. We consider a control problem for nonlinear uncertain singularly impulsive dynamical systems involving a notion of optimality with respect to an *auxiliary cost* which guarantees a bound on the worst-case value of a nonlinear-nonquadratic hybrid cost criterion over a prescribed uncertainty set. Further we specialize result to affine uncertain systems to obtain controllers predicated on an *inverse optimal hybrid control problem*. In particular, to avoid the complexity in solving the steady-state hybrid Hamilton-Jacobi-Bellman equation we parameterize a family of stabilizing hybrid controllers that minimize some *derived* hybrid cost functional that provides flexibility in specifying the control law. Obtained results for nonlinear case are further specialized to linear singularly impulsive dynamical systems with polynomial and multilinear performance functional.

Finally, in this paper we use the following standard notation. Let $\mathbb{R}$ denote the set of real numbers, let $\mathcal{N}$ denote the set of nonnegative integers, let $\mathbb{R}^n$ denote the set of $n \times 1$ real column vectors, let $\mathbb{R}^{n\times m}$ denote the set of $n \times m$ real matrices, let $\mathbb{S}^n$ denote the set of $n \times n$ symmetric matrices, and let $\mathbb{N}^n$ (resp., $\mathbb{P}^n$) denote the set of $n \times n$ nonnegative (resp., positive) definite matrices, and let I_n or I denote the $n \times n$ identity matrix. Furthermore, $A \geq 0$ (resp., $A > 0$) denotes the fact that the Hermitian matrix is nonnegative (resp., positive) definite and $A \geq B$ (resp., $A > B$) denotes the fact that $A - B \geq 0$ (resp., $A - B > 0$). In addition, we write $V'(x)$ for the Fréchet derivative of $V(\cdot)$ at x. Finally, let C^0 denote the set of continuous functions and C^r denote the set of functions with r continuous derivatives.

II. OPTIMAL ROBUST CONTROL FOR NONLINEAR UNCERTAIN SINGULARLY IMPULSIVE DYNAMICAL SYSTEMS

In this section we consider a control problem for nonlinear uncertain singularly impulsive dynamical systems involving a notion of optimality with respect to an *auxiliary cost* which guarantees a bound on the worst-case value of a nonlinear-nonquadratic hybrid cost criterion over a prescribed uncertainty set. The optimal robust hybrid time-invariant feedback controllers are derived as a direct consequence of Theorem 2.1 given in [7] and provide a generalization of the Hamilton-Jacobi-Bellman conditions for state-dependent singularly impulsive dynamical systems with optimality notions over the infinite horizon with an infinite number of resetting times, for addressing robust feedback controllers of nonlinear uncertain singularly impulsive dynamical systems. To address robust optimal control problem let $\mathcal{D} \subset \mathbb{R}^n$ be an open set with $0 \in \mathcal{D}$, and let $\mathcal{C}_\mathrm{c} \subset \mathbb{R}^{m_\mathrm{c}}$, $\mathcal{C}_\mathrm{d} \subset \mathbb{R}^{m_\mathrm{d}}$, where $0 \in \mathcal{C}_\mathrm{c}$ and $0 \in \mathcal{C}_\mathrm{d}$. Furthermore, let $\mathcal{F}_\mathrm{c} \subset \{F_\mathrm{c} : \mathcal{D} \times \mathcal{C}_\mathrm{c} \to \mathbb{R}^n : F_\mathrm{c}(0,0) = 0\}$, and $\mathcal{F}_\mathrm{d} \subset \{F_\mathrm{d} : \mathcal{D} \times \mathcal{C}_\mathrm{d} \to \mathbb{R}^n : F_\mathrm{d}(0,0) = 0\}$. For simplicity of exposition, we also define $(F_\mathrm{c}(\cdot,\cdot), F_\mathrm{d}(\cdot,\cdot)) \in \mathcal{F}_\mathrm{c} \times \mathcal{F}_\mathrm{d} \triangleq \mathcal{F}$. Next, consider the nonlinear uncertain singularly impulsive controlled dynamical system

$$E_\mathrm{c}\dot{x}(t) = F_\mathrm{c}(x(t), u_\mathrm{c}(t)), \quad x(0) = 0, \quad x(t) \notin \mathcal{Z}_x, \quad u_\mathrm{c}(t) \in \mathcal{U}_\mathrm{c}, \qquad \text{(II.1)}$$

$$E_\mathrm{d}\Delta x(t) = F_\mathrm{d}(x(t), u_\mathrm{d}(t)), \quad x(t) \in \mathcal{Z}_x, \quad u_\mathrm{d}(t) \in \mathcal{U}_\mathrm{d}, \qquad \text{(II.2)}$$

where $t \geq 0$, $x(t) \in \mathcal{D}$ is the state vector, $(u_\mathrm{c}(t), u_\mathrm{d}(t_k)) \in \mathcal{U}_\mathrm{c} \times \mathcal{U}_\mathrm{d} \subset \mathcal{C}_\mathrm{c} \times \mathcal{C}_\mathrm{d}$, $k \in \mathcal{N}$, is the hybrid control input,

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where the control constraint sets $\mathcal{U}_c, \mathcal{U}_d$ are given. We assume $(0,0) \in \mathcal{U}_c \times \mathcal{U}_d$, $F_c : \mathcal{D} \times \mathcal{U}_c \to \mathbb{R}^n$ is Lipschitz continuous and satisfies $F_c(0,0) = 0$, $F_d : \mathcal{D} \times \mathcal{U}_d \to \mathbb{R}^n$ is continuous and satisfies $F_d(0,0) = 0$, and $\mathcal{Z}_x \subset \mathbb{R}^n$. To address the robust optimal nonlinear hybrid feedback control problem let $\phi_c : \mathcal{D} \to \mathcal{U}_c$ be such that $\phi_c(0) = 0$ and let $\phi_d : \mathcal{D} \to \mathcal{U}_d$ be such that $\phi_d(0) = 0$. If $(u_c(t), u_d(t)) = (\phi_c(E_c x(t)), \phi_d(E_d x(t)))$, where $x(t)$, $t \geq 0$, satisfies (II.1), (II.2), then $(u_c(\cdot), u_d(\cdot))$ is a *hybrid feedback control*. Given the hybrid feedback control $(u_c(t), u_d(t)) = (\phi_c(E_c x(t)), \phi_d(E_d x(t)))$, the closed-loop state-dependent singularly impulsive dynamical system has the form

$$\begin{aligned} E_c \dot{x}(t) &= F_c(x(t), \phi_c(E_c x(t))), \qquad x(0) = x_0, \\ &\qquad t \geq 0, \quad x(t) \notin \mathcal{Z}_x, \qquad \text{(II.3)} \\ E_d \Delta x(t) &= F_d(x(t), \phi_d(E_d x(t))), \\ &\qquad x(t) \in \mathcal{Z}_x, \qquad \text{(II.4)} \end{aligned}$$

for all $(F_c(\cdot,\cdot), F_d(\cdot,\cdot)) \in \mathcal{F}$.

Next we present sufficient conditions for characterizing robust nonlinear hybrid feedback controllers that guarantee robust stability over a class of nonlinear uncertain singularly impulsive dynamical systems and minimize an auxiliary hybrid performance functional. For the statement of this result let $L_c : \mathcal{D} \times \mathcal{U}_c \to \mathbb{R}$, $L_d : \mathcal{D} \times \mathcal{U}_d \to \mathbb{R}$ and define the set of asymptotically stabilizing controllers for the nominal nonlinear singularly impulsive dynamical system $(F_{c0}(\cdot,\cdot), F_{d0}(\cdot,\cdot))$ by

$$\begin{aligned} \mathcal{C}(x_0) \triangleq \{(u_c(\cdot), u_d(\cdot)) \ : \ & (u_c(\cdot), u_d(\cdot)) \text{ is admissible} \\ & \text{and the zero solution } x(t) \equiv 0 \\ & \text{to (II.1), (II.2)} \\ & \text{is asymptotically stable with} \\ & (F_c(\cdot,\cdot), F_d(\cdot,\cdot)) \\ & = (F_{c0}(\cdot,\cdot), F_{d0}(\cdot,\cdot))\}. \end{aligned}$$

Consider the nonlinear uncertain singularly impulsive dynamical system (II.1), (II.2) with hybrid performance functional

$$\begin{aligned} J(E_c x_0, u_c(\cdot), u_d(\cdot)) &= \int_0^\infty L_c(E_c x(t), u(t)) \mathrm{d}t \\ &+ \sum_{k \in \mathcal{N}_{[0,\infty)}} L_d(E_d x(t_k), u_d(t_k)), \qquad \text{(II.5)} \end{aligned}$$

where $(\mathcal{F}_c(\cdot,\cdot), \mathcal{F}_d(\cdot,\cdot)) \in \mathcal{F}$ and $(u_c(\cdot), u_d(\cdot))$ is an admissible control. Assume there exist functions $V : \mathcal{D} \to \mathbb{R}$, $\Gamma_c : \mathcal{D} \times \mathcal{U}_c \to \mathbb{R}$, $\Gamma_d : \mathcal{D} \times \mathcal{U}_d \to \mathbb{R}$, and a hybrid control law $\phi_c : \mathcal{D} \to \mathcal{U}_c$ and $\phi_d : \mathcal{D} \to \mathcal{U}_d$, where $V(\cdot)$ is a C^1 function, such that

$$\begin{aligned} V(0) &= 0, \qquad \text{(II.6)} \\ V(E_c x) &\geq 0, \quad x \in \mathcal{D},\ x \neq 0, \qquad \text{(II.7)} \\ \phi_c(0) &= 0, \qquad \text{(II.8)} \\ \phi_d(0) &= 0, \qquad \text{(II.9)} \\ V'(E_c x) F_c(x, \phi_c(x)) &\leq V'(E_c x) F_{c0}(x, \phi_c(x)) \\ &+ \Gamma_c(x, \phi_c(x)), \quad x \notin \mathcal{Z}_x, \quad F_c(\cdot,\cdot) \in \mathcal{F}_c, \qquad \text{(II.10)} \end{aligned}$$

$$V'(E_c x) F_{c0}(x, \phi_c(x)) + \Gamma_c(x, \phi_c(x)) < 0,\ x \notin \mathcal{Z}_x,\ x \neq 0, \qquad \text{(II.11)}$$

$$\begin{aligned} V(E_d x + F_d(x, \phi_d(x))) \ - \ & V(E_d x) \leq \\ & V(E_d x + F_{d0}(x, \phi_d(x))) - V(E_d x) \\ & + \Gamma_d(x, \phi_d(x)), \quad x \in \mathcal{Z}_x,\ F_d(\cdot,\cdot) \in \mathcal{F}_d, \qquad \text{(II.12)} \\ V(E_d x + F_{d0}(x, \phi_d(x))) \ - \ & V(E_d x) + \Gamma_d(x, \phi_d(x)) \\ & \leq 0, \qquad x \in \mathcal{Z}_x, \qquad \text{(II.13)} \\ H_c(E_c x, \phi_c(x)) &= 0, \quad x \notin \mathcal{Z}_x, \qquad \text{(II.14)} \\ H_c(E_c x, u_c(x)) &\geq 0, \quad x \notin \mathcal{Z}_x,\ u_c \in \mathcal{U}_c, \qquad \text{(II.15)} \\ H_d(E_d x, \phi_d(E_c x)) &= 0, \quad x \in \mathcal{Z}_x, \qquad \text{(II.16)} \\ H_d(E_d x, u_d(x)) &\geq 0, \quad x \in \mathcal{Z}_x,\ u_d \in \mathcal{U}_d, \qquad \text{(II.17)} \end{aligned}$$

where $(F_{c0}(\cdot,\cdot), F_{d0}(\cdot,\cdot)) \in \mathcal{F}$ defines the nominal singularly impulsive dynamical system and

$$H_c(E_c x, u_c) \triangleq L_c(E_c x, u_c) + V'(E_c x) F_{c0}(x, u_c) + \Gamma_c(x, u_c), \qquad \text{(II.18)}$$

$$\begin{aligned} H_d(E_d x, u_d) \triangleq L_d(E_d x, u_d) &+ V(E_d x + F_{d0}(x, u_d)) \\ &- V(x E_d) + \Gamma_d(x, u_d). \qquad \text{(II.19)} \end{aligned}$$

Then, with the hybrid feedback control $(u_c(\cdot), u_d(\cdot)) = (\phi_c(E_c x(\cdot)), \phi_d(E_d x(\cdot)))$, there exists a neighborhood of the origin $\mathcal{D}_0 \subset \mathcal{D}$ such that if $x_0 \in \mathcal{D}_0$, the zero solution $x(t) \equiv 0$ of the closed-loop system (II.3), (II.4) is locally asymptotically stable for all $(F_c(\cdot,\cdot), F_d(\cdot,\cdot)) \in \mathcal{F}$. Furthermore,

$$\begin{aligned} \sup_{(F_c(\cdot,\cdot), F_d(\cdot,\cdot)) \in \mathcal{F}} \quad & J(E_c x_0, \phi_c(E_c x(\cdot)), \phi_d(E_d x(\cdot))) \\ & \leq \mathcal{J}(E_c x_0, \phi_c(\cdot), \phi_d(\cdot)) \\ & = V(E_c x_0), \quad x_0 \in \mathcal{D}_0, \qquad \text{(II.20)} \end{aligned}$$

where

$$\begin{aligned} \mathcal{J}(E_c x_0, u_c(\cdot), u_d(\cdot)) \ \triangleq \\ \int_0^\infty & [L_c(E_c x(t), u_c(t)) + \Gamma_c(x(t), u_c(t))] \mathrm{d}t \\ + & \sum_{k \in \mathcal{N}_{[0,\infty)}} [L_d(E_d x(t_k), u_d(t_k)) + \Gamma_d(x(t_k), u_d(t_k))], \end{aligned} \qquad \text{(II.21)}$$

and where $(u_c(\cdot), u_d(\cdot))$ is an admissible control and $x(t)$, $t \geq 0$, is a solution of (II.1), (II.2) with $(F_c(x(t), u_c(t)), F_d(x(t), u_d(t))) = (F_{c0}(x(t), u_c(t)), F_{d0}(x(t), u_d(t)))$. In addition, if $x_0 \in \mathcal{D}_0$ then the hybrid feedback control $(u_c(\cdot), u_d(\cdot)) = (\phi_c(E_c x(\cdot)), \phi_d(E_d x(\cdot)))$ minimizes $J(E_c x_0, u_c(\cdot), u_d(\cdot))$ in the sense that

$$\begin{aligned} & J(E_c x_0, \phi_c(E_c x(\cdot)), \phi_d(E_d x(\cdot))) = \\ & \min_{(u_c(\cdot), u_d(\cdot)) \in \mathcal{C}(x_0)} J(E_c x_0, u_c(\cdot), u_d(\cdot)). \qquad \text{(II.22)} \end{aligned}$$

Finally, if $\mathcal{D} = \mathbb{R}^n$, and

$$V(E_{c/d} x) \to \infty \quad \text{as} \quad \|x\| \to \infty, \qquad \text{(II.23)}$$

then the zero solution $x(t) \equiv 0$ of the closed-loop system (II.3), (II.4) is globally asymptotically stable for all $(F_c(\cdot,\cdot), F_d(\cdot,\cdot)) \in \mathcal{F}$, [3] and [7].

Proof: Local and global asymptotic stability is a direct consequence of (II.6)–(II.13) by applying Theorem 2.1 of [7] to the closed-loop system (II.3), (II.4). Next, let

$(u_{\rm c}(\cdot), u_{\rm d}(\cdot)) \in \mathcal{C}(x_0)$ and let $x(\cdot)$ be the solution of (II.1), (II.2) with $(F_{\rm c}(\cdot,\cdot), F_{\rm d}(\cdot,\cdot)) = (F_{\rm c0}(\cdot,\cdot), F_{\rm d0}(\cdot,\cdot))$.

Then it follows that

$$0 = -\dot{V}(E_{\rm c}x(t)) + V'(E_{\rm c}x(t))F_{\rm c}(x(t), u_{\rm c}(t)),\ x(t) \notin \mathcal{Z}_x, \tag{II.24}$$

$$\begin{aligned} 0 = & -\Delta V(E_{\rm d}x(t)) + V(E_{\rm d}x + F_{\rm d}(x(t), u_{\rm d}(t))) \\ & -V(E_{\rm d}x(t)), \qquad x(t) \in \mathcal{Z}_x. \end{aligned} \tag{II.25}$$

Hence,

$$\begin{aligned} & L_{\rm c}(E_{\rm c}x(t), u_{\rm c}(t)) + \Gamma_{\rm c}(E_{\rm c}\tilde{x}(t), u_{\rm c}(t)) = \\ & \quad -\dot{V}(E_{\rm c}x(t)) + L_{\rm c}(E_{\rm c}x(t), u_{\rm c}(t)) \\ & \quad +V'(E_{\rm c}x(t))F_{\rm c0}(x(t), u_{\rm c}(t)) + \Gamma_{\rm c}(E_{\rm c}\tilde{x}(t), u_{\rm c}(t)) \\ & \quad = -\dot{V}(E_{\rm c}x(t)) + H_{\rm c}(E_{\rm c}x(t), u_{\rm c}(t)), \\ & \qquad\qquad x(t) \notin \mathcal{Z}_x. \end{aligned} \tag{II.26}$$

Similarly,

$$\begin{aligned} & L_{\rm d}(E_{\rm d}x(t), u_{\rm d}(t)) + \Gamma_{\rm d}(x(t), u_{\rm d}(t)) = \\ & \quad -\Delta V(E_{\rm d}x(t)) + L_{\rm d}(E_{\rm d}x(t), u_{\rm d}(t)) \\ & \quad +\Delta V(E_{\rm d}x(t)) + \Gamma_{\rm d}(x(t), u_{\rm d}(t)) \\ & \quad = -\Delta V(E_{\rm d}x(t)) + H_{\rm d}(E_{\rm d}x(t), u_{\rm d}(t)), \\ & \qquad x(t) \in \mathcal{Z}_x. \end{aligned} \tag{II.27}$$

Now, over the interval $[0, t)$ yields

$$\begin{aligned} & \int_0^t [L_{\rm c}(E_{\rm c}x(t), u_{\rm c}(t)) + \Gamma_{\rm c}(\tilde{x}(t), u_{\rm c}(t))]{\rm d}t \\ & + \sum_{k\in\mathcal{N}_{[0,t)}} [L_{\rm d}(E_{\rm d}x(t_k), u_{\rm d}(t_k)) + \Gamma_{\rm d}(x(t_k), u_{\rm d}(t_k))] \\ & = \int_0^t [-\dot{V}(E_{\rm c}x(t)) + H_{\rm c}(x(t), u_{\rm c}(t))]{\rm d}t \\ & + \sum_{k\in\mathcal{N}_{[0,t)}} [-\Delta V(E_{\rm d}x(t_k)) + H_{\rm d}(E_{\rm d}x(t_k), u_{\rm d}(t_k))] \\ & = -V(E_{\rm c}x(t)) + V(E_{\rm c}x_0) + \int_0^t H_{\rm c}(E_{\rm c}x(t), u_{\rm c}(t)){\rm d}t \\ & + \sum_{k\in\mathcal{N}_{[0,t)}} H_{\rm d}(E_{\rm d}x(t_k), u_{\rm d}(t_k)) \\ & \geq V(E_{\rm c}x_0) \\ & = \mathcal{J}(E_{\rm c}x_0, \phi_{\rm c}(x(\cdot)), \phi_{\rm d}(x(\cdot))). \end{aligned} \tag{II.28}$$

Letting $t \to \infty$ and noting that $V(E_{\rm c/d}x(t)) \to 0$ for all $x_0 \in \mathcal{D}_0$ yields (II.22). □

Next, we specialize Theorem II to linear uncertain singularly impulsive dynamical systems. Specifically, in this case we consider $\mathcal{F} \triangleq \mathcal{F}_{\rm c} \times \mathcal{F}_{\rm d}$ to be the set of uncertain linear singularly impulsive dynamical systems, where

$$\begin{aligned} \mathcal{F}_{\rm c} = & \{(A_{\rm c} + \Delta A_{\rm c})x + B_{\rm c}u_{\rm c} : x \in \mathbb{R}^n, A_{\rm c} \in \mathbb{R}^{n\times n}, \\ & B_{\rm c} \in \mathbb{R}^{n\times m_{\rm c}}, \Delta A_{\rm c} \in \mathbf{\Delta}_{A_{\rm c}}\}, \\ \mathcal{F}_{\rm d} = & \{(A_{\rm d} + \Delta A_{\rm d} - E_{\rm d})x + B_{\rm d}u_{\rm d} : x \in \mathbb{R}^n, A_{\rm d} \in \mathbb{R}^{n\times n}, \\ & B_{\rm d} \in \mathbb{R}^{n\times m_{\rm d}}, \Delta A_{\rm d} \in \mathbf{\Delta}_{A_{\rm d}}\}, \end{aligned}$$

where $\mathbf{\Delta}_{A_{\rm c}}, \mathbf{\Delta}_{A_{\rm d}} \subset \mathbb{R}^{n\times n}$, are given bounded uncertainty sets of uncertain perturbations $\Delta A_{\rm c}, \Delta A_{\rm d}$ of the nominal system matrices $A_{\rm c}, A_{\rm d}$, such that $0 \in \mathbf{\Delta}_{A_{\rm c}}$ and $0 \in \mathbf{\Delta}_{A_{\rm d}}$. For simplicity of exposition, we also define $(\Delta A_{\rm c}, \Delta A_{\rm d}) \in \mathbf{\Delta}_{A{\rm c}} \times \mathbf{\Delta}_{A{\rm d}} \triangleq \mathbf{\Delta}$. For the following result let $R_{\rm c1} \in \mathbb{P}^n$, $R_{\rm c2} \in \mathbb{P}^{m_{\rm c}}$, $R_{\rm d1} \in \mathbb{N}^n$, $R_{\rm d2} \in \mathbb{N}^{m_{\rm d}}$ be given.

Consider the linear state-dependent uncertain singularly impulsive controlled dynamical system

$$\begin{aligned} E_{\rm c}\dot{x}(t) = & (A_{\rm c} + \Delta A_{\rm c})x(t) + B_{\rm c}u_{\rm c}(t), \quad x(0) = x_0, \\ & t \geq 0, \quad x(t) \notin \mathcal{Z}, \end{aligned} \tag{II.29}$$

$$E_{\rm d}\Delta x(t) = (A_{\rm d} + \Delta A_{\rm d} - E_{\rm d})x(t) + B_{\rm d}u_{\rm d}(t),\ x(t) \in \mathcal{Z}, \tag{II.30}$$

with performance functional

$$\begin{aligned} & J_{\Delta A_{\rm c},\Delta A_{\rm d}}(E_{\rm c}x_0, u_{\rm c}(\cdot), u_{\rm d}(\cdot)) \triangleq \\ & \int_0^\infty [x^{\rm T}(t)E_{\rm c}^{\rm T}R_{\rm c1}E_{\rm c}x(t) + u_{\rm c}^{\rm T}(t)R_{\rm c2}u_{\rm c}(t)]{\rm d}t \\ & + \sum_{k\in\mathcal{N}_{[0,\infty)}} [x^{\rm T}(t_k)E_{\rm d}^{\rm T}R_{\rm d1}E_{\rm d}x(t_k) + u_{\rm d}^{\rm T}(t_k)R_{\rm d2}u_{\rm d}(t_k)], \end{aligned} \tag{II.31}$$

where $(u_{\rm c}(\cdot), u_{\rm d}(\cdot))$ is admissible, $(\Delta A_{\rm c}, \Delta A_{\rm d}) \in \mathbf{\Delta}$. Furthermore, assume there exist $P \in \mathbb{P}^n$, $\Omega_{\rm c} : \mathbb{P}^n \to \mathbb{N}^n$, $\Omega_{{\rm d}xx} : \mathbb{P}^n \to \mathbb{N}^n$, $\Omega_{{\rm d}xu_{\rm d}} : \mathbb{N}^n \to \mathbb{R}^{n\times m_{\rm d}}$, and $\Omega_{{\rm d}u_{\rm d}u_{\rm d}} : \mathbb{N}^n \to \mathbb{N}^{m_{\rm d}}$, such that

$$\begin{aligned} & x^{\rm T}(\Delta A_{\rm c}^{\rm T}E_{\rm c}^{\rm T}P + P\Delta A_{\rm c}E_{\rm c})x \leq x^{\rm T}E_{\rm c}^{\rm T}\Omega_{\rm c}(P)E_{\rm c}x, \\ & \qquad x \notin \mathcal{Z}, \quad \Delta A_{\rm c} \in \mathbf{\Delta}_{A{\rm c}}, \end{aligned} \tag{II.32}$$

$$\begin{aligned} & x^{\rm T}(\Delta A_{\rm d}^{\rm T}PA_{\rm d} + A_{\rm d}^{\rm T}P\Delta A_{\rm d} - \Delta A_{\rm d}PB_{\rm d}(R_{\rm d2} \\ & +B_{\rm d}^{\rm T}PB_{\rm d} + \Omega_{{\rm d}u_{\rm d}u_{\rm d}}(P))^{-1} \\ & (B_{\rm d}^{\rm T}PA_{\rm d} + E_{\rm d}^{\rm T}\Omega_{{\rm d}xu_{\rm d}}^{\rm T}(P)E_{\rm d}) - (B_{\rm d}^{\rm T}PA_{\rm d} \\ & +\Omega_{{\rm d}xu_{\rm d}}^{\rm T}(P))^{\rm T}(R_{\rm d2} + B_{\rm d}^{\rm T}PB_{\rm d} + \Omega_{{\rm d}u_{\rm d}u_{\rm d}}(P))^{-1} \\ & B_{\rm d}^{\rm T}P\Delta A_{\rm d} + \Delta A_{\rm d}^{\rm T}P\Delta A_{\rm d})x \leq x^{\rm T}(E_{\rm d}^{\rm T}\Omega_{{\rm d}xx}(P)E_{\rm d}^{\rm T} \\ & -\Omega_{{\rm d}xu_{\rm d}}(P) \\ & (R_{\rm d2}+B_{\rm d}^{\rm T}PB_{\rm d} +\Omega_{{\rm d}u_{\rm d}u_{\rm d}}(P))^{-1} \\ & (B_{\rm d}^{\rm T}PA_{\rm d}+\Omega_{{\rm d}xu_{\rm d}}^{\rm T}(P)) \\ & -(B_{\rm d}^{\rm T}P+\Omega_{{\rm d}xu_{\rm d}}^{\rm T}(P))^{\rm T}(R_{\rm d2} + B_{\rm d}^{\rm T}PB_{\rm d} + \Omega_{{\rm d}u_{\rm d}u_{\rm d}}(P))^{-1} \\ & \Omega_{{\rm d}xu_{\rm d}}^{\rm T}(P) \\ & +(B_{\rm d}^{\rm T}PA_{\rm d} + \Omega_{{\rm d}xu_{\rm d}}^{\rm T}(P))^{\rm T}(R_{\rm d2} + B_{\rm d}^{\rm T}PB_{\rm d}+ \\ & \Omega_{{\rm d}u_{\rm d}u_{\rm d}}(P))^{-1} \\ & \cdot\Omega_{{\rm d}u_{\rm d}u_{\rm d}}(P)(R_{\rm d2} + B_{\rm d}^{\rm T}PB_{\rm d} \\ & +\Omega_{{\rm d}u_{\rm d}u_{\rm d}}(P))^{-1}(B_{\rm d}^{\rm T}PA_{\rm d} + \Omega_{{\rm d}xu_{\rm d}}^{\rm T}(P)))x, \\ & x \in \mathcal{Z}, \\ & \Delta A_{\rm d} \in \mathbf{\Delta}_{A{\rm d}}. \end{aligned} \tag{II.33}$$

Furthermore, suppose there exists $P \in \mathbb{P}^n$ satisfying

$$\begin{aligned} 0 = & x^{\rm T}(A_{\rm c}^{\rm T}PE_{\rm c} + E_{\rm c}^{\rm T}PA_{\rm c} + E_{\rm c}^{\rm T}R_{\rm c1}E_{\rm c} + \Omega_{\rm c}(P) - \\ & PB_{\rm c}R_{\rm c2}^{-1}B_{\rm c}^{\rm T}P)E_{\rm c}x, \qquad x \notin \mathcal{Z}, \end{aligned} \tag{II.34}$$

$$0 < R_{\rm d2} + B_{\rm d}^{\rm T}PB_{\rm d} + \Omega_{{\rm d}u_{\rm d}u_{\rm d}}(P), \tag{II.35}$$

$$\begin{aligned} 0 = & x^{\rm T}(A_{\rm d}^{\rm T}PA - E_{\rm d}^{\rm T}PE_{\rm d} + E_{\rm d}^{\rm T}R_{\rm d1}E_{\rm d} \\ & +\Omega_{{\rm d}xx}(P) - (B_{\rm d}^{\rm T}PA_{\rm d} + \Omega_{{\rm d}xu_{\rm d}}^{\rm T}(P))^{\rm T} \\ & (R_{\rm d2} + B_{\rm d}^{\rm T}PB_{\rm d} + \Omega_{{\rm d}u_{\rm d}u_{\rm d}}(P))^{-1} \\ & \cdot(B_{\rm d}^{\rm T}PA_{\rm d} + \Omega_{{\rm d}xu_{\rm d}}^{\rm T}(P)))x, \\ & x \in \mathcal{Z}. \end{aligned} \tag{II.36}$$

Then, with hybrid feedback control $(u_{\rm c}, u_{\rm d}) = (\phi_{\rm c}(x), \phi_{\rm d}(x)) = (-R_{\rm c2}^{-1}B_{\rm c}^{\rm T}PE_{\rm c}x,\ -(R_{\rm d2} + B_{\rm d}^{\rm T}PB_{\rm d} + \Omega_{{\rm d}u_{\rm d}u_{\rm d}}(P))^{-1}(B_{\rm d}^{\rm T}PA_{\rm d} + \Omega_{{\rm d}xu_{\rm d}}^{\rm T}(P))x)$ the zero solution $x(t) \equiv 0$ to (II.29), (II.30) is globally asymptotically stable for all $x_0 \in \mathbb{R}^n$, $(\Delta A_{\rm c}, \Delta A_{\rm d}) \in \boldsymbol{\Delta}_{A{\rm c}} \times \boldsymbol{\Delta}_{A{\rm d}}$ and

$$\sup_{(\Delta_{\rm c},\Delta_{\rm d})\in\boldsymbol{\Delta}} J_{(\Delta A_{\rm c},\Delta A_{\rm d})}(E_{\rm c}x_0) \le \mathcal{J}(E_{\rm c}x_0, \phi_{\rm c}(\cdot), \phi_{\rm d}(\cdot)) = x_0^{\rm T}E_{\rm c}^{\rm T}PE_{\rm c}x_0, \quad x_0 \in \mathbb{R}^n, \tag{II.37}$$

where

$$\begin{aligned}\mathcal{J}(E_{\rm c}x_0, u_{\rm c}(\cdot), u_{\rm d}(\cdot)) \triangleq \int_0^\infty [x^{\rm T}(t)E_{\rm c}^{\rm T}R_{\rm c1}E_{\rm c}x(t)\\ +u_{\rm c}^{\rm T}(t)R_{\rm c2}u_{\rm c}(t) + x^{\rm T}(t)\Omega_{\rm c}(P)x(t)]{\rm d}t\\ + \sum_{k\in\mathcal{N}_{[0,\infty)}} [x^{\rm T}(t_k)E_{\rm d}^{\rm T}R_{\rm d1}E_{\rm d}x(t_k)\\ +u_{\rm d}^{\rm T}(t_k)R_{\rm d2}u_{\rm d}(t_k) + x^{\rm T}(t_k)\Omega_{{\rm d}xx}(P)x(t_k)\\ +2x^{\rm T}(t_k)\Omega_{{\rm d}xu_{\rm d}}(P)u_{\rm d}(t_k) + u_{\rm d}^{\rm T}(t_k)\\ \cdot\Omega_{{\rm d}u_{\rm d}u_{\rm d}}(P)u_{\rm d}(t_k)],\end{aligned} \tag{II.38, II.39}$$

and where $(u_{\rm c}, u_{\rm d})$ is admissible and $x(t)$, $t \ge 0$, is a solution to (II.29), (II.30) with $(\Delta A_{\rm c}, \Delta A_{\rm d}) = (0, 0)$. Furthermore,

$$\mathcal{J}(E_{\rm c}x_0, \phi_{\rm c}(x(\cdot)), \phi_{\rm d}(x(\cdot))) = min_{(u_{\rm c}(\cdot),u_{\rm d}(\cdot))\in\mathcal{C}(x_0)}\mathcal{J}(E_{\rm c}x_0, u_{\rm c}(\cdot), u_{\rm d}(\cdot)), \tag{II.40}$$

where $\mathcal{C}(x_0)$ is the set of asymptotically stabilizing hybrid controllers for the nominal singularly impulsive dynamical system and $x_0 \in \mathbb{R}^n$, [3] and [7].

Proof: The detailed proof is given in [7]. □

III. Inverse Optimal Robust Control for Nonlinear Affine Uncertain Singularly Impulsive Dynamical Systems

In this section we specialize Theorem II to affine uncertain systems. The controllers obtained are predicated on an *inverse optimal hybrid control problem*. In particular, to avoid the complexity in solving the steady-state hybrid Hamilton-Jacobi-Bellman equation we do not attempt to minimize a *given* hybrid cost functional, but rather, we parametrize a family of stabilizing hybrid controllers that minimize some *derived* hybrid cost functional that provides flexibility in specifying the control law. The performance integrand is shown to explicitly depend on the nonlinear singularly impulsive system dynamics, the Lyapunov function of the closed-loop system, and the stabilizing hybrid feedback control law wherein the coupling is introduced via the hybrid Hamilton-Jacobi-Bellman equation. Hence, by varying the parameters in the Lyapunov function and the performance integrand, the proposed framework can be used to characterize a class of globally stabilizing hybrid controllers that can meet the closed-loop system response constraints.

Consider the state-dependent affine (in the control) uncertain singularly impulsive dynamical system

$$E_{\rm c}\dot{x}(t) = f_{\rm c}(x(t)) + \Delta f_{\rm c}(x(t)) + G_{\rm c}(x(t))u_{\rm c}(t), \quad x(0) = x_0,\ x(t) \notin \mathcal{Z}_x, \tag{III.41}$$

$$E_{\rm d}\Delta x(t) = f_{\rm d}(x(t)) + \Delta f_{\rm d}(x(t)) + G_{\rm d}(x(t))u_{\rm d}(t), \quad x(t) \in \mathcal{Z}_x, \tag{III.42}$$

where $t \ge 0$, $f_{\rm c0}, f_{\rm d0} : \mathcal{D} \to \mathbb{R}^n$ and satisfies $f_{\rm c0}(0) = 0$, $f_{\rm d0}(0) = 0$, $\mathcal{D} = \mathbb{R}^n$, $\mathcal{U}_{\rm c} = \mathcal{C}_{\rm c} = \mathbb{R}^{m_{\rm c}}$, $\mathcal{U}_{\rm d} = \mathcal{C}_{\rm d} = \mathbb{R}^{m_{\rm d}}$, and $(\Delta f_{\rm c}, \Delta f_{\rm d}) \in \mathcal{F}_{\rm c} \times \mathcal{F}_{\rm d} \triangleq \mathcal{F}$, where

$$\Delta f_{\rm c}(\cdot) \in \mathcal{F}_{\rm c} \subset \{\Delta f_{\rm c} : \mathbb{R}^n \to \mathbb{R}^n : \Delta f_{\rm c}(0) = 0\},$$
$$\Delta f_{\rm d}(\cdot) \in \mathcal{F}_{\rm d} \subset \{\Delta f_{\rm d} : \mathbb{R}^n \to \mathbb{R}^n : \Delta f_{\rm d}(0) = 0\}.$$

In this section no explicit structure is assumed for the elements of $\mathcal{F}$. Furthermore, we consider performance integrands $L_{\rm c}(E_{\rm c}x, u_{\rm c})$ and $L_{\rm d}(E_{\rm d}x, u_{\rm d})$ of the form

$$L_{\rm c}(E_{\rm c}x, u_{\rm c}) = L_{\rm c1}(E_{\rm c}x) + u_{\rm c}^{\rm T}R_{\rm c2}(x)u_{\rm c},\ x \notin \mathcal{Z}, \tag{III.43}$$
$$L_{\rm d}(E_{\rm d}x, u_{\rm d}) = L_{\rm d1}(E_{\rm d}x) + u_{\rm d}^{\rm T}R_{\rm d2}(x)u_{\rm d}, \quad x \in \mathcal{Z}, \tag{III.44}$$

where $L_{\rm c1} : \mathbb{R}^n \to \mathbb{R}$ and satisfies $L_{\rm c1}(E_{\rm c}x) \ge 0$, $x \in \mathbb{R}^n$, $R_{\rm c2} : \mathbb{R}^n \to \mathbb{P}^{m_{\rm c}}$, $L_{\rm d1} : \mathbb{R}^n \to \mathbb{R}$ and satisfies $L_{\rm d1}(E_{\rm d}x) \ge 0$, $x \in \mathbb{R}^n$, and $R_{\rm d2} : \mathbb{R}^n \to \mathbb{P}^{m_{\rm d}}$ so that (II.5) becomes

$$\begin{aligned}J(E_{\rm c}x_0, u_{\rm c}(\cdot), u_{\rm d}(\cdot)) = \int_0^\infty [L_{\rm c1}(E_{\rm c}x(t)) +\\ u_{\rm c}^{\rm T}(t)R_{\rm c2}(x(t))u_{\rm c}(t)]{\rm d}t + \sum_{k\in\mathcal{N}_{[0,\infty)}} [L_{\rm d1}(E_{\rm d}x(t_k))\\ +u_{\rm d}^{\rm T}(t_k)R_{\rm d2}(x(t_k))u_{\rm d}(t_k)].\end{aligned} \tag{III.45}$$

Consider the nonlinear uncertain controlled affine singularly impulsive system (III.41), (III.42) with performance functional (III.45). Assume there exists a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$, and functions $P_{12} : \mathbb{R}^n \to \mathbb{R}^{1\times m_{\rm d}}$, $P_2 : \mathbb{R}^n \to \mathbb{N}^{m_{\rm d}}$, $P_{1f_{\rm d}} : \mathbb{R}^n \to \mathbb{R}^{1\times n}$, $P_{2f_{\rm d}} : \mathbb{R}^n \to \mathbb{N}^{n\times n}$, $P_{u_{\rm d}f_{\rm d}} : \mathbb{R}^n \to \mathbb{R}^{m_{\rm d}\times n}$, $\Gamma_{\rm c} : \mathbb{R}^n \to \mathbb{R}$, $\Gamma_{{\rm d}xx} : \mathbb{R}^n \to \mathbb{R}^n$, $\Gamma_{{\rm d}xu_{\rm d}} : \mathbb{R}^n \to \mathbb{R}^{1\times m_{\rm d}}$, and $\Gamma_{{\rm d}u_{\rm d}u_{\rm d}} : \mathbb{R}^n \to \mathbb{N}^{m_{\rm d}}$ such that

$$P_{12}(0) = 0, \tag{III.46}$$
$$P_{1f_{\rm d}}(0) = 0, \tag{III.47}$$
$$\Gamma_{{\rm d}xu_{\rm d}}(0) = 0, \tag{III.48}$$
$$V(0) = 0, \tag{III.49}$$
$$V(E_{\rm c}x) \ge 0, \quad x \in \mathbb{R}^n, \quad x \ne 0, \tag{III.50}$$

$$V'(E_{\rm c}x)\Delta f_{\rm c}(x) \le \Gamma_{\rm c}(x),\ x \notin \mathcal{Z}_x,\ \Delta f_{\rm c} \in \mathcal{F}_{\rm c}, \tag{III.51}$$

$$V'(E_{\rm c}x)[f_{\rm c0}(x) - \tfrac{1}{2}G_{\rm c}(x)R_{\rm c2}^{-1}(x)G_{\rm c}^{\rm T}(x)V'^{\rm T}(E_{\rm c}x)] +\Gamma_{\rm c}(x) < 0, \quad x \notin \mathcal{Z}_x, \quad x \ne 0, \tag{III.52}$$

$$\begin{aligned}P_{1f_{\rm d}}(x)\Delta f_{\rm d}(x) + \Delta f_{\rm d}^{\rm T}(x)P_{1f_{\rm d}}^{\rm T}(x)\\ +\Delta f_{\rm d}^{\rm T}(x)P_{2f_{\rm d}}(x)\Delta f_{\rm d}(x) + \phi_{\rm d}^{\rm T}(E_{\rm d}x)P_{u_{\rm d}f_{\rm d}}(x)\Delta f_{\rm d}(x)\\ +\Delta f_{\rm d}^{\rm T}(x)P_{u_{\rm d}f_{\rm d}}^{\rm T}(x)\phi_{\rm d}(x)\\ \le \Gamma_{{\rm d}xx}(x) + \Gamma_{{\rm d}xu_{\rm d}}(x)\phi_{\rm d}(x) + \phi_{\rm d}^{\rm T}(x)\Gamma_{{\rm d}u_{\rm d}u_{\rm d}}(x)\phi_{\rm d}(x),\\ x \in \mathcal{Z}_x, \Delta f_{\rm d}(\cdot) \in \mathcal{F}_{\rm d},\end{aligned} \tag{III.53}$$

$$V(E_{\rm d}x+f_{\rm d0}(x))-V(E_{\rm d}x)+P_{12}(x)\phi_{\rm d}(x)+$$
$$\phi_{\rm d}(x)^{\rm T}P_2(x)\phi_{\rm d}(x)$$
$$+\Gamma_{{\rm d}xx}(x)+\Gamma_{{\rm d}xu_{\rm d}}(x)\phi_{\rm d}(x)+$$
$$\phi_{\rm d}^{\rm T}(x)\Gamma_{{\rm d}u_{\rm d}u_{\rm d}}(x)\phi_{\rm d}(x)\le 0,\quad x\in\mathcal{Z}_x,$$

$$V(E_{\rm d}x+f_{\rm d0}(x)+G_{\rm d}(x)u_{\rm d})=V(E_{\rm d}x+f_{\rm d0}(x))$$
$$+P_{12}(x)u_{\rm d}+u_{\rm d}^{\rm T}P_2(x)u_{\rm d},$$

$$V(E_{\rm d}x+f_{\rm d0}(x)+\Delta f_{\rm d}(x)+G_{\rm d}(x)u_{\rm d})-V(E_{\rm d}x)=$$
$$V(E_{\rm d}x+f_{\rm d0}(x)+G_{\rm d}(x)u_{\rm d})-V(E_{\rm d}x)+P_{1f_{\rm d}}(x)\Delta f_{\rm d}(x)+$$
$$\Delta f_{\rm d}^{\rm T}(x)P_{1f_{\rm d}}^{\rm T}(x)+\Delta f_{\rm d}^{\rm T}(x)$$
$$\cdot P_{2f_{\rm d}}(x)\Delta f_{\rm d}(x)+u_{\rm d}^{\rm T}P_{u_{\rm d}f_{\rm d}}(x)\Delta f_{\rm d}(x)$$
$$+\Delta f_{\rm d}^{\rm T}(x)P_{u_{\rm d}f_{\rm d}}^{\rm T}(x)u_{\rm d},$$
$$x\in\mathcal{Z}_x,\quad u_{\rm d}\in\mathbb{R}^{m_{\rm d}},\Delta f_{\rm d}(\cdot)\in\mathcal{F}_{\rm d},$$

and

$$V(E_{{\rm c}/{\rm d}}x)\to\infty \text{ as } \|x\|\to\infty. \tag{III.61}$$

Then the zero solution $x(t)\equiv 0$ to the closed-loop system

$$E_{\rm c}\dot{x}(t) = f_{\rm c}(x(t))+\Delta f_{\rm c}(x(t))+G_{\rm c}(x(t))\phi_{\rm c}(x(t)),\quad x(0)=x_0,\ x(t)\notin\mathcal{Z}_x, \tag{III.62}$$
$$E_{\rm d}\Delta x(t) = f_{\rm d}(x(t))+\Delta f_{\rm d}(x(t))+G_{\rm d}(x(t))\phi_{\rm d}(x(t)),\quad x(t)\in\mathcal{Z}_x, \tag{III.63}$$

is globally asymptotically stable for all $(\Delta f_{\rm c},\Delta f_{\rm d})\in\mathcal{F}$ with the hybrid feedback control law

$$\phi_{\rm c}(x) = -\tfrac{1}{2}R_{\rm c2}^{-1}(x)G_{\rm c}^{\rm T}(x)V'^{\rm T}(E_{\rm c}x),\quad x\notin\mathcal{Z}_x, \tag{III.64}$$
$$\phi_{\rm d}(x) = -\tfrac{1}{2}(R_{\rm d2}(x)+P_2(x)+\Gamma_{{\rm d}u_{\rm d}u_{\rm d}}(x))^{-1}\cdot(P_{12}+\Gamma_{{\rm d}xu_{\rm d}(x)})^{\rm T}(x),\ x\in\mathcal{Z}_x, \tag{III.65}$$

and performance functional (III.45), satisfies

$$J(E_{\rm c}x_0,\phi_{\rm c}(x(\cdot)),\phi_{\rm d}(x(\cdot))) = \min_{(u_{\rm c}(\cdot),u_{\rm d}(\cdot))\in\mathcal{C}(x_0)} J(E_{\rm c}x_0,u_{\rm c}(\cdot),u_{\rm d}(\cdot)),\ x_0\in\mathbb{R}^n, \tag{III.66}$$

where

$$\mathcal{J}(E_{\rm c}x_0,u_{\rm c}(\cdot),u_{\rm d}(\cdot)) \triangleq \int_0^\infty [L_{\rm c}(E_{\rm c}x(t),u_{\rm c}(t)) +\Gamma_{\rm c}(\tilde{x}(t),u_{\rm c}(t))]{\rm d}t+\sum_{k\in\mathcal{N}_{[0,\infty)}}[L_{\rm d}(x(t_k),u_{\rm d}(t_k))+ \Gamma_{\rm d}(x(t_k),u_{\rm d}(t_k))], \tag{III.67}$$

and

$$\Gamma_{\rm c}(x,u_{\rm c}) = \Gamma_{{\rm c}xx}(x),\quad x\notin\mathcal{Z}_x, \tag{III.68}$$
$$\Gamma_{\rm d}(x,u_{\rm d}) = \Gamma_{{\rm d}xx}(x)+\Gamma_{{\rm d}xu_{\rm d}}(x)u_{\rm d}+u_{\rm d}^{\rm T}\Gamma_{{\rm d}u_{\rm d}u_{\rm d}}(x)u_{\rm d},\quad x\in\mathcal{Z}_x, \tag{III.69}$$

and where $(u_{\rm c}(\cdot),u_{\rm d}(\cdot))$ is an admissible control and $x(t)$, $t\ge 0$, is a solution of (III.41), (III.42) with $(\Delta f_{\rm c},\Delta f_{\rm d})=(0,0)$.

(III.54)

In addition, the hybrid performance functional (III.67), with

$$L_{\rm c1}(E_{\rm c}x) = \phi_{\rm c}^{\rm T}(x)R_{\rm c2}(x)\phi_{\rm c}(x)-V'(E_{\rm c}x)f_{\rm c0}(x) -\Gamma_{{\rm c}xx}(x), \tag{III.70}$$
$$L_{\rm d1}(E_{\rm d}x) = \phi_{\rm d}^{\rm T}(x)(R_{\rm d2}(x)+P_2(x)+\Gamma_{{\rm d}u_{\rm d}u_{\rm d}}(x))\phi_{\rm d}(x) -V(E_{\rm d}x+f_{\rm d0}(x))+V(E_{\rm d}x)-\Gamma_{{\rm d}xx}(x), \tag{III.71}$$

(III.55)

(III.56)

(III.57)

is minimized in the sense that

(III.58)

$$\mathcal{J}(E_{\rm c}x_0,\phi_{\rm c}(E_{\rm c}x(\cdot)),\phi_{\rm d}(E_{\rm d}x(\cdot))) = \min_{(u_{\rm c}(\cdot),u_{\rm d}(\cdot))\in\mathcal{C}(x_0)} \mathcal{J}(E_{\rm c}x_0,u_{\rm c}(\cdot),u_{\rm d}(\cdot)). \tag{III.72}$$

[3] and [7].

Proof: The result is a direct consequence of Theorem II with $\mathcal{D}=\mathbb{R}^n$, $\mathcal{U}_{\rm c}=\mathbb{R}^{m_{\rm c}}$, $\mathcal{U}_{\rm d}=\mathbb{R}^{m_{\rm d}}$, $F_{\rm c}(x,u_{\rm c})=f_{\rm c0}(x)+\Delta f_{\rm c}(x)+G_{\rm c}(x)u_{\rm c}$, $F_{\rm c0}(x,u_{\rm c})=f_{\rm c0}(x)+G_{\rm c}(x)u_{\rm c}$, (III.59) $L_{\rm c}(E_{\rm c}x,u_{\rm c})$ given by (III.43), $\Gamma_{\rm c}(x,u_{\rm c})$ given by (III.68), (III.60) for $x\notin\mathcal{Z}_x$, $F_{\rm d}(x,u_{\rm d})=f_{\rm d0}(x)+\Delta f_{\rm d}(x)+G_{\rm d}(x)u_{\rm d}$, $F_{\rm d}(x,u_{\rm d})=f_{\rm d0}(x)+G_{\rm d}(x)u_{\rm d}$, $L_{\rm d}(E_{\rm d}x,u_{\rm d})$ given by (III.44), $\Gamma_{\rm d}(x,u_{\rm d})$ given by (III.69), for $x\in\mathcal{Z}$. Specifically, with (III.41)–(III.44), (III.68), and (III.69), the Hamiltonian have the form

$$H_{\rm c}(E_{\rm c}x,u_{\rm c}) = L_{\rm c1}(E_{\rm c}x)+u_{\rm c}^{\rm T}R_{\rm c2}(x)u_{\rm c} +V'(E_{\rm c}x)(f_{\rm c0}(x)+G_{\rm c}(x)u_{\rm c})+\Gamma_{{\rm c}xx}(x),\quad x\notin\mathcal{Z}_x,\ u_{\rm c}\in\mathcal{U}_{\rm c}, \tag{III.73}$$
$$H_{\rm d}(E_{\rm d}x,u_{\rm d}) = L_{\rm d1}(E_{\rm d}x)+u_{\rm d}^{\rm T}R_{\rm d2}(x)u_{\rm d} +V(E_{\rm d}x+f_{\rm d0}(x)+G_{\rm d}(x)u_{\rm d})-V(E_{\rm d}x) +\Gamma_{{\rm d}xx}(x)+\Gamma_{{\rm d}xu_{\rm d}}(x)u_{\rm d}+u_{\rm d}^{\rm T}\Gamma_{{\rm d}u_{\rm d}u_{\rm d}}(x)u_{\rm d},\quad x\in\mathcal{Z}_x,\ u_{\rm d}\in\mathcal{U}_{\rm d}. \tag{III.74}$$

Now, the hybrid feedback control law (III.64), (III.65) is obtained by setting $\frac{\partial H_{\rm c}}{\partial u_{\rm c}}=0$ and $\frac{\partial H_{\rm d}}{\partial u_{\rm d}}=0$. With (III.64) and (III.65) it follows that (III.51)–(III.60) imply (II.10)–(II.13). Next, since $V(\cdot)$ is C^1 and $x=0$ is a local minimum of $V(\cdot)$, it follows that $V'(0)=0$, and hence, since by assumption $P_{12}(0)=0$ and $\Gamma_{{\rm d}xu_{\rm d}}(0)=0$, it follows that $\phi_{\rm c}(0)=0$ and $\phi_{\rm d}(0)=0$ which proves (II.8), (II.9). Next, with $L_{\rm c1}(E_{\rm c}x)$ and $L_{\rm d1}(E_{\rm d}x)$ given by (III.70) and (III.71), respectively, and $\phi_{\rm c}(x)$, $\phi_{\rm d}(x)$ given by (III.64) and (III.65), (II.14) and (II.16) hold. Finally, since

$$H_{\rm c}(E_{\rm c}x,u_{\rm c}) = H_{\rm c}(E_{\rm c}x,u_{\rm c})-H_{\rm c}(E_{\rm c}x,\phi_{\rm c}(x)) = [u_{\rm c}-\phi_{\rm c}(x)]^{\rm T}R_{\rm c2}(x)[u_{\rm c}-\phi_{\rm c}(x)],\quad x\notin\mathcal{Z}_x, \tag{III.75}$$
$$H_{\rm d}(E_{\rm d}x,u_{\rm d}) = H_{\rm d}(E_{\rm d}x,u_{\rm d})-H_{\rm d}(E_{\rm d}x,\phi_{\rm d}(x)) = [u_{\rm d}-\phi_{\rm d}(x)]^{\rm T}(R_{\rm d2}(x) +P_2(x+\Gamma_{{\rm d}u_{\rm d}u_{\rm d}}(x)))[u_{\rm d}-\phi_{\rm d}(x)],\quad x\in\mathcal{Z}_x, \tag{III.76}$$

where $R_{\rm c2}(x)>0$, $x\notin\mathcal{Z}_x$, and $R_{\rm d2}(x)+P_2(x)+\Gamma_{{\rm d}u_{\rm d}u_{\rm d}}(x)>0$, $x\in\mathcal{Z}_x$, conditions (II.15) and (II.17) hold. The result now follows as a direct consequence of Theorem II. □

IV. Robust Nonlinear Hybrid Control with Polynomial Performance Functional

In this section we specialize the results of Section IV to linear singularly impulsive systems controlled by inverse optimal nonlinear hybrid controllers that minimize a derived polynomial cost functional. Specifically, assume $\mathcal{F} \triangleq \mathcal{F}_{\rm c} \times \mathcal{F}_{\rm d}$ to be the set of uncertain systems, where

$$\mathcal{F}_{\rm c} = \{(A_{\rm c} + \Delta A_{\rm c})x + B_{\rm c}u_{\rm c} : x \in \mathbb{R}^n, A_{\rm c} \in \mathbb{R}^{n\times n}, B_{\rm c} \in \mathbb{R}^{n\times m_{\rm c}}, \Delta A_{\rm c} \in \mathbf{\Delta}_{A_{\rm c}}\}, \quad (IV.77)$$

$$\mathcal{F}_{\rm d} = \{(A_{\rm d} + \Delta A_{\rm d})x : x \in \mathbb{R}^n, A_{\rm d} \in \mathbb{R}^{n\times n}, \Delta A_{\rm d} \in \mathbf{\Delta}_{A_{\rm d}}\}, \quad (IV.78)$$

where $\mathbf{\Delta}_{A_{\rm c}}, \mathbf{\Delta}_{A_{\rm d}} \subset \mathbb{R}^{n\times n}$ are given bounded uncertainty sets of uncertain perturbations $\Delta A_{\rm c}, \Delta A_{\rm d}$ of the nominal asymptotically stable system matrices $A_{\rm c}, A_{\rm d}$ such that $0 \in \mathbf{\Delta}_{A_{\rm c}}$ and $0 \in \mathbf{\Delta}_{A_{\rm d}}$. For simplicity of exposition, we also define $(\Delta A_{\rm c}, \Delta A_{\rm d}) \in \mathbf{\Delta}_{A{\rm c}} \times \mathbf{\Delta}_{A{\rm d}} \triangleq \mathbf{\Delta}$. For the results in this section we assume $u_{\rm d}(t) \equiv 0$. Furthermore, let $R_{1{\rm c}} \in \mathbb{P}^n$, $R_{1{\rm d}} \in \mathbb{N}^n$, $R_{2{\rm c}} \in \mathbb{P}^{m_{\rm c}}$, $\hat{R}_q, \hat{\hat{R}}_q \in \mathbb{N}^n$, $q = 2, \ldots, r$, be given, where r is a positive integer, and define $S_{\rm c} \triangleq B_{\rm c}R_{2{\rm c}}^{-1}B_{\rm c}^{\rm T}$.

Consider the linear uncertain controlled singularly impulsive system

$$E_{\rm c}\dot{x}(t) = (A_{\rm c} + \Delta A_{\rm c})x(t) + B_{\rm c}u_{\rm c}(t), \quad x(0) = x_0, \quad x(t) \notin \mathcal{Z}_x, \quad (IV.79)$$

$$E_{\rm d}\Delta x(t) = (A_{\rm d} + \Delta A_{\rm d} - E_{\rm d})x(t), \quad x(t) \in \mathcal{Z}_x, \quad (IV.80)$$

where $u_{\rm c}$ is admissible and $(\Delta A_{\rm c}, \Delta A_{\rm d}) \in \mathbf{\Delta}$. Let $\Omega_{\rm c}, \Omega_{\rm d} : \mathcal{N}_{\rm P} \subseteq \mathbb{S}^n \to \mathbb{N}^n$, $P \in \mathcal{N}_{\rm P}$, be such that

$$x^{\rm T}(\Delta A_{\rm c}^{\rm T} PE_{\rm c} + E_{\rm c}^{\rm T} P\Delta A_{\rm c})x \le x^{\rm T}\Omega_{\rm c}(P)x, \quad x \notin \mathcal{Z}, \quad \Delta A_{\rm c} \in \mathbf{\Delta}_{A{\rm c}}, \quad (IV.81)$$

$$x^{\rm T}(\Delta A_{\rm d}^{\rm T} P\Delta A_{\rm d} + \Delta A_{\rm d}^{\rm T} PA_{\rm d} + \Delta A_{\rm d}^{\rm T} P\Delta A_{\rm d})x \le x^{\rm T}\Omega_{\rm d}(P)x, \quad x \in \mathcal{Z}, \quad \Delta A_{\rm d} \in \mathbf{\Delta}_{A{\rm d}}. \quad (IV.82)$$

Assume there exist $P \in \mathbb{P}^n$ and $M_q \in \mathbb{N}^n$, $q = 2, \ldots, r$, such that

$$0 = x^{\rm T}(A_{\rm c}^{\rm T} PE_{\rm c} + E_{\rm c}^{\rm T} PA_{\rm c} + E_{\rm c}^{\rm T} R_{1{\rm c}}E_{\rm c} + \Omega_{\rm c}(P) - PS_{\rm c}P)x, \quad x \notin \mathcal{Z}_x, \quad (IV.83)$$

$$0 = x^{\rm T}[(A_{\rm c} - S_{\rm c}P)^{\rm T} M_qE_{\rm c} + E_{\rm c}^{\rm T} M_q(A_{\rm c} - S_{\rm c}P) + \hat{R}_q]x, \quad x \notin \mathcal{Z}_x, q = 2, \ldots, r, \quad (IV.84)$$

$$0 = x^{\rm T}(A_{\rm d}^{\rm T} PA_{\rm d} - E_{\rm d}^{\rm T} PE_{\rm d} + E_{\rm d}^{\rm T} R_{1{\rm d}}E_{\rm d} + \Omega_{\rm d}(P))x, \quad x \in \mathcal{Z}_x, \quad (IV.85)$$

$$0 = x^{\rm T}(A_{\rm d}^{\rm T} M_qA_{\rm d} - E_{\rm d}^{\rm T} M_qE_{\rm d} + \hat{\hat{R}}_q)x, \quad x \in \mathcal{Z}_x, q = 2, \ldots, r. \quad (IV.86)$$

Then the zero solution $x(t) \equiv 0$ of the uncertain closed-loop system

$$E_{\rm c}\dot{x}(t) = (A_{\rm c} + \Delta A_{\rm c})x(t) + B_{\rm c}\phi_{\rm c}(x(t)), \; x(0) = x_0, \quad x(t) \notin \mathcal{Z}_x, \quad (IV.87)$$

$$E_{\rm d}\Delta x(t) = (A_{\rm d} + \Delta A_{\rm d} - E_{\rm d})x(t), \quad x(t) \in \mathcal{Z}_x, \quad (IV.88)$$

is globally asymptotically stable with the feedback control law

$$\phi_{\rm c}(x) = -R_{2{\rm c}}^{-1}B_{\rm c}^{\rm T}(P + \sum_{q=2}^{r}(x^{\rm T}E_{\rm c}^{\rm T} M_qE_{\rm c}x)^{q-1}M_q)E_{\rm c}x, \quad x \notin \mathcal{Z}_x, \quad (IV.89)$$

and the performance functional (III.45) satisfies

$$\sup_{(\Delta A_{\rm c},\Delta A_{\rm d})\in\mathbf{\Delta}} J_{\Delta A_{\rm c},\Delta A_{\rm d}}(E_{\rm c}x_0, \phi_{\rm c}(x_0)) \le \mathcal{J}(E_{\rm c}x_0, \phi_{\rm c}(x_0)) = x_0^{\rm T}E_{\rm c}^{\rm T}PE_{\rm c}x_0 + \sum_{q=2}^{r}\frac{1}{q}(x_0^{\rm T}E_{\rm c}^{\rm T}M_qE_{\rm c}x_0)^q, \; x_0 \in \mathbb{R}^n, \quad (IV.90)$$

where

$$\mathcal{J}(E_{\rm c}x_0, u_{\rm c}(\cdot)) \triangleq \int_0^\infty [L_{\rm c}(E_{\rm c}x(t), u_{\rm c}(t)) + \Gamma_{\rm c}(\tilde{x}(t), u_{\rm c}(t))]{\rm d}t + \sum_{k\in\mathcal{N}_{[0,\infty)}} [L_{\rm d}(E_{\rm d}x(t_k)) + \Gamma_{\rm d}(x(t_k))], \quad (IV.91)$$

and where $u_{\rm c}$ is admissible, and $x(t)$, $t \ge 0$, is a solution to (IV.79), (IV.80) with $(\Delta A_{\rm c}, \Delta A_{\rm d}) = (0, 0)$, and

$$\Gamma_{\rm c}(x, u_{\rm c}) = x^{\rm T}(\Omega_{\rm c}(P) + \sum_{q=2}^{r}(x^{\rm T}E_{\rm c}^{\rm T}M_qE_{\rm c}x)^{q-1}\Omega_{\rm c}(M_q))E_{\rm c}x, \quad x \notin \mathcal{Z}_x \quad (IV.92)$$

$$\Gamma_{\rm d}(x) = x^{\rm T}\Omega_{\rm d}(P)x + \sum_{q=2}^{r}\frac{1}{q}[(x^{\rm T}\hat{\hat{R}}_qx)\sum_{j=1}^{q}(x^{\rm T}E_{\rm d}^{\rm T}M_qE_{\rm d}x)^{j-1} \cdot(x^{\rm T}(A_{\rm d}^{\rm T}M_qA_{\rm d} + \Omega_{\rm d}(x))x)^{q-j} - (x^{\rm T}A_{\rm d}^{\rm T}M_qA_{\rm d}x)^{q-j}], \quad x \in \mathcal{Z}_x, \quad (IV.93)$$

where $u_{\rm c}$ is admissible and $(\Delta A_{\rm c}, \Delta A_{\rm d}) \in \mathbf{\Delta}$. In addition, the performance functional (III.45), with $R_{2{\rm c}}(x) = R_{2{\rm c}}$ and

$$L_{1{\rm c}}(E_{\rm c}x) = x^{\rm T}(E_{\rm c}^{\rm T}R_{1{\rm c}}E_{\rm c} + \sum_{q=2}^{r}(x^{\rm T}E_{\rm c}^{\rm T}M_qE_{\rm c}x)^{q-1}\hat{R}_q + [\sum_{q=2}^{r}(x^{\rm T}E_{\rm c}^{\rm T}M_qE_{\rm c}x)^{q-1}M_q]^{\rm T}S_{\rm c} [\sum_{q=2}^{r}(x^{\rm T}E_{\rm c}^{\rm T}M_qE_{\rm c}x)^{q-1}M_q])x, \quad (IV.94)$$

$$L_{1{\rm d}}(E_{\rm d}x) = x^{\rm T}E_{\rm d}^{\rm T}R_{1{\rm d}}E_{\rm d}x + \sum_{q=2}^{r}\frac{1}{q}[(x^{\rm T}\hat{\hat{R}}_qx) \sum_{j=1}^{q}(x^{\rm T}E_{\rm d}^{\rm T}M_qE_{\rm d}x)^{j-1} \cdot(x^{\rm T}A_{\rm d}^{\rm T}M_qA_{\rm d}x)^{q-j}], \quad (IV.95)$$

is minimized in the sense that $\mathrm{J}(\mathrm{E}_{\mathrm{c}}x_0, \phi_{\mathrm{c}}(x(\cdot))) = \min_{u_{\mathrm{c}}(\cdot)\in\mathcal{C}(x_0)} J(E_{\mathrm{c}}x_0, u_{\mathrm{c}}(\cdot)), \quad x_0 \in \mathbb{R}^n, (IV.95)$where $\mathcal{C}(x_0)$ is the set of asymptotically stabilizing controllers for the nominal system and $x_0 \in \mathbb{R}^n$, [3] and [7].

Proof: The result is a direct consequence of Corollary III. □

V. Robust Nonlinear Hybrid Control with Multilinear Performance Functional

Finally, we specialize the results of Section VI to linear singularly impulsive systems controlled by inverse optimal hybrid controllers that minimize a derived multilinear functional. First, however, we give several definitions involving multilinear forms. A scalar function $\psi : \mathbb{R}^n \to \mathbb{R}$ is q-*multilinear* if q is a positive integer and $\psi(x)$ is a linear combination of terms of the form $x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}$, where i_j is a nonnegative integer for $j = 1, \ldots, n$, and $i_1 + i_2 + \ldots + i_n = q$. Furthermore, a q-multilinear function $\psi(\cdot)$ is nonnegative definite (resp., positive definite) if $\psi(x) \geq 0$ for all $x \in \mathbb{R}^n$ (resp., $\psi(x) > 0$ for all nonzero $x \in \mathbb{R}^n$). Note that if q is odd then $\psi(x)$ cannot be positive definite. If $\psi(\cdot)$ is a q-multilinear function then $\psi(\cdot)$ can be represented by means of Kronecker products, that is, $\psi(x)$ is given by $\psi(x) = \Psi x^{[q]}$, where $\Psi \in \mathbb{R}^{1\times n^q}$ and $x^{[q]} \triangleq x \otimes x \otimes \cdots \times x$ (q times), where $\otimes$ denotes Kronecker product. For the next result recall the definition of S_{c}, let $R_{1\mathrm{c}} \in \mathbb{P}^n$, $R_{1\mathrm{d}} \in \mathbb{P}^n$, $R_{2\mathrm{c}} \in \mathbb{P}^{m_{\mathrm{c}}}$, $\hat{R}_{2q}, \hat{\hat{R}}_{2q} \in \mathcal{N}^{(2q,n)}$, $q = 2, \ldots, r$, be given, where $\mathcal{N}^{(2q,n)} \triangleq \{\Psi \in \mathbb{R}^{1\times n^{2q}} : \Psi x^{[2q]} \geq 0, \ x \in \mathbb{R}^n\}$, and define the repeated (q times) Kronecker sum as $\overset{q}{\oplus} A \triangleq A \oplus A \oplus \cdots \oplus A$.

Consider the linear controlled singularly impulsive system (IV.79), (IV.80). Assume there exist $P \in \mathbb{P}^n$ and $\hat{P}_q \in \mathcal{N}^{(2q,n)}$, $q = 2, \ldots, r$, such that

$$
\begin{aligned}
0 &= x^{\mathrm{T}}(A_{\mathrm{c}}^{\mathrm{T}} P E_{\mathrm{c}} + E_{\mathrm{c}}^{\mathrm{T}} P A_{\mathrm{c}} + E_{\mathrm{c}}^{\mathrm{T}} R_{1\mathrm{c}} E_{\mathrm{c}} - P B_{\mathrm{c}} R_{2\mathrm{c}}^{-1} B_{\mathrm{c}}^{\mathrm{T}} P)x, \\
&\quad x \notin \mathcal{Z}_x, \qquad \text{(V.96)} \\
0 &= x^{\mathrm{T}}(\hat{P}_q[\overset{2q}{\oplus}(E_{\mathrm{c}}^{\mathrm{T}} A_{\mathrm{c}} - S_{\mathrm{c}} P)] + \hat{R}_{2q})x, \\
&\quad x \notin \mathcal{Z}_x, \quad q = 2, \ldots, r, \qquad \text{(V.97)} \\
0 &= x^{\mathrm{T}}(A_{\mathrm{d}}^{\mathrm{T}} P A_{\mathrm{d}} - E_{\mathrm{d}}^{\mathrm{T}} P E_{\mathrm{d}} + E_{\mathrm{d}}^{\mathrm{T}} R_{1\mathrm{d}} E_{\mathrm{d}})x, \\
&\quad x \in \mathcal{Z}_x, \qquad \text{(V.98)} \\
0 &= x^{\mathrm{T}}(\hat{P}_q[A_{\mathrm{d}}^{[2q]} - E_{\mathrm{d}}^{[2q]}] + \hat{\hat{R}}_{2q})x, \\
&\quad x \in \mathcal{Z}_x, \qquad q = 2, \ldots, r. \qquad \text{(V.99)}
\end{aligned}
$$

Then the zero solution $x(t) \equiv 0$ of the closed-loop system (IV.79), (IV.80) is globally asymptotically stable with the feedback control law

$$
\begin{aligned}
\phi_{\mathrm{c}}(x) &= -R_{2\mathrm{c}}^{-1} B_{\mathrm{c}}^{\mathrm{T}}(P E_{\mathrm{c}} x + \tfrac{1}{2} g'^{\mathrm{T}}(E_{\mathrm{c}} x)), \\
&\quad x \notin \mathcal{Z}_x, \qquad \text{(V.100)}
\end{aligned}
$$

where $g(x) \triangleq \sum_{q=2}^{r} \hat{P}_q E_{\mathrm{c}} x^{[2q]}$, and the performance functional (III.45), with $R_{2\mathrm{c}}(x) = R_{2\mathrm{c}}$ and

$$
\begin{aligned}
L_{1\mathrm{c}}(E_{\mathrm{c}}x) &= x^{\mathrm{T}} E_{\mathrm{c}} R_{1\mathrm{c}} x E_{\mathrm{c}} + \sum_{q=2}^{r} \hat{R}_{2q} E_{\mathrm{c}} x^{[2q]} \\
&\quad + \tfrac{1}{4} g'(E_{\mathrm{c}}x) S_{\mathrm{c}} g'^{\mathrm{T}}(E_{\mathrm{c}}x), \qquad \text{(V.101)} \\
L_{1\mathrm{d}}(x) &= x^{\mathrm{T}} E_{\mathrm{d}}^{\mathrm{T}} R_{1\mathrm{d}} E_{\mathrm{d}} x + \sum_{q=2}^{r} \hat{\hat{R}}_{2q} E_{\mathrm{d}} x^{[2q]} \qquad \text{(V.102)}
\end{aligned}
$$

is minimized in the sense that

$$
\begin{aligned}
J(E_{\mathrm{c}}x_0, \phi_{\mathrm{c}}(x(\cdot))) &= \\
&\min_{u_{\mathrm{c}}(\cdot)\in\mathcal{C}(x_0)} J(E_{\mathrm{c}}x_0, u_{\mathrm{c}}(\cdot)), \quad x_0 \in \mathbb{R}^n. \qquad \text{(V.103)}
\end{aligned}
$$

Finally,

$$
\begin{aligned}
J(E_{\mathrm{c}}x_0, \phi_{\mathrm{c}}(x(\cdot))) &= x_0^{\mathrm{T}} E_{\mathrm{c}}^{\mathrm{T}} P E_{\mathrm{c}} x_0 + \sum_{q=2}^{r} \hat{P}_q E_{\mathrm{c}} x_0^{[2q]}, \\
&\quad x_0 \in \mathbb{R}^n. \qquad \text{(V.104)}
\end{aligned}
$$

[3] and [7].

Proof: The result is a direct consequence of Theorem II with $f_{\mathrm{c}}(x) = A_{\mathrm{c}}x$, $f_{\mathrm{d}}(x) = (A_{\mathrm{d}} - E_{\mathrm{d}})x$, $G_{\mathrm{c}}(x) = B_{\mathrm{c}}$, $G_{\mathrm{d}}(x) = 0$, $u_{\mathrm{d}} = 0$, $R_{2\mathrm{c}}(x) = R_{2\mathrm{c}}$, $R_{2\mathrm{d}}(x) = I_{m_{\mathrm{d}}}$, and $V(E_{\mathrm{c/d}}x) = x^{\mathrm{T}} E_{\mathrm{c/d}}^{\mathrm{T}} P E_{\mathrm{c/d}} x + \sum_{q=2}^{r} \hat{P}_q E_{\mathrm{c}} x^{[2q]}$. Specifically, for $x \notin \mathcal{Z}_x$ it follows from (IV.81), (V.97), and (V.100) that

$$
\begin{aligned}
V'(E_{\mathrm{c}}x)[f_{\mathrm{c}}(x) &- \tfrac{1}{2} G_{\mathrm{c}}(x) R_{2\mathrm{c}}^{-1}(x) G_{\mathrm{c}}^{\mathrm{T}}(x) V'^{\mathrm{T}}(E_{\mathrm{c}}x)] = \\
&- x^{\mathrm{T}} E_{\mathrm{c}}^{\mathrm{T}} R_{1\mathrm{c}} E_{\mathrm{c}} x - \sum_{q=2}^{r} \hat{R}_{2q} E_{\mathrm{c}} x^{[2q]} \\
&- \phi_{\mathrm{c}}^{\mathrm{T}}(x) R_{2\mathrm{c}} \phi_{\mathrm{c}}(x) - \tfrac{1}{4} g'(E_{\mathrm{c}}x) S_{\mathrm{c}} g'^{\mathrm{T}}(x),
\end{aligned}
$$

which implies (2.2.13). For $x \in \mathcal{Z}_x$ it follows from (V.98) and (V.99) that

$$
\begin{aligned}
\Delta V(E_{\mathrm{d}}x) &= V(E_{\mathrm{d}}x + f_{\mathrm{d}}(x)) - V(E_{\mathrm{d}}x) = \\
&\quad - x^{\mathrm{T}} E_{\mathrm{d}}^{\mathrm{T}} R_{1\mathrm{d}} E_{\mathrm{d}} x - \sum_{q=2}^{r} \hat{\hat{R}}_{2q} x^{[2q]},
\end{aligned}
$$

which implies (2.2.14) with $G_{\mathrm{d}}(x) = 0$. Finally, with $u_{\mathrm{d}} = 0$, condition is automatically satisfied so that all the conditions of Corollary V are satisfied. □

VI. Conclusion

In this paper we have developed optimal robust control and inverse optimal robust control results for the class of nonlinear uncertain singularly impulsive dynamical systems [5]. Results are based on Lyapunov and asymptotic stability theorems developed in [6], and results presented in [7].

VII. Future Work

Further work will specialize results of this paper to time-delay systems.

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