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An Investigation on the Design and Performance Assessment of double-PID and LQR Controllers for the Inverted Pendulum

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Abstract— **The widespread application of inverted pendulum principles requires better dynamic performance and steady state performance of the inverted pendulum system. The objective of this paper is to design and investigate the time specification performance of the inverted pendulum controllers. Two control methods are proposed in this paper, an innovative double PID control method and a modern LQR (liner quadratic regulator) control method. Dynamic performance and steady state performance are investigated and compared of the two controllers. This paper proves that the LQR controller can guarantee the inverted pendulum a faster and smoother stabilizing process and with better robustness and less oscillation than the double-PID controller. The novelty of this paper is the design of the two controllers, and the adoption of limits cycles as the performance assessment method for the inverted pendulum, which not only makes the steady state performance assessment available, but also, provides an effective way for the evaluation of any equilibrium control problem with friction involved.**

Keywords- inverted pendulum; performance assessment; double-PID controller; LQR controller; limits cycles

I. INTRODUCTION (HEADING 1)

The inverted pendulum system, as a typical experimental device for the research and application of control theory, is a single input and multiple outputs, nonlinear and unstable system. The inverted pendulum problem is one of the most important problems in control theory and has always been appealing among researchers [1]. Because the control of the inverted pendulum is similar to other position and equilibrium control tasks, such as the control of the electric two-wheeled self-balancing vehicles, the stabilization of aircraft in the turbulent air flow, the arm position control of open loop robots etc. All these tasks bring new challenges to control engineering, and therefore, much attention has been given to explore better solutions for the inverted pendulum system and to acquire better control performance, include dynamic performance and steady state performance.

In this paper, an innovative double-PID control method and a modern LQR (liner quadratic regulator) control method are designed respectively, and the dynamic performance and steady state performance are assessed and compared based on virtual prototype simulation and experimental setup. The virtual prototype is built which consists of control model, mechanical model, and comprehensive friction model with obtained parameters to fully investigate the steady state performance. Experimental setup validates the virtual prototype simulation and especially the predicted limit circle, which is caused by friction as is proposed in this paper and is adopted as steady state performance assessment indices. The paper proposes a comprehensive assessment on the two designed controllers for an inverted pendulum based on both traditional performance indices and the novel limits cycles.

II. SESTEM DESCRIPTION AND MODELING

A. Sestem Description

A schematic of the inverted pendulum is shown in Fig. 1. In this paper, the simulation and experiment are both based on a direct driven inverted pendulum system, in which a pendulum is mounted on a stage driven by an ironless permanent magnet linear motor. Linear motor is a new type driving device which can directly transform electric energy to mechanical linear motion. Sensors are attached to the cart and the pivot in order to measure the cart position and pendulum inclined angle respectively.

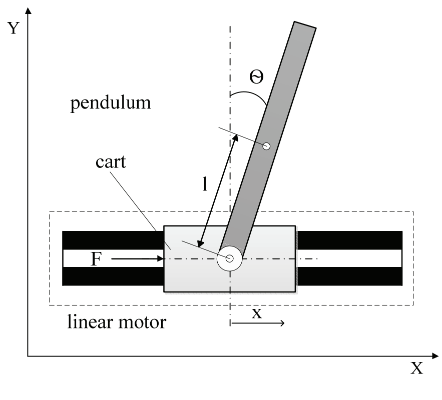

Figure 1. Schematic of the inverted pendulum

Usually the pendulum is in its pendant configuration and there are many methods to swing up the pendulum from its pendant position [2]; when the pendulum approaches the upright unstable equilibrium the control is switched to using a stabilizing controller. This paper focuses on designing the stabilizing controllers. A control force exerted by the linear motor is required on the cart to maintain the pendulum in its upright equilibrium. The control objective in this paper is to maintain the upright unstable equilibrium of the inverted pendulum system with better dynamic qualities, robustness and steady state performance.

B. Nonlinear Differential Equations of Seytem Motion

The motion equations of the cart and pendulum in Fig. 1 can be obtained using Hamilton's principle, shown in (1), (2):

$$(M+m)\ddot{x}(t)+ml\ddot{\theta}(t)\cos\theta(t)-ml\dot{\theta}^2(t)\sin\theta(t)=F_M(t)+F_{fric}(t) \quad (1)$$

$$\frac{4}{3}ml^2\ddot{\theta}(t)+ml\ddot{x}(t)\cos\theta(t)-mgl\sin\theta(t)=f_{fric}(t) \quad (2)$$

where x is the position of the cart, θ is the pendulum angle, measuring from the upright position, F is the force applied to the cart, F_{fric}(t)is the friction force between cart and the track, and f_{fric}(t) is the friction force between the pendulum and the pivot, which is very small thus ignored. In this section, since we concern the design of the inverted pendulum controllers, especially the dynamic performance of the controllers, the friction forces are simplified as viscous frictions, as in (3).

$$F_{fric}(t)=-\varepsilon\dot{x} \quad (3)$$

In (3) ε is coefficient of viscous friction. The definition and value of the inverted pendulum system parameters are given in Table 1.

TABLE I. PARAMETERS IN THE INVERTED PENDULUM SYSTEM

Parameter	Description	Value
M	Mass of the cart	1.336kg
m	Mass of the pendulum	0.083kg
l	Distance from the pivot to mass center of the pendulum	0.1685m
g	Gravitational constant	9.8m/s^2
K_f	Current to force conversion factor	1N/A

C. Mathematical Model of Sestem Input Force

In experimental setup, the effective electromagnetic thrust applies by the linear motor to the cart is given by (4)

$$F_M(t)=\frac{3\pi}{2\tau}\left[\lambda_{PM}i_q+(L_d-L_q)i_qi_d\right] \quad (4)$$

where λ_{PM} is flux linkage generated by permanent magnet, L_d and L_q is flux linkage of d-axis and q-axis respectively. The linear motor adopts the zero d-axis current control method, thus the applied force can be simplified as (5).

$$F_M(t)=K_fi_q \quad (5)$$

In (5) i_q is the q-axis current after coordinate conversion from the external current supplied to the linear motor, and K_f is current to force conversion factor, which is given in Table 1 in experimental apparatus.

D. Mathematical Model of Inverted Pendulum System

Combine the equations (1), (2) and the input force equation (5), solve for $\ddot{x}$ and $\ddot{\theta}$ from the differential equations, and introduce the variable: $y=(y_1,y_2,y_3,y_4)^T=(x,\dot{x},\theta,\dot{\theta})$ then after linearization the first-order mathematical model of the inverted pendulum system can be obtained as described in (6).

$$\dot{\mathbf{y}}(t)=\begin{pmatrix}0 & 1 & 0 & 0\\ 0 & \frac{-4\varepsilon}{(4M+m)} & \frac{-3mg}{(4M+m)} & 0\\ 0 & 0 & 0 & 1\\ 0 & \frac{3\varepsilon}{(4M+m)l} & \frac{3(M+m)g}{(4M+m)l} & 0\end{pmatrix}\mathbf{y}(t)+\begin{pmatrix}0\\ \frac{4K_f}{4M+m}\\ 0\\ \frac{-3K_f}{4M+ml}\end{pmatrix}i_q(t) \quad (6)$$
$$=A\mathbf{y}+\mathbf{b}i_q(t)$$

III. DESIGN AND TUNING OF DOUBLE-PID CONTROLLER

Taking the Laplace transform of the system first-order equation, we can obtain the transfer function of the system as

$$G_A(s)=\frac{\Theta(s)}{I(s)}=\frac{mlK_fs}{[m^2l^2-\frac{4}{3}ml(M+m)]s^3-\frac{4}{3}\varepsilon ml^2s^2+ml(M+m)gs+\varepsilon mgl} \quad (7)$$

$$G_P(s)=\frac{X(s)}{I(s)}=\frac{4ml^2K_fs^2-3mglK_f}{[4m(M+m)l^2-3m^2l^2]s^4+4\varepsilon ml^2s^3\text{-}3m(M+m)gls^2-3\varepsilon mgls} \quad (8)$$

Several methods have been proposed to control the inverted pendulum, such as traditional PID control [3], fuzzy control [4], genetic algorithm optimizing control [5], and linear quadratic regulator (LQR) control [6]. Although a lot of control algorithms are researched in the designing of the inverted pendulum system controller, PID controller is still the most widely used controller structure in the realization of a control system. However, the inverted pendulum system is a one input and two output system which contradicts to the one input and one output control characteristic of the single PID controller. The steady state error of the pendulum angle using single PID controller results in the one direction displacement of the cart. In [3], the displacement of the cart cannot be controlled very well because the PID controller can control only one variable, and in [7], the cart position control problem is ignored and only pendulum angle is focused. In this paper, we attempt to use double PID control structure to solve the multi-output problem. Block diagram of the designed double-PID control method is shown as follows in Fig. 2.

The angle reference signal is zero at the upright equilibrium position. The position reference signal is given by a pulse signal, with pulse amplitude 0.5m, pulse period 20s and pulse width 50% of period, to test the position tracking ability of the controller. After tuning the control parameters, we obtain a group of parameters for double-PID controller, with which the controller is capable of controlling the inverted pendulum system and providing a very robust performance.

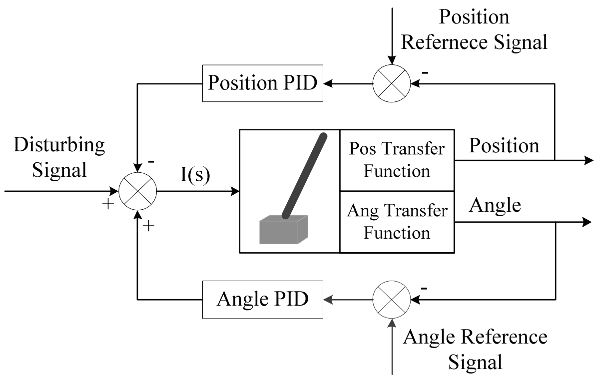

Figure 2. Block diagram of double-PID control method

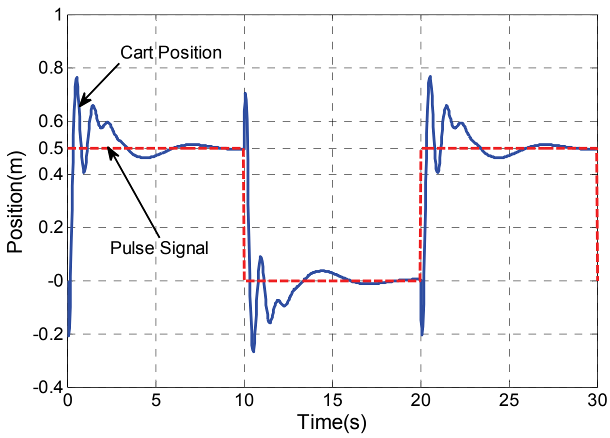

Figure 3. Cart position when tracking pulse signal

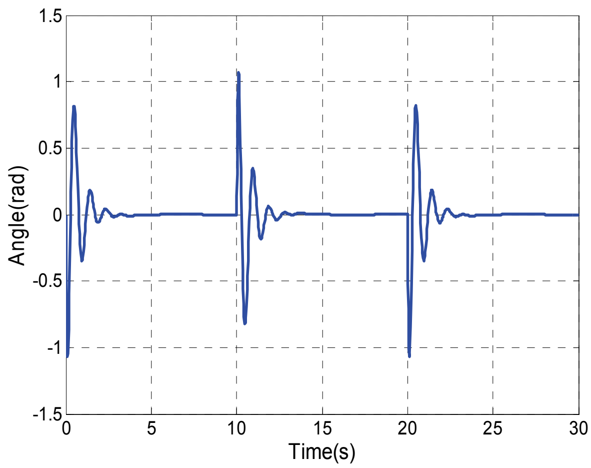

Figure 4. Pendulum angle keep balancing while cart tracking reference signal

The control parameters are:

[K_{pp}=20, K_{pi}=30, K_{pd}=14; K_{ap}=50, K_{ai}=90, K_{ad}=8].

Simulation result shows that the cart can track the position reference signal while the pendulum angle still balancing at the upright equilibrium point with occasional tracking disturbing, as shown in Fig. 3 and Fig. 4. Dynamic performance of the inverted pendulum using double-PID controller will be discussed later.

IV. Design and Tuning of LQR Controller

A. *Design of LQR Controller*

Still considering the first-order mathematical model of inverted pendulum system (6), using linear quadratic control theory to design the control i_q(t) so that the upright position becomes a stable equilibrium point as described in [8]. Consider the linear quadratic cost function (9)

$$J = \inf_{i_q(t)\in L(0,\infty)} \int_0^\infty [\mathbf{y}(t)^T Q\mathbf{y}(t) + r i_q^2(t)]dt \qquad (9)$$

The weights $Q \ge 0$, $r \ge 0$ are then chosen to reflect the relative importance error loss of the state **y**, and energy loss of the current i_q(t). Q and r are weighting matrixes that penalize certain states and control inputs of the system. Q is positive semi-definite matrix, while r is positive definite matrix. If this system is disturbed and offset the zero state, the control i_q(t) can make the system come back to zero state and J is minimal at the same time [9]. The solution to this problem is the feedback law as shown in (10)

$$i_q(t) = -r^{-1}\mathbf{b}^T P(t)\mathbf{y}(t) \qquad (10)$$

where P(t) is the solution of Riccati equation, which is

$$A^T P + PA - P\mathbf{b}r^{-1}\mathbf{b}^T P + Q = 0 \qquad (11)$$

Then we can obtain the value of P, and the linear optimal feedback matrix can be obtained using (12)

$$K = r^{-1}B^T P = [k_1, k_2, k_3, k_4]^T \qquad (12)$$

The Riccati equation may be solved numerically for given values for A, **b**, r, and Q. Fig. 5 shows the full states feedback representation of inverted pendulum system.

B. *Controller Solving and System Simulation*

According to the LQR optimal control law, its optimality is totally depended on the selection of Q and r. However, there is no resolving method to choose these two matrices. The widespread method used to choose Q and r is by means of simulation and trial [10, 11, and 12]. Since we consider the control of the cart position and pendulum angle, so in this section, the LQR controller is tuned by changing the nonzero elements of the Q_{11} and Q_{33} elements, which, corresponding to and closely influence the two concerned states: cart position and pendulum angle. We choose the weighting matrix Q=diag[4000, 0, 5100, 0], and r=1 after trial and error.

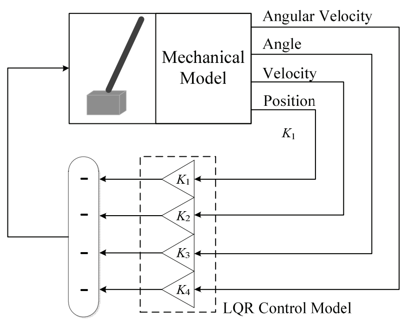

Figure 5. Block diagram of LQR controller possesses full states feedback

The linear optimal feedback matrix K can be calculated using the weighting matrix Q, which is

$$K = [-63.2456, -40.0071, 137.2913, 17.6830]$$

Dynamic performance of LQR controller can be assessed by means of step response of the cart and the corresponding disturbing response of the pendulum. Suppose that the cart position reference input is 0.5m (step signal), and the pendulum angle reference input is zero, Simulation Result in Fig. 6 shows that the cart position can be controlled and pendulum can be balanced smoothly using LQR optimal controller, and the system possess good performance.

V. DYNAMIC PERFORMANCE ASSESSMENT ON THE DESIGNED CONTROLLERS

Dynamic performance indices are chosen to reflect practical control effect. Performance indices include *rise time*, *transition time*, *steady state error*, and *maximum overshoot*, for the response of cart position and pendulum angle.

For comparison of the system dynamic response, plot the step response (cart position reference input is 0.5m for double-PID and LQR) of the cart and the corresponding disturbing response of the pendulum about the double-PID and LQR controller in one figure, see Fig. 7.

Table 2 and Table 3 show the summary of the dynamic performance indices for the double-PID controller and LQR controller.

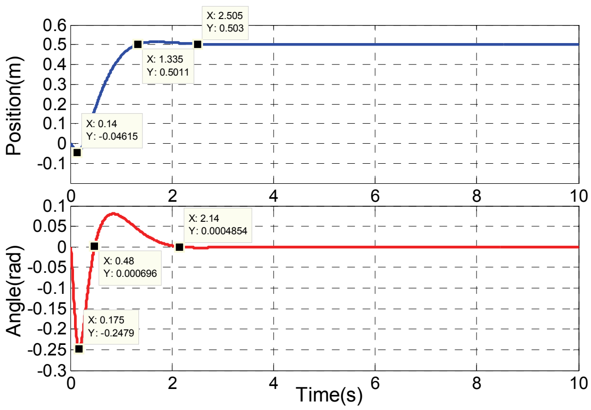

Figure 6. Dynamic response of the system using LQR controller

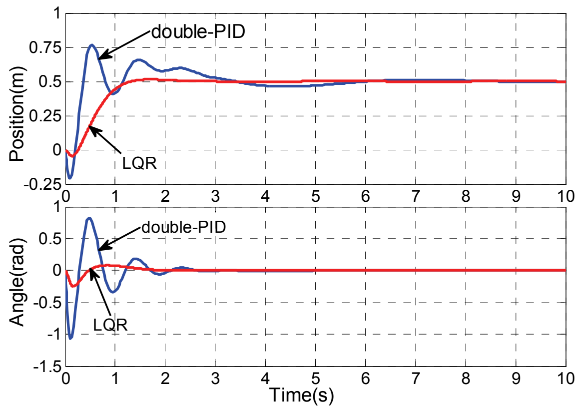

Figure 7. Dynamic response comparison of designed controllers

TABLE II. PERFORMANCE INDICES OF CART POSITION

Time Response Specification	Double-PID	LQR
Rise Time	0.36s	1.34s
Transition Time	8.65s	2.51s
Steady State Error	≈0.00	≈0.00
Maximum Overshoot	0.265m	0.046m

TABLE III. PERFORMANCE INDICES OF PENDULUM ANGLE

Time Response Specification	Double-PID	LQR
Rise Time	0.28s	0.48s
Transition Time	3.13s	2.14s
Steady State Error	≈0.00	≈0.00
Maximum Overshoot	1.07rad	0.248rad

Based on the performance indices tabulated in the two tables, LQR controller has the faster transition time both in cart position and pendulum angle, and with the longer rise time of cart position and pendulum angle, the balancing process will be more smooth. Furthermore, the pendulum angle maximum overshoot using double-PID method is bigger than using LQR method, which is because of the shorter rise time of the cart position when using double-PID contributes to larger pendulum angle shocking and maximum overshoot.

In conclusion, for dynamic response, the inverted pendulum controlled by LQR controller 1) balances faster because of the shorter settling time; 2) has a smoother response because of longer rise time; 3) has better robustness due to less maximum overshoot and shocking numbers. So, the LQR controller can guarantee the inverted pendulum system possess better dynamic performance.

VI. ASSESSMENT ON STEADY STATE PERFORMANCE OF THE DESIGNED CONTROLLERS

In the former section, friction is simplified as viscous friction, and this is valid in the designing of the double-PID and LQR controllers, especially for evaluating the dynamic performance of the designed controllers. However, this kind of simplification leads to near zero steady state error, and obviously it is not the case. In this section, the steady state performance, with the influence of friction, can be investigated by means of virtual prototype and experimental setup.

A. Comprehensive Friction Model

The viscous friction, as described before, is described as $F_{fric}(t) = -\varepsilon \dot{x}$. The Coulomb friction F_C is proportional to the normal load, i.e. $F_C = \mu_s F_N$. The static friction F_{static} counteracts the external applied linear motor forces $F_M(t)$. Define F_S as the maximum static friction, which is higher than the Coulomb friction, however we simplify it as $F_S=F_C$. Thus the static friction can be described as in (13).

$$F_{static} = \begin{cases} -F_M & \text{if } \dot{x} = 0 \text{ and } |F_{applied}| < F_S \\ \mu_s F_N \operatorname{sgn}(F_M) & \text{if } \dot{x} = 0 \text{ and } |F_{applied}| \geq F_S \end{cases} \quad (13)$$

The friction components (viscous friction, Coulomb friction and static friction) can be combined in different ways and such combination is referred as the classical model. However, the classical combination results of a discontinuous friction force at the transition segment from static friction to the Coulomb and viscous friction. The discontinuity contradicts to experimental observation [13]. A common way that can smooth the transition segment is to add an exponential factor [14]. This leads to the comprehensive friction model in (14).

$$F_{fric} = \begin{cases} F_{static} & \text{if } \dot{x} = 0 \\ -(\mu_c + (\mu_s - \mu_c)e^{-(\dot{x}/v_s)^{\gamma}})F_N \,\mathrm{sgn}(\dot{x}) - \varepsilon\dot{x} & \text{if } \dot{x} \neq 0 \end{cases} \quad (14)$$

where ε , μ_s and μ_c are the coefficient of viscous, Coulomb, and static frictions, respectively. F_N is the magnitude of the normal force equal to gravitation of the cart plus the pendulum, v_s is called the Stribeck velocity, and r is the form factor. The using of the comprehensive friction models confronts the problem of detecting zero velocity. Karnopp proposed the creation of a range of values and within the range the movement velocity is zero [15]. For velocities within this range, the friction is static friction. In the simulation we set ZR=0.005m/s.

To estimate μ_c and ε we applied two constant current values i_q to the linear motor and let the cart to reach a steady state, and the corresponding constant final velocity, $\dot{x}$, can be obtained. Then we can use the steady state equation of motion to solve for μ_c and ε . When estimating the value of th static friction coefficient μ_s , the Armstrong-Helouvry procedure [16] was used, that is to gradually increase the current applied to the linear motor and measure the current (and hence force) necessary to make the cart start to move. The Stribeck parameters, v_s and r can be chose by trial and error, so that the simulations of the model can best match experimental data. The value of the friction parameters are given in Table 4.

TABLE IV. PARAMETERS OF THE COMPREHENSIVE FRICTION MODEL

Parameter	Description	Value
μ_s	Coefficient of static friction	0.042
μ_c	Coefficient of Coulomb friction	0.027
ε	Coefficient of viscous friction	0.1N/m/sec
υ_s	Stribeck velocity	0.083m/s
γ	Form factor	2

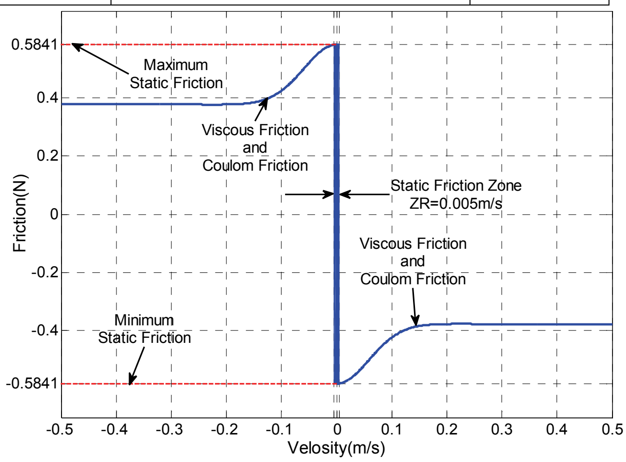

Figure 8. Force-Velocity plot of the friction model

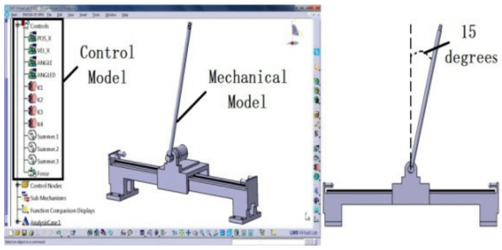

Figure 9. Virtual prototype of the inverted pendulum system

A Force-Velocity plot for the comprehensive friction model with parameters obtained from experiment is shown in Fig. 8. The figure shows that the maximum friction is 0.5841N.

B. Assessment on the Steady State Performance based on Virtual Prototype Simulation

A virtual prototype of the inverted pendulum system is developed in Virtul.Lab environment, as shown in Fig. 9. The virtual prototype allows us easily inputting the conditions in real environment, such as frictions. The virtual prototype consists of mechanical model and control model. The mechanical model simulates the experimental setup of the inverted pendulum in size and mass. The control model simulates the double-PID and LQR controller respectively. The initial condition is that the pendulum starts with an inclined angle 15 degrees from upright position.

The steady state performance indices, as well as the dynamic performance indices, of the two controllers are shown in Fig. 10 and Fig. 11. The figures show that the steady-state response of the system has oscillation, the so called limit cycle, which is validated both in theory and in experiment, and can be seen as steady state error. Reference [17] proposed that the presence of friction produces limits cycles. Considering one of the practical use of inverted pendulum, the electric two-wheeled self-balancing vehicle, lots of efforts have been spent to reduce the existed vibration by improved control algorithm [18] or available friction compensation model. Thus for practical application of the inverted pendulum, the smaller oscillation amplitude is the better.

Table 5 shows a comparison between the performance indices of double-PID controller and LQR controller. The oscillation amplitude of cart position is 0.095m and pendulum angle is 5.73deg when using double-PID controller, while using LQR controller the amplitude is 0.025m and 1.15deg respectively, which means that the LQR controller can guarantee smaller steady state error. The oscillation frequency of cart position and pendulum angle is 0.417Hz when using double-PID controller, while using LQR controller the frequency is 0.250Hz, which means that the oscillation is slower, this indicates that the cart or the pendulum will oscillate with more smooth.

TABLE V. STEADY STATE PERFORMANCE INDICES

Performance Index	double-PID	LQR
Cart Position *Oscillation Amplitude*	0.095m	0.025m
Pendulum Angle *Oscillation Amplitude*	5.73deg	1.15deg
Cart Position *Oscillation Frequency*	0.417Hz	0.250Hz
Pendulum Angle *Oscillation Frequency*	0.417Hz	0.250Hz

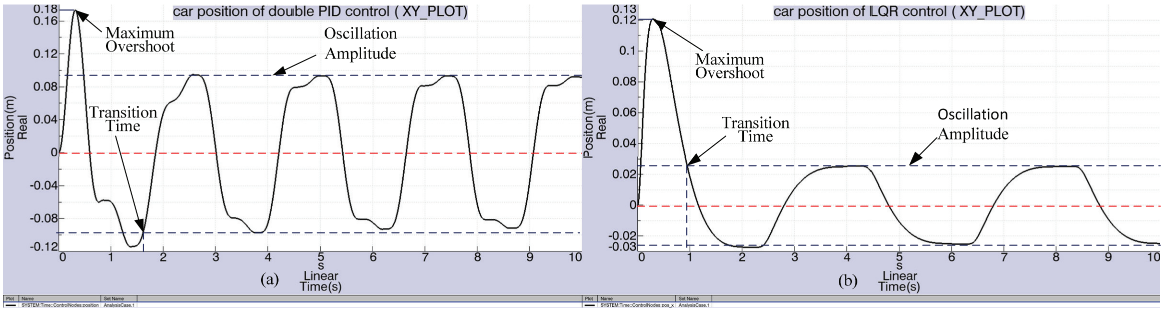

Figure 10. Comparison of cart position between double-PID controller and LQR controller based on virtual prototype

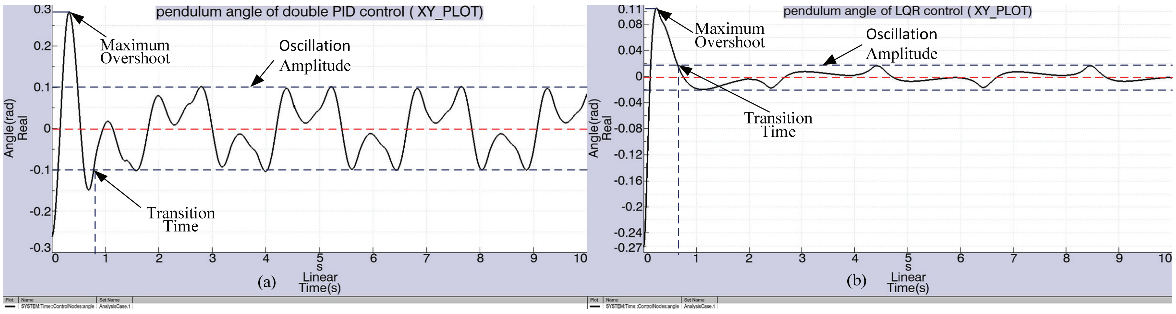

Figure 11. Comparison of pendulum angle between double-PID controller and LQR controller based on virtual prototype

C. Assessment on the Steady State Performance based on Experimental Setup

Experimental setup based on real-time control in dSPACE environment provides the interface between the Simulink environment variables and the real system. Experimental setup, as shown in Fig. 12, intends to investigate the steady state performance of the two controllers. Results are shown in Fig. 13, Fig. 14 and Fig. 15. Table 5 summarizes the steady state performance indices of the two double-PID and LQR controllers. Note that Figure 15 plots the control error signal which equals the control force applied by linear motor. We can see that smaller control force is needed for maintaining the equilibrium position at steady state when using LQR controller. Thus we can conclude that the steady state performance of the LQR controller is better for its smaller oscillation amplitude, oscillation frequency and maintaining force. The data also shows that the simulation results of virtual prototype and that of the experiments is very close.

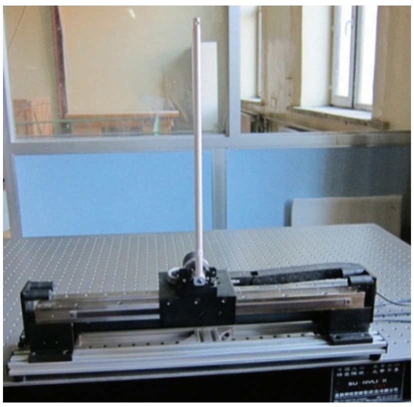

Figure 12. Experimental setup of inverted pendulum

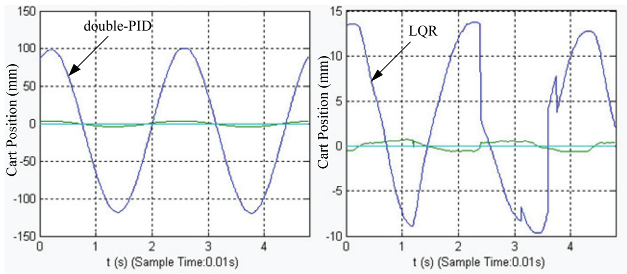

Figure 13. Cart position in experimental setup

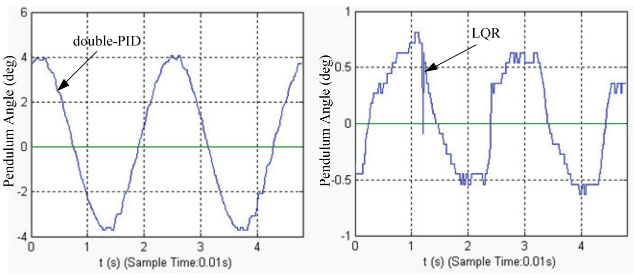

Figure 14. Pendulum angle in experimental setup

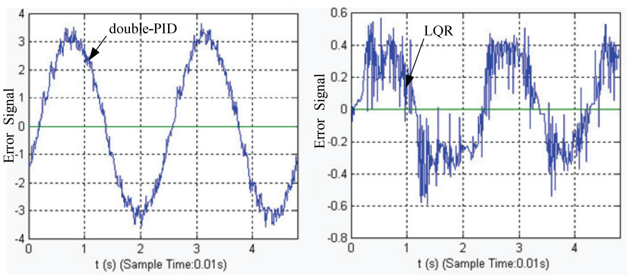

Figure 15. Control error signal in experimental setup

TABLE VI. STEADY STATE PERFORMANCE OF THE DESIGNED CONTROLLERS BASED ON EXPERIMENTAL SETUP

Performance Index	double-PID	LQR
Cart Position *Oscillation Amplitude*	110mm	10mm
Pendulum Angle *Oscillation Amplitude*	4deg	0.6deg
Cart Position *Oscillation Frequency*	0.455Hz	0.431Hz
Pendulum Angle *Oscillation Frequency*	0.455Hz	0.431Hz
Control *Error (Force) Amplitude*	3N	0.6N

VII. CONCLUSIONS

This paper proposes two control methods for the inverted pendulum, an innovative double-PID control method and modern LQR control method. Control parameters are tuned based on simulation, and results show the two proposed controllers are all capable of controlling the cart position and pendulum angle of the inverted pendulum system with acceptable performance. Dynamic performance and steady state performance are investigated of the two proposed control methods. Dynamic performance proves that the proposed LQR controller can guarantee the inverted pendulum a faster and smoother stabilizing process and with better robustness than the double-PID controller. Steady state performance adopts limits cycles as assessment indices, which not only makes the assessment available, but also, provides an effective way for the evaluation of any equilibrium control problem with friction involved. Results show that the steady state performance of the proposed LQR controller has smaller oscillation amplitude than that of the double-PID controller. Further improvement need to be done for both of the controllers. The double-PID controller should be improved to guarantee better dynamic and steady state performance, and the LQR controller can be improved, such as using friction compensation, to further reduce the oscillation amplitude and frequency.

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REFERENCES

[1] X. M. Chen, H. X. Zhou, R. H. Ma, F. C. Zuo, G. F. Zhai, M. L. Gong, “Linear motor driven inverted pendulum and LQR controller design,” Proceedings of IEEE, International Conference on Automation and Logistics. Jinan. China, pp. 1750–1754, August 2007.

[2] D. Chatterjee, A. Patra, H. K.Joglekar, “Swing-up and stabilization of a cart-pendulum system under restricted cart track length,” system & control letters, 2002, vol.47, pp. 335-364.

[3] Y. Xin, B. Xu, H. Xin, J. Xu, L. Y. Hu, “The computer simulation and real-time control for the inverted pendulum system based on PID,” Communication Systems and Information Technology, Lecture Notes in Electrical Engineering, 2011, vol.100, pp. 729-736.

[4] J. Yi, N. Yubazaki, “Stabilization fuzzy control of inverted pendulum systems,” Artificial Intelligence in Engineering, 2000, pp.153–163.

[5] T. K. Liu, C. H. Chen, Z. S. Li, J. H. Chou, “Method of inequalities-based multi-objective genetic algorithm for optimizing a cart-double-pendulum system,” International Journal of Automation and Computing, 2009, vol. 6, pp. 29-37.

[6] H. M. Wang, S. J. Jiang, “LQR control of single inverted pendulum based on square root filter,” Advanced Materials Research, 2012, pp. 433-440.

[7] J. J. Wang, “Simulation studies of inverted pendulum based on PID controllers,” Simulation Modelling Practice and Theory, 2011, vol. 19, pp. 440-449.

[8] M. Ge, M. S. Chiu, Q. G. Wang, “Robust PID controller design via LMI approach,” Journal of Process Control, 2002, vol.12, pp. 3-13.

[9] Y. Z. Yin, H. S. Zhang, “Linear quadratic regulation for discrete-time systems with single input delay,” Proceedings of IEEE, China Control Conference, Shenzhen, China, pp. 672-677, Angust 2006.

[10] K. Barya, S. Tiwari, R. Jha, “Comparison of LQR and robust controllers for stabilizing inverted pendulum system,” Proceedings of IEEE, International Conference on Communication Control and Computing Technologies, Jalandhar, India, pp. 300-304, October 2010.

[11] Z. Y. Xu, X. D. Huang, “Based on linear quadratic regulator optimal control of single inverted pendulum system Co-simulation of ADAMS and SIMULINK,” Advanced Materials Research, 2011, vol. 383-390, pp. 7258-7264.

[12] L. Y. Hu, G. P. Liu, X. P. Liu, H. Zhang, “The computer simulation and real-time stabilization control for the inverted pendulum system based on LQR,” Proceedings of IEEE, Fifth International Conference on Natural Computation, Nanchang, China, pp. 438-442, August 2009.

[13] S. A. Campbell, S. Crauford, K. Morris. “Friction and the Inverted Pendulum Stabilization Problem,” Journal of Dynamic system, Measurement, and Control, 2008, vol. 130, pp. 054502 (1-7).

[14] B. Armstrong Helouvry, P. Dupont, C. Canudas de Wit, “A survey of Models, Analysis Tools and Compensation Methods for the control of Machines with Friction,” Automatica, 1994, vol. 30, pp. 1083-1138.

[15] H. Olsson, K. J. Astrom, C. Canudas de Wit, M. Cafvert, P. Lischinsky, “Friction Models and Friction Compensation,” European Journal of Control, 1998, vol. 3, pp. 176-195.

[16] B. A. Helouvry, “Control of Machines with Friction,” Journal of Tribology, 1992, vol. 114.

[17] H. T. Teixeira, Victor Semedo de Mattos Siqueira, Celso Jose Munaro, “Comparison of Methods for Estimation and Compensation of Friction Applied to an Inverted Pendulum,” Proceedings of the IEEE, International Conference on Control and Automation, Santiago, Chile, pp. 818-823, December 2011.

[18] D. Chol, J. Oh, “Human-friendly Motion Control of a Wheeled Inverted Pendulum by Reduced-order Disturbance Observer,” Proceedings of IEEE, International Conference on Robotics and Automation, Pasadena, USA, pp. 2521-2526May 2008.