COMPARISON OF DIFFERENT METHODS FOR ROBUST CONTROLLER DESIGN

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Abstract: The presented paper deals with simple and practical methods of discrete controller design using conventional approach based on robust stability analysis in frequency domain, and the new approach using the reflection vectors techniques. The control structure consists of feed-forward and feedback part. Proposed algorithms were tested in illustrated examples for stable, unstable and oscillating systems. Simulations were realized in MATLAB-Simulink, Version 7.0. Obtained results show applicability of the theoretical principles for control of processes subject to parametrical model uncertainty.

Keywords: robust control, robust stability, parametrical uncertainty, diofantine equation, pole-placement, time-optimal controller, PID controller, quadratic programming, reflection vectors

1 INTRODUCTION AND PRELIMINARIES

During last ten years, development of robust control elementary principles and evolution of new robust control methods for different model uncertainty types are visible. Progress in new techniques and theories in control of processes with model uncertainty is necessary because of performance requirements on control of complex processes containing large number of loops, activities coordination of a many agents in hybrid and stochastic control of systems containing large plant model uncertainties. Based on theoretical assumptions, modeling and simulation methods, an effective approach to the control of processes with strong and undefined uncertainties is designed. Such uncertainties are typical for biotechnology processes, chemical plants, automobile industry, aviation etc. For such processes is necessary to design robust and practical algorithms which ensures the high performance and robust stability using proposed mathematical techniques with respect the parametric and unmodelled uncertainties. Solution to such problems is possible using robust predictive methods and „soft-techniques" which include fuzzy sets, neuron networks and genetic algorithms.

Robust control is used to guarantee stability of plants with parameter changes. The robust controller design consists of two steps:

- analysis of parameter changes and their influence for closed-loop stability,
- robust control synthesis.

In hybrid control structures that combine the discrete controller and continuous plant, it is difficult to assess the closed-loop stability. One possibility is transformation of the controller and the continuous plant to the discrete-time region and specifying requirements for the discrete controller design. The problem of the robust controller design can be solved as:

- Time-optimal robust controller design,

- Design of the robust controller based on the pole-placement.

In both parts of the robust controller design it is possible to evolve from the solution of Diophantine equations.

2 PROBLEM STATEMENT

Consider the robust control synthesis of a scalar discrete-time control loop. Transfer function of the original continuous-time system is described by the transfer function

$$G_P(s) = \frac{\bar{B}(s)}{\bar{A}(s)} e^{-Ds} = \frac{\bar{b}_m s^m + \bar{b}_{m-1} s^{m-1} + \ldots + \bar{b}_0}{\bar{a}_n s^n + \bar{a}_{n-1} s^{n-1} + \ldots + \bar{a}_0} e^{-Ds} \tag{1}$$

Transfer function of (1) can be converted to its discrete-time counterpart

$$G_P\left(z^{-1}\right) = \frac{b_0 + b_1 z^{-1} + \ldots + b_n z^{-n}}{1 + a_1 z^{-1} + \ldots + a_n z^{-n}} z^{-d} \tag{2}$$

For the plant (2) a discrete-time controller is to be designed in form

$$G_R(z) = \frac{q_0 + q_1 z^{-1} + \ldots + q_\upsilon z^{-\upsilon}}{1 + p_1 z^{-1} + \ldots + p_\mu z^{-\mu}} = \frac{Q(z)}{P(z)} \tag{3}$$

The corresponding closed-loop characteristic equation is

$$1 + G_P(z^{-1}) G_R(z^{-1}) = 0 \tag{4}$$

Substituting (3) and (2) in (4) after a simple manipulation yield the characteristic equation

$$1 + G_P G_R = (1 + p_1 z^{-1} + \ldots + p_\mu z^{-\mu})(1 + a_1 z^{-1} + \ldots + a_n z^{-n}) + (q_0 + q_1 z^{-1} + \ldots + q_\upsilon z^{-\upsilon})(b_1 z^{-1} + \ldots + b_n z^{-n}) z^{-d} = 0 \tag{5}$$

Unknown coefficients of the discrete controller can be designed using various methods. In this paper robust controller design method based on reflection vectors is used.

The pole assignment problem is as follows: find a controller $G_R(z)$ such that $C(z)=e(z)$ where $e(z)$ is a given (target) polynomial of degree k. It is known [8] that when $\mu = n - 1$, the above problem has a solution for arbitrary $e(z)$ whenever the plant has no common pole-zero pairs. In general for $\mu < n - 1$ exact attainment of a desired target polynomial $e(z)$ is impossible.

Let us relax the requirement of attaining the target polynomial $e(z)$ exactly and enlarge the target region to a polytope V in the polynomial space containing the point e representing the desired closed-loop characteristic polynomial. Without any restriction we can assume that $a_n = p_0 = 1$ and deal with monic polynomials $C(z)$, i.e. $\alpha_0 = 1$.

Let us introduce the stability measure as $\rho = c^T c$, where

$$c = S^{-1} C \tag{6}$$

and S is a matrix of dimensions $(n + \mu + 1) \times (n + \mu + 1)$ representing vertices of the target polytope V. For monic polynomials holds

$$\sum_{i=1}^{k+1} c_i = 1 \tag{7}$$

where $k = n + \mu$. If all coefficients are positive, i.e. $c_i > 0$, $i = 1, \ldots, k + 1$, then the point C is placed inside the polytope V.

The minimum ρ is attained if

$$c_1 = c_2 = \ldots = c_{k+1} = \frac{1}{k+1} \tag{8}$$

Then the point C is placed in centre of the polytope V.

In matrix form we have

$$C = G\,x \tag{9}$$

where G is the Sylvester matrix of the plant with dimensions $(n+\mu+d+1)\times(\mu+\upsilon+2)$ and x is the $(\mu+\upsilon+2)$-vector of controller parameters: $x = \left[p_\mu, \ldots, p_1, 1, q_\upsilon, \ldots, q_0\right]^T$.

$$G = \begin{bmatrix}
0 & 0 & \ldots & 0 & 0 & b_{n+\mu-\upsilon} & b_{n+\mu-\upsilon+1} & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 & b_{n+\mu-\upsilon-1} & b_{n+\mu-\upsilon} & \ldots & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . \\
0 & 0 & \ldots & 0 & 0 & b_{n+\mu-\upsilon-d+1} & b_{n+\mu-\upsilon-d+2} & \ldots & 0 & 0 \\
a_n & 0 & \ldots & 0 & 0 & b_{n+\mu-\upsilon-d} & b_{n+\mu-\upsilon-d+1} & \ldots & 0 & 0 \\
a_{n-1} & a_n & \ldots & 0 & 0 & b_{n+\mu-\upsilon-d-1} & b_{n+\mu-\upsilon-d} & \ldots & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . \\
a_1 & a_2 & \ldots & a_\mu & a_{\mu+1} & b_{\mu-\upsilon-d+1} & b_{\mu-\upsilon-d+2} & \ldots & b_{\mu-d} & b_{\mu-d+1} \\
1 & a_1 & \ldots & a_{\mu-1} & a_\mu & b_{\mu-\upsilon-d} & b_{\mu-\upsilon-d+1} & \ldots & b_{\mu-d-1} & b_{\mu-d} \\
0 & 1 & \ldots & a_{\mu-2} & a_{\mu-1} & b_{\mu-\upsilon-d-1} & b_{\mu-\upsilon-d} & \ldots & b_{\mu-d-2} & b_{\mu-d-1} \\
. & . & . & . & . & . & . & . & . & . \\
0 & 0 & \ldots & a_{d+1} & a_{d+2} & 0 & 0 & \ldots & b_1 & b_2 \\
0 & 0 & \ldots & a_d & a_{d+1} & 0 & 0 & \ldots & 0 & b_1 \\
0 & 0 & \ldots & a_{d-1} & a_d & 0 & 0 & \ldots & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . \\
0 & 0 & \ldots & 1 & a_1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 1 & 0 & 0 & \ldots & 0 & 0
\end{bmatrix}$$

Now we can formulate the following control design problem: find a discrete controller $G_R(z)$ such that the closed-loop characteristic polynomial $C(z)$ is placed:

a. In a stable target polytope V, $C(z) \in V$ (to guarantee stability),

b. As close as possible to a target polynomial $e(z)$, $e(z) \in V$ (to guarantee performance).

Let the polytope V denote the $(k+1)\times N$ matrix composed of coefficient vectors v_j, $j = 1, \ldots, N$ corresponding to vertices of the polytope V.

Then we can formulate the above controller design problem as an optimization task: Find x that minimizes the cost function

$$J_1 = \min_x \; x^T G^T G x - 2e^T G x = \min_x \left\| Gx - e \right\|^2 \tag{10}$$

subject to the linear constraints

$$G\,x = V\,w(x), \tag{11}$$

$$w_j(x) > 0, \quad j = 1, \ldots, N, \tag{12}$$

$$\sum_j w_j(x) = 1. \tag{13}$$

Here *w(x)* is the vector of weights of the polytope *V* vertices to obtain the point *C* = *G x*. Fulfillment of the latter two constraints (12), (13) guarantees that the point *C* is indeed located inside the polytope *V*. Then, finding the robust pole-placement controller coefficients represents an optimization problem that can be solved using the Matlab Toolbox OPTIM (quadprog) with constraints [9].

Generally J_1 is a kind of distance to the centre of the target polytope *V*. Is it better to use another criterion J_2, which measures the distance to the Schur polynomial *E(z)*

$$J_2 = (C-E)^T (C-E) = (Gx-E)^T (Gx-E). \tag{14}$$

It is possible to use the weighted combination of J_1 and J_2

$$J = (1-\alpha)J_1 + \alpha J_2, \quad 0 \le \alpha \le 1 \tag{15}$$

and to solve the following quadratic programming task

$$\begin{aligned} &J = \min_x \left\{ x^T G^T \left[(1-\alpha)(SS^T)^{-1} + \alpha I_{k+1} \right] Gx - 2\alpha E^T Gx \right\}, \\ &S^{-1}Gx < 0. \end{aligned} \tag{16}$$

Assume the discrete robust controller design task with parametrical uncertainties in system description. Let us also assume that coefficients of the discrete-time system transfer functions $a_n, \ldots, a_1$ and $b_n, \ldots, b_1$ are placed in polytope *W* with the vertices $d^j = \left[a_n^j, \ldots a_1^j, b_n^j, \ldots, b_1^j\right]$:

$$W = conv\{ d_j, j = 1,\ldots,M \} \tag{17}$$

As (9) is linear in system parameters, it is possible to claim that for arbitrary vector of the controller coefficients *x* is the vector of the characteristic polynomial coefficients *C(z)* placed in the polytope *A* with vertices $a^1, \ldots, a^M$:

$$A = conv\{ a^j, j = 1,\ldots,M \} \tag{18}$$

where $a^j = D^j x$ and D^j is the Sylvester matrix of dimensions $(n + \mu + d + 1)$ x $(\mu + \upsilon + 2)$, composed of vertices set d^j , as in case of the exact model (9).

A) Problem Formulation

The digital controller $x = \left[p_\mu, \ldots, p_1, 1, q_\upsilon, \ldots, q_0\right]^T$ is to be designed such that all its vertices a^j, $j = 1, \ldots, M$ are placed inside a stable desired target polytope *V*.

This problem can be effectively solved using quadratic programming procedure. It is necessary to find the controller *x* by minimization of the cost function

$$J = \min\left\{ x^T \bar{D}^T \begin{bmatrix} (1-\alpha)(I_M \otimes (S^T)^{-1})(I_M \otimes S^{-1}) + \\ +\alpha I_{(k+1)M} \end{bmatrix} \bar{D}x - 2\alpha E^T \bar{D}x \right\}, S^{-1}D^j x > 0,\ j = 1,\ldots,M \tag{19}$$

where I_M is identity matrix of dimension *M*, $\otimes$ is the Kronecker product and $\bar{D}^T = \left[D_1^T, \ldots, D_M^T\right]$.

B) Stable Region Computation via Reflection Coefficients

Polynomials are usually specified by their coefficients or roots. They can be characterized also by their reflection coefficients using Schur-Cohn recursion.

Let $C_k(z^{-1})$ be a monic polynomial of degree k with real coefficients $c_i \in R$, $i = 0, ..., k$,

$$C(z^{-1}) = 1 + c_1 z^{-1} + ... + c_k z^{-k}. \tag{20}$$

Reciprocal polynomial $C_k^*(z^{-1})$ of the polynomial $C_k(z^{-1})$ is defined in [11] as follows

$$C_k^*(z^{-1}) = c_k + c_{k-1} z^{-1} + \ldots + c_1 z^{-k+1} + z^{-k} \tag{21}$$

Reflection coefficients r_i, $i = 1, ..., k$, can be obtained from the polynomial $C_k(z^{-1})$ using backward Levinson recursion [12]

$$z^{-1} C_{i-1}(z^{-1}) = \frac{1}{1 - \left| r_i^2 \right|} \left[C_i(z^{-1}) - r_i C_i^*(z^{-1}) \right] \tag{22}$$

where $r_i = -c_i$ and c_i is the last coefficient of $C_i(z^{-1})$ of degree i. From (22) we obtain in a straightforward way:

$$C_i(z^{-1}) = z^{-1} C_{i-1}(z^{-1}) + r_i C_{i-1}^*(z^{-1}). \tag{23}$$

Expressions for polynomial coefficients $C_{i-1}(z^{-1})$ and $C_i(z^{-1})$ result from equations (22,23):

$$C_{i-1}(z^{-1}) = \frac{1}{1 - \left| r_i^2 \right|} \left[\sum_{j=0}^{i-1} \left(c_{i,j+1} - r_i c_{i,i-j-1} \right) z^{-j} \right] \tag{24}$$

$$C_i(z^{-1}) = \sum_{j=0}^{i} \left(c_{i-1,j-1} + r_i c_{i-1,i-j-1} \right) z^{-j}. \tag{25}$$

The reflection coefficients r_i are also known as Schur-Szegö parameters [11], partial correlation coefficients [6] or k-parameters [13]. Presented forms and structures were effectively used in many applications of signal processing [13] and system identification [6]. A complete characterization and classification of polynomials using their reflection coefficients instead of roots (zeros) of polynomials is given in [11].

The main advantage of using reflection coefficients is that the transformation from reflection to polynomial coefficients is very simple. Indeed, according to (23) and (25), polynomial coefficients c_i depend multilinearly on the reflection coefficients r_i. If the coefficients $c_i \in R$ are real, then also the reflection coefficients $r_i \in R$ are real.

Transformation from reflection coefficients r_i, $i = 1, ..., k$, to polynomial coefficients c_i, $i = 1, ..., k$, is as follows

$$c_i = c_i^{(k)}, \quad c_i^{(i)} = -r_i,$$

$$c_j^{(i)} = c_j^{(i-1)} - r_i c_{i-j}^{(i-1)}, \quad i = 1, ..., k; \quad j = 1, ..., i-1 \tag{26}$$

or in the matrix form

$$C = R(r) c^{(t)}, \ \mathrm{t} = 1, ..., \mathrm{k} - 1, \tag{27}$$

where

$$C = [c_k, \ldots, c_1, 1]^T, \quad C^{(t)} = \left[0, c_t^{(t)}, \ldots, c_1^{(t)}, 1\right]^T, \; C^{(0)} = [0, 1]^T,$$

$$R(r) = R_k(r_k) \begin{bmatrix} 0^T \\ R_{k-1}(r_{k-1}) \end{bmatrix} \cdots \begin{bmatrix} 0^T \\ R_t(r_t) \end{bmatrix},$$

$$R_j(r_j) = I_{j+1} - r_j E_{j+1},$$

where I_k is a k x k identity matrix, $E_k = \begin{bmatrix} 0 \ldots 1 \\ \cdot \;\; \cdot \;\; \cdot \\ 1 \ldots 0 \end{bmatrix}$ and 0^T is a row vector of zeros.

Lemma 1. A linear discrete-time dynamic system is stable if its characteristic polynomial is Schur stable, i.e., if all its poles lie inside the unit circle.

The stability criterion in terms of reflection coefficients is as follows [11].

Lemma 2. A polynomial $C(z^{-1})$ has all its roots inside the unit disk if and only if $|r_i| < 1$, $i = 1, \ldots, k$.

A polynomial $C(z^{-1})$ lies on the stability boundary if some $r_i = \pm 1$, $i = 1, \ldots, k$. For monic Schur polynomials there is a one-to-one correspondence between $C = [c_k, \ldots, c_1]^T$ and $r = [r_1, \ldots, r_k]^T$.

Stability region in the reflection coefficient space is simply the k-dimensional unit hypercube $R = \{r_i \in (-1, 1), i = 1, \ldots, k\}$.. The stability region in the polynomial coefficient space can be found starting from the hypercube R.

C) Stable Polytope of Reflection Vectors

It will be shown that for a family of polynomials the linear cover of the so-called reflection vectors is Schur stable.

Definition 1. The reflection vectors of a Schur stable monic polynomial $C(z^{-1})$ are defined as the points on stability boundary in polynomial coefficient space generated by changing a single reflection coefficient r_i of the polynomial $C(z^{-1})$.

Let us denote the positive reflection vectors of $C(z^{-1})$ as $v_i^+(C) = (C | r_i = 1), i = 1, \ldots, k,$ and the negative reflection vectors of $C(z^{-1})$ as $v_i^-(C) = (C | r_i = -1), i = 1, \ldots, k.$

The following assertions hold:

1. every Schur polynomial has $2k$ reflection vectors $v_i^+(C)$ and $v_i^-(C), i = 1, \ldots, k;$

2. all reflection vectors lie on the stability boundary $(r_i^v = \pm 1);$

3. the line segments between reflection vectors $v_i^+(C)$ and $v_i^-(C)$ are Schur stable.

In the following theorem a family of stable polynomials is defined such that the polytope generated by reflection vectors of these polynomials is stable.

Theorem 1. Consider $r_1^C \in (-1, 1)$, $r_k^C \in (-1, 1)$ and $r_2^C = \ldots = r_{k-1}^C = 0$. Then the inner points of the polytope $V(C)$ generated by the reflection vectors of the point C

$$V(C) = conv\left\{v_i^{\pm}(C), i = 1,\dots,k\right\} \tag{28}$$

are Schur stable.

D) Roots of Reflection Vectors

In this section we study the root placement of reflection vectors. It is useful for selecting a stable target simplex to solve the robust output control problem.

By definition, at least one root of a reflection vector $v_i(C)$ (i.e. root of $V_i(z^{-1}) = \begin{bmatrix} z^{-k} & \dots & z^{-1} & 1 \end{bmatrix}\begin{bmatrix} v_i(C) \\ 1 \end{bmatrix}$) must lie on the unit circle, and the number of unit circle roots is determined by the number i of the reflection vector $v_i(C)$ and the character of the roots (real or complex) is determined from the sign of the boundary reflection coefficient $(r_i^V = \pm 1)$.

E) Robust Controller Design

A robust controller is to be designed such that the closed-loop characteristic polynomial is placed in the stable polytope (linear cover) of reflection vectors. It means that the following problems have to be solved:

1. choice of initial polynomial $C(z^{-1})$ for generating the polytope $V(C)$,

2. choice of $k + 1$ most suitable vertices of $V(C)$ to build a target simplex S,

3. choice of a target polynomial $E(z^{-1})$.

In the following section some "thumb rules" are given for choosing a stable target simplex S.

To choose $k + 1$ vertices of the target simplex S we use the well known fact that poles with positive real parts are preferred to those with negative ones [1]. The positive reflection vectors $v_i^+(C)$ with i odd and negative reflection vectors $v_i^-(C)$ with i even are chosen yielding k vertices. The $(k+1)$th vertex of the target simplex S is chosen as the mean of the remaining reflection vectors.

The target polynomial $E(z^{-1})$ of order k is reasonable to be chosen inside the stable polytope of reflection vectors $V(C)$. A common choice is $E(z^{-1})=C(z^{-1})$.

For higher-order polynomials the size of the target simplex S is considerably less than the volume of the polytope of reflection vectors V. That is why the above quadratic programming method with a preselected target simplex S works only if uncertainties are sufficiently small. Otherwise it is reasonable to use some search procedure to find a robust controller such that the polytope of closed-loop characteristic polynomial is placed inside the stable polytope of reflection vectors $V(C)$.

3 EXAMPLES

Consider a system described by the second order transfer function

$$G_P(s) = \frac{B(s)}{A(s)} e^{-Ds} = \frac{b_0}{s^2 + a_1 s + a_0} e^{-Ds} \tag{29}$$

with individual coefficients varying within uncertainty intervals

$$b_0 \in \langle 3;4 \rangle, a_1 \in \langle 0.4;0.8 \rangle, a_0 \in \langle 0.6;1 \rangle, D = 0.6 \tag{30}$$

After some modification of (29) we obtain

$$G_P(s) = \frac{1}{\frac{1}{b_0}s^2 + \frac{a_1}{b_0}s + \frac{a_0}{b_0}} e^{-Ds} = \frac{1}{a_2' s^2 + a_1' s + a_0'} e^{-Ds} \tag{31}$$

with individual coefficients varying within uncertainty intervals

$$a_2' \in \left\langle \frac{1}{4}; \frac{1}{3} \right\rangle, a_1' \in \left\langle \frac{1}{10}; \frac{4}{15} \right\rangle, a_0' \in \left\langle \frac{3}{20}; \frac{1}{3} \right\rangle, D = 0.6 \tag{32}$$

To assess stability, four continuous-time Charitonov transfer functions are considered. They have been converted to the discrete region with sampling period *T=0.6s*:

$$G_{+-}\left(z^{-1}\right) = \frac{0.5023z^{-2} + 0.4729z^{-3}}{1 - 1.689z^{-1} + 0.8353z^{-2}} \quad G_{++}\left(z^{-1}\right) = \frac{0.457z^{-2} + 0.3892z^{-3}}{1 - 1.492z^{-1} + 0.6188z^{-2}}$$

$$G_{-+}\left(z^{-1}\right) = \frac{0.566z^{-2} + 0.456z^{-3}}{1 - 1.187z^{-1} + 0.5273z^{-2}} \quad G_{--}\left(z^{-1}\right) = \frac{0.6399z^{-2} + 0.59z^{-3}}{1 - 1.377z^{-1} + 0.7866z^{-2}} \tag{33}$$

A) Time-optimal controller

For each of these discrete transfer functions (33) were designed time-optimal controllers by the solution of the diofantine equation

$$A(z)P(z) + B(z)Q(z) = C(z) \tag{34}$$

where the target closed-loop characteristic polynomial by the time-optimal robust controller design is: *C(z) = 1.*

$$G_{R+-}\left(z^{-1}\right) = \frac{Q\left(z^{-1}\right)}{P\left(z^{-1}\right)} = \frac{4.0113 - 2.94z^{-1}}{1 + 1.84z^{-1} + 1.0864z^{-2}}$$

$$G_{R++}\left(z^{-1}\right) = \frac{Q\left(z^{-1}\right)}{P\left(z^{-1}\right)} = \frac{3.3208 - 1.8976z^{-1}}{1 + 1.562z^{-1} + 0.8019z^{-2}} \tag{35}$$

$$G_{R-+}\left(z^{-1}\right) = \frac{Q\left(z^{-1}\right)}{P\left(z^{-1}\right)} = \frac{0.189 - 0.1948z^{-1}}{1 + 0.8144z^{-1} + 0.2515z^{-2}}$$

$$G_{R--}\left(z^{-1}\right) = \frac{Q\left(z^{-1}\right)}{P\left(z^{-1}\right)} = \frac{-0.1111 - 0.4326z^{-1}}{1 + 1.045z^{-1} + 0.3912z^{-2}}$$

Stability analysis based on the designation of the phase and magnitude margin was realized per Bode logarithmic-frequency characteristics (Fig.1) for each individual closed-loop with appropriate polynomial controller in feedback loop. Found phase margins:

$$\Delta\varphi_{+-} = 101°, \ \Delta\varphi_{++} = 108°, \ \Delta\varphi_{-+} = -37.5°, \ \Delta\varphi_{--} = -20° \tag{36}$$

Based on the absolute value minimization of phase margin *min(abs(Δφ))* was selected discrete polynomial controller G_{R--} and later applied in feedback loop with other plants. Selected controller did not succeed to control the system (29) in whole range of parameter variety and only in case of transfer functions $G_{--}(s)$ and $G_{-+}(s)$ is the closed-loop stable.

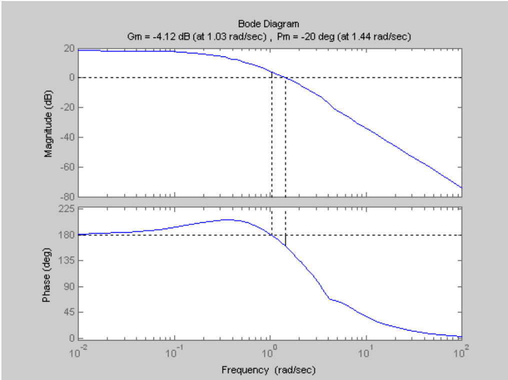

Figure 1: Bode logarithmic-frequency characteristic (closed-loop with controller G_{R--})

B) Robust pole-placement algorithm

One possible way how to design a stable controller is to design it for the plant model with the lowest phase margin (the worst case) and apply it in all other plant models.

Based on the solution of the Diophantine equation the following controller was designed for the continuous-time transfer function $G_{+-}(s)$ with the worst phase margin value:

$$G_{FB}\left(z^{-1}\right)=\frac{Q(z^{-1})}{P(z^{-1})}=\frac{0.94-0.84z^{-1}-0.19z^{-2}}{1+1.7z^{-1}+0.73z^{-2}} \tag{37}$$

with the corresponding control law:

$$u_2(k)=-1.7u_2(k-1)-0.73u_2(k-2)+0.94y(k)-0.84y(k-1)-0.19y(k-2) \tag{38}$$

The target closed-loop characteristic polynomial according to previous consideration is:

$$C(z)=(1+0.86z^{-1})(1+0.67z^{-1})(1-0.21z^{-1})(1-0.71z^{-1})(1-0.76z^{-1}) \tag{39}$$

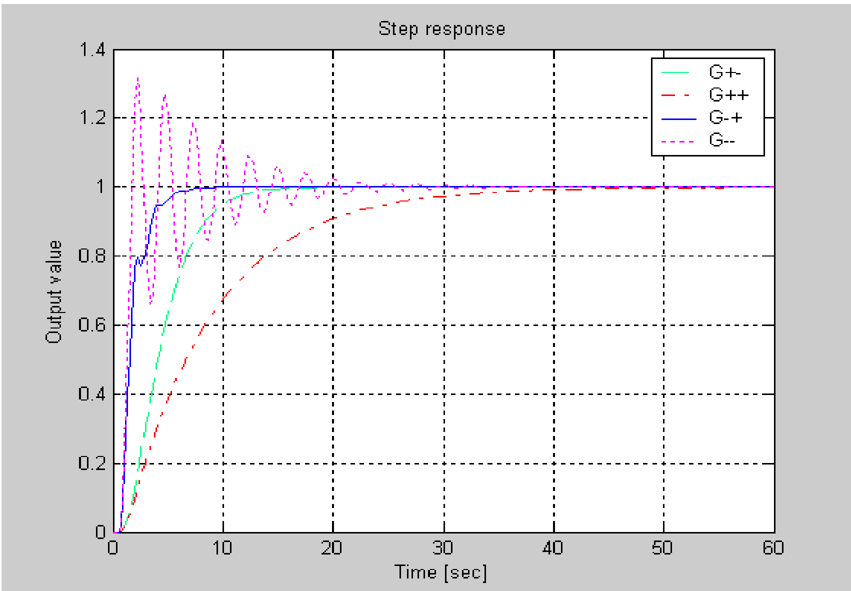

Figure 2: Closed-loop step responses under robust controller

Closed-loop step responses under the discrete-time feedback controller (37) and feed-forward controllers $G_{FF}\left(z^{-1}\right)=S(z^{-1})/P(z^{-1})=\frac{1}{13.6}$ in case of $G_{+-}(s)$ and $G_{++}(s)$ transfer functions, respectively, and $G_{FF}\left(z^{-1}\right)=\frac{1}{2.11}$ in case of $G_{-+}(s)$ and $G_{--}(s)$ transfer functions are illustrated in Fig.2.

C) Controller Design via Reflection Coefficients

Consider the nominal continuous-time transfer function (29) with $b_0=3.5, a_1=0.6, a_0=0.8$ converted to the discrete-time with the sampling period *T=0.6s:*

$$G_{nom}\left(z^{-1}\right)=\frac{0.5477z^{-2}+0.4853z^{-3}}{1-1.462z^{-1}+0.6977z^{-2}} \tag{40}$$

The task is to design a discrete-time controller (3), $\upsilon=\mu=2$.

From the transfer function (40) and matrix form of (9) results

$$C=\begin{bmatrix} 0 & 0 & 0 & 0.4853 & 0 & 0 \\ 0.6977 & 0 & 0 & 0.5477 & 0.4853 & 0 \\ -1.462 & 0.6977 & 0 & 0 & 0.5477 & 0.4853 \\ 1 & -1.462 & 0.6977 & 0 & 0 & 0.5477 \\ 0 & 1 & -1.462 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} p_2 \\ p_1 \\ 1 \\ q_2 \\ q_1 \\ q_0 \end{bmatrix} \tag{41}$$

Let us choose the initial polynomial *C(z^-1^)* for generating the polytope *V(C)* as follows

$$C(z)=[1-(0.3\pm 0.2i)z^{-1}][1-0.2z^{-1}][1+0.3z^{-1}][1+0.3z^{-1}] \tag{42}$$

with reflection coefficients $r_1=0.2,\ r_2=0.14,\ r_3=-0.052,\ r_4=-0.0069,\ r_5=0.00234.$

Now we can find the reflection vectors $v_i(C)$ of the initial polynomial $C(z^{-1})$ leading to the matrix form of the target simplex S (vertex polynomial coefficients)

$$S=\begin{bmatrix} 0 & 0 & 0 & -0.3 & 0.5 & 0.6 \\ 0 & 0 & -0.3 & 0.5 & 0 & 0 \\ 0 & -0.3 & 0.5 & 0 & 0 & 0 \\ -0.3 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 & -0.3 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \tag{43}$$

The discrete-time controller design task for the nominal transfer function (40) has been solved via quadratic programming taking $\alpha=0.1$ in the cost function J (16).

For the selected target simplex S we have obtained the following discrete-time feedback controller

$$G_{FB}\left(z^{-1}\right)=\frac{Q(z^{-1})}{P(z^{-1})}=\frac{0.048-0.0607z^{-1}+0.0366z^{-2}}{1+0.0425z^{-1}+0.00134z^{-2}} \tag{44}$$

with the control law

$$u_2(k) = -0.0425u_2(k-1) - 0.00134u_2(k-2) + 0.048y(k) - 0.0607y(k-1) + 0.0366y(k-2) \quad (45)$$

Corresponding closed-loop step responses under the feedback controller (44) and feed-forward controller $G_{FF}(z^{-1}) = S(z^{-1})/P(z^{-1}) = 7.6$ in case of $G_{+-}(s)$ and $G_{++}(s)$ transfer functions, respectively, and $G_{FF}(z^{-1}) = 15.7$ in case of $G_{-+}(s)$ and $G_{--}(s)$ transfer functions are in Fig.3.

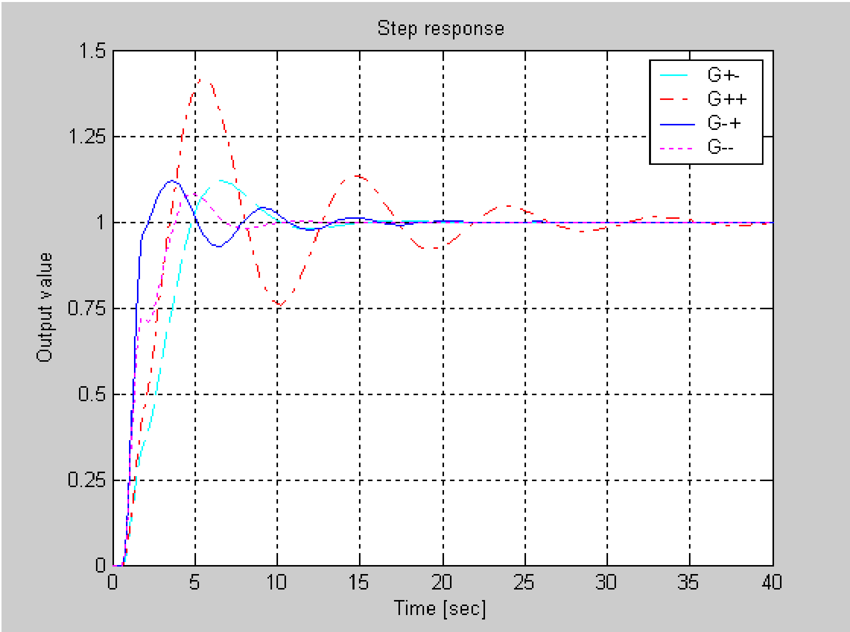

Figure 3: Closed-loop step responses under robust controller

D) PID Controller

The optimal module method is one of the best methods for PID controller design and was applied also for solving example. The PID-controller continuous transfer function calculated by the optimal module method to the nominal plant (29):

$$G_{PID}(s) = \frac{0.2216\,s^2 + 0.1299\,s + 0.1693}{s} = 0.1299\left(1 + \frac{1}{0.7673\,s} + 1.7052\,s\right) \quad (46)$$

PID-controller (46) was tested in whole range of the parameter uncertainty variety with individual continuous transfer functions (33). Designed PID-controller do not stabilize all systems in required quality and stability. In case of G_{+-} transfer function is closed-loop unstable.

4 CONCLUSION

The paper deals with the development of robust methods based on reflection vectors methodology for computation of control law coefficients guaranteeing stability, robustness and high performance with respect to parameter uncertainties. Theoretical results were verified on the examples for feedback and feedforward control structures. Proposed methods were tested for both stable and unstable processes.

The paper proposes theoretical principles and design methodology of robust discrete-time controllers for systems with parametric uncertainties.

The illustrative example was solved using quadratic programming for suitably defined cost function. Simulation results prove applicability of the proposed robust controller design theory for systems with parametric uncertainty.

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