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On Anisotropy-Based Control Problem with Regional Pole Assignment for Descriptor Systems*

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***Abstract*—In this paper, anisotropy-based control problem with regional pole assignment for descriptor systems is investigated. The purpose is to find a state-feedback control law, which guarantees desirable disturbance attenuation level from stochastic input with unknown covariance to controllable output of the closed-loop system, and ensures, that all finite eigenvalues of the closed-loop system belong to the given region inside the unit disk. The proposed control design procedure is based on solving convex optimization problem with strict constraints. The numerical effectiveness is illustrated by numerical example.**

I. Introduction

The problem of pole placement for linear descriptor systems is discussed in different works, i.e. [1]–[3]. Pole placement is a well-known technique for shaping desired transient performance. The exact pole placement problem deals with designing of a state-feedback control law, that provides desired exact finite eigenvalues of the closed-loop system [1]. However, sensitivity to parametric uncertainties and impossibility to apply the additional quality criteria are substantial disadvantages of this approach.

These disadvantages can be overcome by using regional pole assignment technique. In this case, finite eigenvalues of the closed-loop system are supposed to belong to some convex region inside the unit disk on the complex plane. The most useful LMI region is described by the interior of the circle inside the unit disk with given center and radius [2], [3]. In continuous and discrete time cases, there are some results on control of descriptor systems with regional pole placement including additional $\mathcal{H}_2$ and $\mathcal{H}_\infty$criteria [4]–[7]. While pole placement control provides transient response performance, this additional criteria provide guaranteed disturbance attenuation level of the closed-loop system.

In this paper we deal with anisotropic norm of closed-loop descriptor system as an additional criterion. In anisotropy-based control theory, the system is considered to be affected by a random disturbance with unknown covariance. According to [8] the a-anisotropic norm of a system is a particular case of the stochastic norm and is defined as the supremum of the ratio of the root mean square value of the system output to that of the input over all stationary Gaussian inputs with the mean anisotropy upper-bounded by a nonnegative parameter a. For the absolutely continuously distributed Gaussian random vector the anisotropy is defined as a difference between the differential entropy of the Gaussian random vector with zero mean and constant diagonal covariance matrix and the differential entropy of this vector, and can be considered as a measure of distinction between the covariance matrix of a random vector and the identity matrix; see [8]. Moreover, the scaled $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of a system are the limiting cases of the a-anisotropic norm for $a \to 0$ and $a \to \infty$ respectively.

The aim of this paper is to solve anisotropy-based control problem with regional pole placement for discrete-time descriptor systems. The solution of this problem makes possible to find a state-feedback control law such that closed-loop system is admissible, its transient response satisfy the desired performance, and the anisotropic gain from input disturbance to the controllable output does not exceed specified level.

The paper is organized as follows. In section II, problem statement is proposed. Section III provides necessary background on the descriptor systems and anisotropy-based control theory. Main results and a numerical example are presented in sections IV and V respectively.

Notations. The following notations will be used throughout the paper. $\mathbb{R}$ and $\mathbb{C}$ denote real and complex sets respectively; I_n is an identity $(n \times n)$ matrix; Z^{T} is transpose of matrix Z; $\mathrm{sym}(Z) = Z + Z^{\mathrm{T}}$; Z^* is the Hermitian conjugate of the matrix $Z = [z_{ij}] \in \mathbb{C}^{m\times n}$: $Z^* = [z^*_{ji}] \in \mathbb{C}^{n\times m}$.

II. Problem statement

Consider the following discrete-time descriptor system:

$$E_d x(k+1) = A_d x(k) + B_{wd} w(k) + B_{ud} u(k), \quad (1)$$

$$z(k) = C_d x(k) + D_{wd} w(k) \quad (2)$$

where $x(k) \in \mathbb{R}^n$ is the state, $w(k) \in \mathbb{R}^{m_1}$, $z(k) \in \mathbb{R}^p$ is an observable output, $u(k) \in \mathbb{R}^{m_2}$ is a control sequence, E_d, A_d, B_{wd}, B_{ud}, C_d, D_{wd} are constant real matrices of appropriate dimensions; rank $(E_d) = r < n$.

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The behavior of discrete-time descriptor systems is radically different from the behavior of standard ones. Unlike standard systems, the regularity of descriptor system is required as it provides the existence and uniqueness of system's solution.

Definition 1: The system (1) is called regular if

$$\exists \lambda \neq 0 : \det(\lambda E_d - A_d) \neq 0.$$

In the paper, system (1)–(2) is assumed to be regular. If not, several regularization techniques can be found in [9], [10].

Descriptor systems may also have noncausal behavior. It means that the current state may depend on future values of the input signal. Obviously, this undesired property should not appear in the closed-loop system. Causality can be checked by the following expression. The system (1) is causal if

$$\deg \det(zE_d - A_d) = \operatorname{rank} E_d.$$

Stability of the descriptor system is defined in Lyapunov sense, i.e. descriptor system (1) is called stable if

$$\rho(E_d, A_d) = \max |\lambda|_{\lambda \in \{z | \det(zE_d - A_d) = 0\}} < 1.$$

In other words, system (1) is stable if all finite eigenvalues of matrix pencil $(zE_d - A_d)$ lie inside the unit circle.

Definition 2: Regular, stable, and causal descriptor system is called admissible.

So, control of discrete-time descriptor system consists of stabilization and causalization procedures.

Definition 3: Consider the region on the complex plane, defined by

$$\mathfrak{D} = \{z \in \mathbb{C} : d + 2b\mathrm{Re}(z) + c|z|^2 < 0\}. \tag{3}$$

The pair (E_d, A_d) is called $\mathfrak{D}$–admissible if it is admissible and its finite eigenvalues lie inside region $\mathfrak{D}$.

We assume that

1) the whole state vector is observable;
2) $p \leqslant m_1$;
3) $\operatorname{rank} E_d = \operatorname{rank} [\ E_d \quad B_{wd}\ ]$;
4) $\operatorname{rank} E_d = \operatorname{rank} [\ E_d^{\mathrm{T}} \quad C_d^{\mathrm{T}}\ ]$;
5) system (1) is causally controllable;
6) system (1) is stabilizable.

In our problem $w(k)$ is supposed to be a random stationary sequence with known mean anisotropy level $\overline{\mathbf{A}}(W) \leqslant a$ ($a \geqslant 0$). The concept of mean anisotropy is discussed below.

The transfer function of unforced system (1)–(2) is defined as $P(z) = C_d(zE_d - A_d)^{-1}B_{wd} + D_{wd}$, $z \in \mathbb{C}$.

The problem considered in this paper is formulated as follows.

Problem 1. For system (1)–(2) and given scalar numbers a and γ the problem is to find a state-feedback control law

$$u(k) = F_d x(k), \tag{4}$$

such that the closed-loop system

$$E_d x(k+1) = (A_d + B_{ud}F_d)x(k) + B_{wd}w(k), \tag{5}$$
$$z(k) = C_d x(k) + D_{wd}w(k) \tag{6}$$

with transfer function
$P_{cl}(z) = C_d(zE_d - A_d - B_{ud}F_d)^{-1}B_{wd} + D_{wd}$

1) is $\mathfrak{D}$-admissible;
2) its a-anisotropic norm (system's gain from the input disturbance to the controllable output) satisfies the condition
$$|\!|\!| P_{cl}(z) |\!|\!|_a < \gamma.$$

In other words, anisotropy-based suboptimal control problem with pole placement constraints is to find state-feedback control law, such that closed-loop system is stable, causal, its transient response satisfies desirable requirements, and its disturbance attenuation level of the stochastic input signal is bounded by γ.

III. Background

In this section, we provide some preliminary material on descriptor systems [1], [11], anisotropy-based control theory [12], [13], and $\mathfrak{D}$-admissibility conditions [2].

A. Descriptor systems

In this section, we recall some basics of descriptor systems theory necessary for the following investigation, which were not mentioned above.

System (1)–(2) is assumed to be regular. It means that there exist two nonsingular matrices $\overline{W}$ and $\overline{V}$ such that $\overline{W}E_d\overline{V} = \operatorname{diag}(I_r, 0)$.

Consider the following change of variables

$$\overline{V}^{-1}x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \tag{7}$$

where $x_1(k) \in \mathbb{R}^r$ and $x_2(k) \in \mathbb{R}^{n-r}$.

By left multiplying the system (1)–(2) on the matrix $\overline{W}$ and using the change of variables (7), one can rewrite the system (1)–(2) in the form [1]

$$\begin{aligned} x_1(k+1) &= A_{11}x_1(k) + A_{12}x_2(k) + B_{w1}w(k) + B_{u1}u(k), \\ 0 &= A_{21}x_1(k) + A_{22}x_2(k) + B_{w2}w(k) + B_{u2}u(k), \\ y(k) &= C_1x_1(k) + C_2x_2(k) + D_w w(k) \end{aligned}$$

where

$$\overline{W}A_d\overline{V} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \overline{W}B_{ud} = \begin{bmatrix} B_{u1} \\ B_{u2} \end{bmatrix},$$
$$\overline{W}B_{wd} = \begin{bmatrix} B_{w1} \\ B_{w2} \end{bmatrix}, \quad C_d\overline{V} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}. \tag{8}$$

Matrices $\overline{W}$ and $\overline{V}$ can be obtained using singular value decomposition (SVD)

$$E_d = U \operatorname{diag}(S, 0) H^{\mathrm{T}}.$$

Here U and H are real orthogonal matrices, S is a diagonal $(r \times r)$-matrix, it is formed by nonzero singular values of the matrix E_d

$$\overline{W} = \operatorname{diag}(S^{-1}, I_{n-r})U^{\mathrm{T}}, \qquad \overline{V} = H.$$

Hereinafter we use the following notations:
$E = \overline{W}E_d\overline{V} = \operatorname{diag}(I_r, 0)$, $A = \overline{W}A_d\overline{V}$, $B_u = \overline{W}B_{ud}$, $B_w = \overline{W}B_{wd}$, $C = C_d\overline{V}$, $D_w = D_{wd}$.

Note that the system is causal if $\det(A_{22}) \neq 0$, and stable if $\rho(A_{11} - A_{12}A_{22}^{-1}A_{21}) < 1$ [11].

The following lemma, introduced in [2], will be useful below.

Lemma 1: [2] Let $\mathfrak{D}$ be a disc centered around the origin and of radius ω, i.e. $d = -\omega^2$, $b = 0$, and $c = 1$. The pair (E_d, A_d) has g poles inside $\mathfrak{D}$ and $(n-g)$ poles outside $\mathfrak{D}$ if and only if there exist $X = X^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ with g positive, $(n-g)$ negative, and 0 zero eigenvalues satisfying inequality

$$-\omega^2 E_d X E_d^{\mathrm{T}} + A_d X A_d^{\mathrm{T}} < 0. \tag{9}$$

Lemma 1 provides a procedure for checking $\mathfrak{D}$–admissibility of the system (1)–(2). This lemma will be used in future in order to ensure that all finite eigenvalues belong to the selected region $\mathfrak{D}$.

B. Mean anisotropy of the sequence and anisotropic norm of the system

In problem statement, it has been mentioned that the systems affected by exogenous disturbance with nonnegative mean anisotropy level a. This section provides brief introduction on anisotropy of signals and anisotropic norm of systems. Full information on the anisotropy-based robust performance analysis developed originally in [12], [13] can be found in more detail in [14], [15].

Let $W = \{w(k)\}_{k\in\mathbb{Z}}$ be a stationary sequence of square-integrable random m_1-dimensional vectors. The sequence W can be generated from the Gaussian white noise sequence V with zero mean and identity covariance matrix by an admissible shaping filter with a transfer function $G(z) = C_G(zE_G - A_G)^{-1}B_G + D_G$. Mean anisotropy of the signal is Kullback-Leibler information divergence from probability density function (p.d.f.) of the signal to p.d.f. of the Gaussian white noise sequence.

Mean anisotropy of the sequence can be defined by the filter's parameters, using the expression

$$\overline{\mathbf{A}}(W) = -\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\det\frac{m_1 S(\omega)}{\|G(z)\|_2^2}d\omega$$

where $S(\omega) = \widehat{G}(\omega)\widehat{G}^*(\omega)$, $(-\pi \leqslant \omega \leqslant \pi)$, $\widehat{G}(\omega) = \lim_{l\to 1} G(l\mathrm{e}^{i\omega})$ is a boundary value of the transfer function $G(z)$, and $\|G(z)\|_2$ is $\mathcal{H}_2$ norm of shaping filter $G(z)$ defined by the expression

$$\|G\|_2 = \left(\frac{1}{2\pi}\int_0^{2\pi} \mathrm{tr}\left(G^*(e^{i\omega})G(e^{i\omega})\right)d\omega\right)^{\frac{1}{2}}.$$

Remark 1: Since the probability law of the sequence W is completely determined by the shaping filter $G(z)$, the alternative notation $\overline{\mathbf{A}}(G)$ is also used instead of $\overline{\mathbf{A}}(W)$.

Mean anisotropy of the signal characterizes its "spectral color", i.e. the difference between the signal and the Gaussian white noise sequence. If $\overline{\mathbf{A}}(W) = 0$, then the signal is the Gaussian white noise sequence. If $\overline{\mathbf{A}}(W) \to \infty$, the signal is a determinate sequence. For more information, see [12], [16].

Let $Z = PW$ be an output of the linear discrete-time descriptor system $P \in \mathcal{H}_\infty{}^{p\times m_1}$ with a transfer function $P(z)$, which is analytic in the identity circle $|z| < 1$, $P(z)$ has a finite $\mathcal{H}_\infty$-norm, i.e.

$$\|P(z)\|_\infty = \sup_{\omega\in[0,2\pi]} \sigma_{max}\left(P(e^{i\omega})\right) < \infty$$

where $\sigma_{max}\left(P(e^{i\omega})\right)$ is the maximum singular value of the transfer function $P(z)$.

Definition 4: For a given constant value $a \geqslant 0$ a-anisotropic norm of the system P is defined as

$$|\!|\!|P(z)|\!|\!|_a = \sup\left\{\|P(z)G(z)\|_2/\|G(z)\|_2 :\ G(z) \in \mathbf{G}_a\right\}, \tag{10}$$

i.e. the maximum value of the system's gain with respect to the class of shaping filters

$$\mathbf{G}_a = \left\{G(z) \in \mathcal{H}_2{}^{m_1\times m_1} :\ \overline{\mathbf{A}}(G) \leqslant a\right\}.$$

So, a-anisotropic norm $|\!|\!|P(z)|\!|\!|_a$ describes the stochastic gain of the system P with respect to the random input sequence W with mean anisotropy $a \geqslant 0$.

Definition 4 sets a maximum value of anisotropic gain of the system. In practical applications, it is enough to check the condition $|\!|\!|P(z)|\!|\!|_a \leqslant \gamma$ for given scalar $\gamma > 0$ and known mean anisotropy level of the input disturbance $a \geqslant 0$. The following lemma allows to check a-anisotropic norm boundedness using set of convex constraints [17].

Theorem 1: [17] Suppose that

$$\mathrm{rank}\, E_d = \mathrm{rank}\left[\begin{array}{cc} E_d & B_{wd}\end{array}\right].$$

For given scalar values $a \geqslant 0$ and $\gamma > 0$ the unforced system (1)–(2) with a transfer function $P(z) \in \mathcal{H}_\infty{}^{p\times m_1}$ is admissible and its a-anisotropic norm is bounded by γ, i.e.

$$|\!|\!|P(z)|\!|\!|_a < \gamma$$

if there exist matrices $L \in \mathbb{R}^{r\times r}$, $L > 0$, $Q \in \mathbb{R}^{r\times r}$, $R \in \mathbb{R}^{r\times(n-r)}$, $S \in \mathbb{R}^{(n-r)\times(n-r)}$, scalar values $\eta > \gamma^2$ such that

$$\eta - (\mathrm{e}^{-2a}\det(\eta I_{m_1} - B_w^{\mathrm{T}}\Theta B_w - D_w^{\mathrm{T}} D_w))^{1/m_1} < \gamma^2, \tag{11}$$

$$\left[\begin{array}{ccccc} \Phi_{11} & \Gamma A & \Gamma B_w & \Phi_{41}^{\mathrm{T}} & 0 \\ A^{\mathrm{T}}\Gamma^{\mathrm{T}} & \Phi_{22} & \Pi B_w & A^{\mathrm{T}}\Gamma^{\mathrm{T}} & \Phi_{52}^{\mathrm{T}} \\ B_w^{\mathrm{T}}\Gamma^{\mathrm{T}} & B_w^{\mathrm{T}}\Pi^{\mathrm{T}} & -\gamma^2 I_{m_1} & B_w^{\mathrm{T}}\Gamma^{\mathrm{T}} & \Phi_{53}^{\mathrm{T}} \\ \Phi_{41} & \Gamma A & \Gamma B_w & -Q - Q^{\mathrm{T}} & 0 \\ 0 & \Phi_{52} & \Phi_{53} & 0 & -I_p \end{array}\right] < 0 \tag{12}$$

where
$\Phi_{11} = -\frac{1}{2}Q - \frac{1}{2}Q^{\mathrm{T}}$, $\Phi_{22} = \Pi A + A^{\mathrm{T}}\Pi^{\mathrm{T}} - \Theta$,
$\Phi_{41} = L - Q - \frac{1}{2}Q^{\mathrm{T}}$, $\Phi_{52} = C + \alpha C\Pi A$,
$\Phi_{53} = D_w + \alpha C\Pi B_w$,

$$\Theta = \left[\begin{array}{cc} L & 0 \\ 0 & 0\end{array}\right], \Pi = \left[\begin{array}{cc} 0 & 0 \\ 0 & S\end{array}\right], \Gamma = \left[\begin{array}{cc} Q & R\end{array}\right].$$

A scalar $\alpha > 0$ is selected sufficiency large.

In theorem 1, a scalar value α is supposed to be sufficiency large and is selected by designer. It follows from the theorem that calculation of a-anisotropic norm of descriptor system

(1)–(2) is formulated as convex optimization problem as follows

$$\text{find: } \min \gamma^2 \text{ on the set } L, Q, R, S, \eta.$$

In this case we can calculate a-anisotropic norm of descriptor system (1)–(2) with a given precision.

IV. Main Result

In this section, we obtain main results of the paper. Firstly, we derive conditions to check $\mathfrak{D}$-admissibility and a-anisotropic norm boundedness of the unforced descriptor system (1)–(2). Secondly, we obtain a control law which makes closed-loop system $\mathfrak{D}$-admissible with given bounded a-anisotropic norm.

In this section, we consider an admissible unforced system (1)–(2). The problem is to check its a-anisotropic norm boundedness and $\mathfrak{D}$-admissibility.

Theorem 2: Suppose that

$$\operatorname{rank} E_d = \operatorname{rank} [\, E_d \quad B_{wd} \,].$$

For given scalar values $\gamma > 0$, $0 < \omega < 1$, and $a \geqslant 0$ a-anisotropic norm of the system is bounded by the value γ, i.e. $\|P(z)\|_a < \gamma$, and the system P is $\mathfrak{D}$-admissible with radius ω, if there exist matrices $L \in \mathbb{R}^{r\times r}$, $L > 0$, $Q \in \mathbb{R}^{r\times r}$, $R \in \mathbb{R}^{r\times(n-r)}$, $S \in \mathbb{R}^{(n-r)\times(n-r)}$, $X = X^{\mathrm{T}} \in \mathbb{R}^{n\times n}$, and scalar value $\eta > \gamma^2$, satisfying inequalities (11), (12), and

$$\begin{bmatrix} -\omega^2 X & 0 \\ 0 & X \end{bmatrix} + \operatorname{sym}\left(\begin{bmatrix} A \\ -E \end{bmatrix} G\Delta\right) < 0 \tag{13}$$

with

$$G = \begin{bmatrix} Q & R \\ R^{\mathrm{T}} & S \end{bmatrix}, \tag{14}$$

and

$$\Delta = \begin{bmatrix} 0 & 0 & I_r & 0 \\ 0 & I_{n-r} & 0 & 0 \end{bmatrix}, \tag{15}$$

A scalar $\alpha > 0$ is supposed to be sufficiency large.

Proof: The proof of a-anisotropic norm boundedness can be found in [17]. Now we need to prove (13), that guarantees, that all finite eigenvalues lie inside $\mathfrak{D}$-region.

If the unforced system (1)–(2) is $\mathfrak{D}$-admissible, then inequality (9) holds true for some matrix $\overline{X}$.

Left and right multiplying (9) by $\overline{W}$ and $\overline{W}^{\mathrm{T}}$ respectively, we get

$$-\omega^2 \overline{W} E_d \overline{X} E_d^{\mathrm{T}} \overline{W}^{\mathrm{T}} + \overline{W} A_d \overline{X} A_d^{\mathrm{T}} \overline{W}^{\mathrm{T}} < 0. \tag{16}$$

Let $\overline{X} = \overline{V} X \overline{V}^{\mathrm{T}}$. It is possible because of $\overline{V}$ is nonsingular. Taking into account this notation, inequality (16) can be represented as

$$-\omega^2 E X E^{\mathrm{T}} + A X A^{\mathrm{T}} < 0. \tag{17}$$

Noting that pair (E, A) is admissible, hence, A_{22} is invertible. Introduce matrices:

$$\overline{\mathcal{W}} = \begin{bmatrix} I_r & -A_{12}A_{22}^{-1} \\ 0 & I_{n-r} \end{bmatrix}, \quad \overline{\mathcal{V}} = \begin{bmatrix} I_r & 0 \\ -A_{22}^{-1}A_{21} & A_{22}^{-1} \end{bmatrix},$$

Defining $\widehat{X} = \overline{\mathcal{V}} X \overline{\mathcal{V}}^{\mathrm{T}}$ and by left and right multiplying (17) on $\overline{\mathcal{W}}$ and $\overline{\mathcal{W}}$ respectively, we get

$$-\omega^2 \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \widehat{X} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} + \\ + \begin{bmatrix} \widehat{A} & 0 \\ 0 & I_{n-r} \end{bmatrix} \widehat{X} \begin{bmatrix} \widehat{A}^{\mathrm{T}} & 0 \\ 0 & I_{n-r} \end{bmatrix} < 0 \tag{18}$$

with $\widehat{A} = A_{11} - A_{12}A_{22}^{-1}A_{21}$.

Let $\widehat{X}$ be divided as $\widehat{X} = \begin{bmatrix} \widehat{X}_{11} & \widehat{X}_{12} \\ \widehat{X}_{12}^{\mathrm{T}} & \widehat{X}_{22} \end{bmatrix}$, $\widehat{X}_{11} \in \mathbb{R}^{r\times r}$.

It follows from (18) that $\widehat{X}_{11} > 0$. The expression (18) is is equivalent to

$$-\omega^2 \widehat{X}_{11} + \widehat{A}\widehat{X}_{11}\widehat{A}^{\mathrm{T}} < 0, \tag{19}$$

$$\widehat{X}_{22} < 0. \tag{20}$$

We are interested in inequality (19). This inequality is strict, hence, there exist a sufficiently small μ such that

$$-\omega^2 \widehat{X}_{11} + \widehat{A}\widehat{X}_{11}\widehat{A}^{\mathrm{T}} + \mu\omega^2 A_{12}A_{12}^{\mathrm{T}} < 0. \tag{21}$$

Introduce the next matrices

$$\mathcal{Y} = \begin{bmatrix} 0 & 0 & I_r & 0 \\ 0 & I_{n-r} & 0 & 0 \end{bmatrix}^{\mathrm{T}},$$

$$\mathcal{Z} = \begin{bmatrix} A_{11}^{\mathrm{T}} & A_{21}^{\mathrm{T}} & -I_r & 0 \\ A_{12}^{\mathrm{T}} & I_{n-r} & 0 & 0 \end{bmatrix}.$$

One can check that

$$\operatorname{Ker}\mathcal{Y} = \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n-r} \end{bmatrix}^{\mathrm{T}},$$

$$\operatorname{Ker}\mathcal{Z} = \begin{bmatrix} I_r & -A_{12} & \widehat{A} & 0 \\ 0 & 0 & 0 & I_{n-r} \end{bmatrix}^{\mathrm{T}}.$$

Under (21) the following inequalities hold true

$$\begin{cases} \operatorname{Ker}\mathcal{Y}\Upsilon\operatorname{Ker}\mathcal{Y}^{\mathrm{T}} < 0, \\ \operatorname{Ker}\mathcal{Z}^{\mathrm{T}}\Upsilon\operatorname{Ker}\mathcal{Z} < 0 \end{cases} \tag{22}$$

with

$$\Upsilon = \begin{bmatrix} -\omega^2\widehat{X}_{11} & -\mu\omega^2 A_{12} & 0 & 0 \\ -\mu\omega^2 A_{12}^{\mathrm{T}} & -\mu\omega^2 I_{n-r} & 0 & 0 \\ 0 & 0 & \widehat{X}_{11} & 0 \\ 0 & 0 & 0 & -\mu I_{n-r} \end{bmatrix}.$$

By Projection Lemma [18] there exist a matrix $\mathcal{G}$ such that

$$\Upsilon + \operatorname{sym}(\mathcal{Z}^{\mathrm{T}}\mathcal{G}\mathcal{Y}^{\mathrm{T}}) < 0 \tag{23}$$

or

$$\begin{bmatrix} -\omega^2\widehat{X}_{11} & 0 & 0 & 0 \\ 0 & -\mu\omega^2 I_{n-r} & 0 & 0 \\ 0 & 0 & \widehat{X}_{11} & 0 \\ 0 & 0 & 0 & -\mu I_{n-r} \end{bmatrix} + \\ + \operatorname{sym}\left(\mathcal{Z}^{\mathrm{T}}\left(\begin{bmatrix} 0 & 0 \\ 0 & -\mu\omega^2 I_{n-r} \end{bmatrix} + \mathcal{G}\right)\mathcal{Y}^{\mathrm{T}}\right) < 0. \tag{24}$$

Denote

$$G = \left[\begin{array}{cc} 0 & 0 \\ 0 & -\mu\omega^2 I_{n-r} \end{array} \right] + \mathcal{G}$$

and

$$X = \left[\begin{array}{cc} X_{11} & X_{12} \\ X_{12}^{\mathrm{T}} & X_{22} \end{array} \right] = \left[\begin{array}{cc} -\omega^2 \widehat{X}_{11} & 0 \\ 0 & -\mu\omega^2 I_{n-r} \end{array} \right].$$

Then, (24) can be rewritten as

$$\left[\begin{array}{cc} -\omega^2 X & 0 \\ 0 & X \end{array} \right] + \mathcal{Z}^{\mathrm{T}} G \mathcal{Y}^{\mathrm{T}} + \mathcal{Y} G \mathcal{Z} < 0. \tag{25}$$

By choosing G as (14) and substituting it into (25) we get (13). Note that $\mathfrak{D}$-admissibility is stronger than admissibility property for $\omega < 1$. Taking into account that (12) guarantees admissibility of the system, selection (14) does not contradict (12).

Finally, we need to prove that G is invertible. If G is not invertible, there exists a nonzero vector $c = \left[\begin{array}{cc} c_1 & c_2 \end{array} \right]$ such that $Gc = 0$. Let $c_1 \in \mathbb{R}^r$. Then left and right multiplication of (25) on $\left[\begin{array}{cccc} 0 & c_2 & c_1 & 0 \end{array} \right]$ and its transpose respectively yields $-c_2 X_{22} c_2^{\mathrm{T}} + c_1 X_{11} c_1^{\mathrm{T}} < 0$ which is impossible since $X_{11} > 0$ and $X_{22} < 0$.

■

Remark 2: It must be noted that strict LMI are used all along the paper. Indeed, many conditions in the literature involve non strict inequalities such as $E^{\mathrm{T}} X E \geqslant 0$, which are known to lead to numeric complications. Although these constraints can be transformed into strict inequalities by the use of additional developments [19], straightforwardly established strict inequalities which avoid those developments are here preferred. Therefore, LMI processes are more accurate.

While solving problem 1, we need to apply theorem 2 to the closed-loop system (5)–(6). Direct implementation of the conditions of theorem 2 to the system (1)–(2), closed by the control law in the form (4), leads to nonlinear terms for which implementation of inequality (12) as LMI is not possible.

To solve the control problem a better way is to deal with system dual to (5)–(6). A state-space representation of closed-loop dual system is

$$E^{\mathrm{T}} x'(k+1) = (A + B_u F)^{\mathrm{T}} x'(k) + C^{\mathrm{T}} w'(k), \tag{26}$$

$$z'(k) = B_w^{\mathrm{T}} x'(k) + D_w^{\mathrm{T}} w'(k), \tag{27}$$

It's obvious that $\mathcal{H}_2$ and $\mathcal{H}_\infty$norms of the closed-loop system coincide with the same ones of dual system (26)–(27). Being a semi-norm, a-anisotropic norm doesn't satisfy this property. However, in the case of $p \leqslant m_1$ the design specification is satisfied. To show this fact we recall that a-anisotropic norm of the admissible system is convex and monotonic function over a. In addition, when $a = 0$ we get

$$|\!|\!| P_{cl}(z) |\!|\!|_0 = \frac{\|P_{cl}\|_2}{\sqrt{m_1}} \leqslant \frac{\|P_{cl}\|_2}{\sqrt{p}} = |\!|\!| P_{cl}^{dual}(z) |\!|\!|_0. \tag{28}$$

It should be pointed out that $|\!|\!| P_{cl}(z) |\!|\!|_a = |\!|\!| P_{cl}^{dual}(z) |\!|\!|_a$ when $p = m_1$.

Introduce the following linear change of variables

$$\left[\begin{array}{cc} Q & R \\ 0 & S \end{array} \right] F^{\mathrm{T}} = Z. \tag{29}$$

The expression (29) implies that $\left[\begin{array}{cc} Q & R \end{array} \right] F^{\mathrm{T}} = \left[\begin{array}{cc} I_r & 0 \end{array} \right] Z$ and $\left[\begin{array}{cc} 0 & 0 \\ 0 & S \end{array} \right] F^{\mathrm{T}} = \left[\begin{array}{cc} 0 & 0 \\ 0 & I_{n-r} \end{array} \right] Z.$

Theorem 3: For a given scalar values $\gamma > 0$, $0 < \omega < 1$, and mean anisotropy level $a \geqslant 0$ the control design problem is solvable if there exist scalars $\eta > \gamma^2$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and matrices $X = X^{\mathrm{T}} \in \mathbb{R}^{n \times n}$, $Q \in \mathbb{R}^{r \times r}$, $R \in \mathbb{R}^{r \times (n-r)}$, $S \in \mathbb{R}^{(n-r)\times(n-r)}$, $L \in \mathbb{R}^{r \times r}$, $L > 0$, and $Z \in \mathbb{R}^{n \times m_2}$ such that

$$\eta - (\mathrm{e}^{-2a} \det(\eta I_p - C\Theta C^{\mathrm{T}} - D_w D_w^{\mathrm{T}})))^{1/p} < \gamma^2, \tag{30}$$

$$\left[\begin{array}{cc} -\omega^2 X & 0 \\ 0 & X \end{array} \right] + \\ + \operatorname{sym}\left(\left(\left[\begin{array}{c} A \\ -E \end{array} \right] G^{\mathrm{T}} + \left[\begin{array}{c} B_u \\ 0 \end{array} \right] Z^{\mathrm{T}} \right) \Delta \right) < 0, \tag{31}$$

$$\left[\begin{array}{ccccc} \Lambda_{11} & \Lambda_{21}^{\mathrm{T}} & \Lambda_{31}^{\mathrm{T}} & \Lambda_{41}^{\mathrm{T}} & 0 \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{32}^{\mathrm{T}} & \Lambda_{21} & \Lambda_{52}^{\mathrm{T}} \\ \Lambda_{31} & \Lambda_{32} & -\eta I_p & \Lambda_{31} & \Lambda_{53}^{\mathrm{T}} \\ \Lambda_{41} & \Lambda_{21}^{\mathrm{T}} & \Lambda_{31}^{\mathrm{T}} & -(Q + Q^{\mathrm{T}}) & 0 \\ 0 & \Lambda_{52} & \Lambda_{53} & 0 & -I_{m_1} \end{array} \right] < 0 \tag{32}$$

with
$\Lambda_{11} = -\frac{1}{2}Q - \frac{1}{2}Q^{\mathrm{T}}$, $\Lambda_{21} = A\Gamma^{\mathrm{T}} + B_u Z^{\mathrm{T}} \Omega^{\mathrm{T}}$,
$\Lambda_{31} = C\Gamma^{\mathrm{T}}$, $\Lambda_{41} = L - Q - \frac{1}{2}Q^{\mathrm{T}}$,
$\Lambda_{22} = \Pi A^{\mathrm{T}} + A\Pi^{\mathrm{T}} + \Phi Z B_u^{\mathrm{T}} + B_u Z^{\mathrm{T}} \Phi - \Theta$,
$\Lambda_{32} = C\Pi^{\mathrm{T}}$, $\Lambda_{52} = B_w^{\mathrm{T}}$, $\Lambda_{53} = D_w^{\mathrm{T}}$.

$$\Theta = \left[\begin{array}{cc} L & 0 \\ 0 & 0 \end{array} \right], \Pi = \left[\begin{array}{cc} 0 & 0 \\ 0 & S \end{array} \right], \Phi = \left[\begin{array}{cc} 0 & 0 \\ 0 & I_{n-r} \end{array} \right],$$

$$\Omega = \left[\begin{array}{cc} I_r & 0 \end{array} \right], \Gamma = \left[\begin{array}{cc} Q & R \end{array} \right].$$

The gain matrix can be obtained as

$$F_d = Z^{\mathrm{T}} \left[\begin{array}{cc} Q^{-\mathrm{T}} & 0 \\ -S^{-\mathrm{T}} R^{\mathrm{T}} Q^{-\mathrm{T}} & S^{-\mathrm{T}} \end{array} \right] \overline{V}^{-1}. \tag{33}$$

Proof: Taking into account linear change of variables (29) and substituting it into (13) we get Σ_{11} and Σ_{12} in (31) for the closed-loop system (26)–(27). By analogy, substitution (29) into (12) gives us Λ_{21} and Λ_{22} entries from (32), which coincide with the conditions of Theorem 2 for the system (26)–(27). So, according to Theorem 2, the closed-loop system (1)–(2) is $\mathfrak{D}$-admissible, and a-anisotropic norm of its transfer function is bounded by the given scalar γ.

In addition, as the inequality (32) holds, the Λ_{11} entry implies matrix Q is invertible. The invertibility of S is guaranteed by (31) (see proof of Theorem 2). So the feedback gain F_d for the closed-loop system is defined as $F = Z^{\mathrm{T}} \left[\begin{array}{cc} Q^{-\mathrm{T}} & 0 \\ -S^{-\mathrm{T}} R^{\mathrm{T}} Q^{-\mathrm{T}} & S^{-\mathrm{T}} \end{array} \right]$. Note that $F = F_d \overline{V}$. By the inverse change of variables we get F_d from (33). ■

V. Numerical Example

Consider the system with parameters:

$$E_d = \begin{bmatrix} 3 & 0 & 2 & -5 \\ 0 & 3 & -2 & 2 \\ 2 & 2 & 0 & -2 \\ 2 & -4 & 4 & -6 \end{bmatrix},$$

$$A_d = \begin{bmatrix} 4.7 & -3.25 & -0.7 & 0 \\ 0.8 & 0.4 & -6.4 & 2.6 \\ 1 & -1.9 & -5.4 & 2.4 \\ -0.6 & -2.7 & 5.4 & -2.8 \end{bmatrix},$$

$$B_{ud} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad B_{wd} = \begin{bmatrix} 3.2 & -3.5 \\ 2.5 & -7.9 \\ 3.8 & -7.6 \\ -1.2 & 8.2 \end{bmatrix},$$

$$C_d = \begin{bmatrix} 1 & 1 & 0 & -1 \end{bmatrix}, \quad D_{wd} = \begin{bmatrix} 1.2 & 1.3 \end{bmatrix},$$

The system, considered in example, is causal, but not stable. Its finite eigenvalues are $\lambda_i = \{1.2523; \quad 0.5994\}$, $i = \overline{1,2}$.

The goal is to design a state-feedback control minimizing a-anisotropic norm of the closed-loop system, such that finite eigenvalues of the closed-loop system lie inside a circle with radius $\omega = 0.5$. We choose mean anisotropy level $a = 0.2$.

The state-feedback gain is

$$F_d^{(1)} = \begin{bmatrix} -2.9193 & 3.4906 & -3.8471 & 2.3996 \end{bmatrix}.$$

One can check that the closed-loop system is admissible. Its finite eigenvalues are $\lambda_i^{(1)} = \{-0.4744; \quad 0.4997\}$. $\|P_{cl}^{(1)}(z)\|_a = 4.4558$.

Applying anisotropy-based control design procedure without pole placement constraint gives us the following result

$$F_d^{(2)} = \begin{bmatrix} -6.2855 & 5.7999 & 12.8111 & -6.0253 \end{bmatrix}.$$

Finite eigenvalues of the closed-loop system are $\lambda_i^{(2)} = \{0.1443; \quad 0.6134\}$. $\|P_{cl}^{(2)}(z)\|_a = 4.1198$.

A solution of pole placement problem without anisotropic quality criterion [2] is

$$F_d^{(3)} = \begin{bmatrix} -3.0863 & 4.6807 & -1.8345 & 1.2492 \end{bmatrix}.$$

Finite eigenvalues of the closed-loop system are $\lambda_i^{(3)} = \{0.0001; \quad 0.4532\}$. $\|P_{cl}^{(3)}(z)\|_a = 4.9568$.

An illustrative example demonstrates an effectiveness of the developed control design procedure. It is shown that using one of the criterion may not satisfy the designer's requirements. Taking into account both criteria we can achieve better performance of the closed-loop system while solving control problems.

VI. Conclusion

In this paper a novel design procedure for discrete-time descriptor systems is derived. The procedure consists of pole placement control with anisotropic gain constraint. It is shown that the procedure is numerically effective. The proposed algorithm allows to reach both desired transient response performance and disturbance attenuation level of the closed-loop system. In future, this algorithm can be extended on a class of uncertain descriptor systems with norm-bounded uncertainties.

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