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Generalized Output Synchronization of Heterogeneous Linear Multi-agent Systems

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***Abstract*—This paper investigates output synchronization of heterogeneous linear time-invariant systems. Agents distributively communicate measured outputs and synchronize on regulated outputs. Necessary structure of single-agents' drift dynamics is used. Relations between single-agent dynamics, measured outputs and regulated outputs are investigated. Cooperative stability conditions reduce to requirements depending separately on single-agents' structure and interconnecting graph topology, allowing for a distributed control design. Sufficient condition is given based on coordinate transformations which reveal the effects of distributed control on single-agents. It is shown that identical subsystem state synchronization and robustness to interconnections guarantee regulated output synchronization.**

***Keywords*—output synchronization; heterogeneous multi-agent systems; distributed control; output regulation**

I. Introduction

The last two decades have witnessed an increasing interest in multi-agent cooperative systems, [2][4][6][7][10][16]. Early work, [2][6][7][10], refers to consensus without a leader where the asymptotic consensus state depends on precise initial conditions. By adding a command generator leader that pins to a group of agents one can obtain synchronization to a reference trajectory for all initial conditions; this is termed pinning control [13][16][19][20]. For identical agents necessary and sufficient conditions for state synchronization are given by the master stability function [9][13][22] and the related concept of a synchronizing region [9][13][14][16][22], guaranteeing local stability. Global results are obtained by contraction analysis, *i.e.* incremental stability approaches [5][8][11][22], or Lyapunov methods, [18][21][34].

For heterogeneous agents passivity, [1][35][36], and internal model principle (IMP), [17][25], are used for output synchronization. The work in [17] introduces identical local reference generators that synchronize; thus producing a common output reference for each agent to track. This results in a hierarchical structure of augmented single-agents. The internal model principle is shown to be necessary and sufficient for output synchronization in linear heterogeneous multi-agent systems. The use of identical distributed reference generators that synchronize stems from [12], where the same approach is applied to identical linear time-invariant (LTI) agent state synchronization. Similar construction is found in [29]. Recent work [25] extends this to nonlinear systems. The passivity based approach in [36] touches on the necessary conditions involving the IMP from [17]; however, assuming passivity additionally restricts the single-agents. On the other hand, papers on distributed output regulation [27]-[30][33] propose local dynamic regulators, under fixed distributed feedback, thereby also imposing *a priori* structure on the resulting augmented single-agent drift dynamics and distributed control. The no-loop assumption on communication graphs is assumed in [27][31]. Looking at the seminal paper [2], one finds a similar structure; estimator or dynamic regulator augmenting the original single-agent systems. The subsequent development of cooperative control for identical agents, suggested by the synchronizing region, removes special such assumptions on single-agents. This line is pursued for heterogeneous agents in [31], albeit with the no-loop assumption.

This paper brings a sufficient condition for general LTI systems to synchronize over one set of outputs, *regulated outputs,* while communicating some other set of outputs, *measured outputs*. This allows achieving synchronization of a larger set of outputs while communicating fewer signals, thereby possibly reducing the communication burden. Set like this the problem includes state synchronization of identical agents by using either state [14][16][19] or output-feedback [3], and the output synchronization (regulation) of heterogeneous agents [1][18][27]-[31]. Our approach does not use observers nor dynamic compensators, in contrast to [17][33][35]. All single-agent control loops, bringing systems to the required form, by *e.g.* classical eigenvalue-eigenvector placement, are assumed closed, in line with [31]. Focus is instead on distributed control, required structure of single-agent systems and their interactions. The main contribution is a sufficient condition for regulated output synchronization. A novel stabilization scheme is introduced based on identical system state-synchronization, [15]. A coordinate transformation is used, related to the necessary structure of single-agents, revealing how a general measured output distributed feedback affects and interconnects different parts of single-agent systems. This novel cooperative stabilization approach treats interconnections of single-agents, induced by a general measured output distributed feedback, both as cooperatively stabilizing controls and as detrimental disturbances. This approach removes the no-loop requirement on graph topology, [27][31], and requires no *a priori* imposed, specially constructed, single-agent drift dynamics. It however leads to additional constraints on distributed control.

The outline of the paper is as follows; Section II gives graph preliminaries and notation. Section III presents the system and defines the control problem. Section IV restates necessary conditions, familiar from the literature for specially constructed systems, in a geometrical language applicable to our general settings. It sheds light on required geometrical structure of the total multi-agent system as well as of the single-agents. Section V brings sufficient conditions guaranteeing regulated output synchronization. Coordinate transformations are used, inspired by Section IV, together with known identical agent state synchronization results, to design controls for regulated output synchronization. Conclusions are given in Section VI.

II. Graph Theory and Notational Conventions

Consider a graph, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with a nonempty finite set of N nodes, $\mathcal{V} = \{v_1, \cdots, v_N\}$, and a set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Directed graphs are considered, and information propagates through the graph along the edges. Two nodes v_j, v_k connected by an edge $(v_j, v_k) \in \mathcal{E}$ are termed *parent* node v_k and *child* node v_j, *i.e.* the edge leaves the parent node and connects into the child node. Denote the adjacency matrix as $E = \left[e_{ij}\right]$ with $e_{ij} > 0$ if $(v_i, v_j) \in \mathcal{E}$ and $e_{ij} = 0$ otherwise. Note that diagonal elements satisfy $e_{ii} = 0$. The set of neighbors of node v_i is $\mathcal{N}_i = \{v_j : (v_i, v_j) \in \mathcal{E}\}$, *i.e.* set of nodes with arcs connecting into v_i. Define the in-degree matrix as a diagonal matrix, $H = diag(h_1 \dots h_N)$, with $h_i = \sum_j e_{ij}$, the (weighted) in-degree of node i. Define the graph Laplacian matrix as $L = H - E$, which has all row sums equal to zero. A *directed path* is a sequence of edges joining two nodes. A graph is said to be *strongly connected* if any two nodes can be joined by a directed path. Node is termed *isolated* if it has no incoming edges. Hence in strongly connected graphs there are no isolated nodes. A *directed tree* is a subgraph having a single isolated node v_0, such that all other nodes except v_0 have only one parent and are joined to v_0 by a directed path. Node v_0 is called a *root node*. A graph is said to contain a directed *spanning tree* if there exists a directed tree containing every node in $\mathcal{V}$. The Laplacian matrix L has a simple zero eigenvalue iff its directed graph contains a spanning tree. A graph is said to be *reducible* if its Laplacian matrix is cogredient, *i.e.* can be transformed, to the block triangular form

$$T^T L T = \begin{bmatrix} L_{11} & L_{12} \\ 0 & L_{22} \end{bmatrix}, \tag{1}$$

where T is a permutation matrix. If the graph is not reducible it is said to be *irreducible*. A directed graph is irreducible if and only if it is strongly connected. The symbol $\underline{1}_N$ stands for the vector $\left[1 \dots 1\right]^T \in \mathbb{R}^N$.

III. System Description and the Control Problem

Let the multi-agent system be comprised of N agents

$$\dot{x}_i = A_i x_i + B_i u_i,\ y_i = C_i x_i,\ z_i = D_i x_i. \tag{2}$$

Agents are assumed heterogeneous with $x_i \in \mathbb{R}^{n_i}$, $y_i \in \mathbb{R}^p$, $z_i \in \mathbb{R}^q$, $u_i \in \mathbb{R}^{m_i}$. Matrices C_i, D_i are assumed to have full row rank, implying no redundant outputs. It is also taken that the y_i-output vector, as a linear function on single-agent state-space $\mathbb{R}^{n_i}$, is linearly independent of the measured z_i-outputs. When present, the leader is given as

$$\dot{x}_0 = A_0 x_0,\ y_0 = C_0 x_0,\ z_0 = D_0 x_0. \tag{3}$$

with $y_0 \in \mathbb{R}^p$, $z_0 \in \mathbb{R}^q$. Control goal is to reach regulated y-output consensus asymptotically, $\|y_i - y_j\| \to 0$ as $t \to \infty$, by communicating measured z-outputs. With an isolated leader the control goal is asymptotic output reference tracking, $\|y_i - y_0\| \to 0$ as $t \to \infty$.

The local neighborhood errors in measured z-outputs are

$$\varepsilon_{zi} = \sum\nolimits_{j=1}^N e_{ij}(z_j - z_i) + g_i(z_0 - z_i), \tag{4}$$

where $g_i \geq 0$ are the *pinning gains* with $g_i > 0$ for a small percentage of nodes having direct access to the leader, (3). If there is no isolated leader (4) takes the form

$$\varepsilon_{zi} = \sum\nolimits_{j=1}^N e_{ij}(z_j - z_i). \tag{5}$$

In the form of total state-space vectors

$$x = \left[x_1^T .. x_N^T\right]^T \in \mathbb{R}^{\sum_{i=1}^N n_i}, y = \left[y_1^T .. y_N^T\right]^T \in \mathbb{R}^{Np}, z = \left[z_1^T .. z_N^T\right]^T \in \mathbb{R}^{Nq},$$

the total local neighborhood error vector (4) reads

$$\varepsilon_z = -(L + G) \otimes I_q (z - \overline{z}_0), \tag{6}$$

where $G = diag(g_1 \dots g_N) \geq 0$ is a diagonal matrix of pinning gains and $\overline{z}_0 = \left[z_0^T \dots z_0^T\right]^T \in \mathbb{R}^{Nq}$, while (5) reads

$$\varepsilon_z = -L \otimes I_q z. \tag{7}$$

In this paper the leader (3), when present, is by convention included with the other agents (2) in an augmented graph as an isolated root of a spanning tree. Thus, the local neighborhood error signal in this case takes the form of (5) or (7) as well, with an augmented graph adjacency matrix and Laplacian. The distributed feedback is then formed from (4) or (5) as

$$u_i = K_i \varepsilon_{zi}, \tag{8}$$

where the feedback gains K_i are designed later. Note however that, different from [17], z-consensus at this point is not required. The closed-loop system with controls (8) is then

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i K_i \sum\nolimits_{j=1}^N e_{ij}(D_j x_j - D_i x_i), \\ y_i &= C_i x_i. \end{aligned} \tag{9}$$

Special instances of system (9) are studied as state consensus of identical agents, $A_i = A,\ B_i = B,\ K_i = K$, communicating their states, $D_i = I_n$ [12][14][19], or outputs, $D_i = C$ [3], and the output consensus of heterogeneous agents with outputs communicated, $D_i = C_i$ [17][27][28][30][31]. This paper aims at revealing relations between the y-, z-outputs (C_i, D_i) and system's structural properties (A_i, B_i) that are necessary and sufficient for y-output synchronization.

IV. NECESSARY CONDITIONS FOR OUTPUT SYNCHRONIZATION

This section restates the necessary conditions for special systems, appearing in [17], in general geometrical terms applicable to our settings. Assumptions on single-agent systems and their interconnections, under which the presented conditions are indeed necessary, are highlighted and motivated. For y-consensus in the total state-space,

$$\mathcal{X} := \times_{k=1}^{N} \mathbb{R}^{n_k} = \mathbb{R}^{\sum_{k=1}^{N} n_k}, \quad (10)$$

there must exist an invariant subspace of the closed-loop system (9) where $y_i = y_j, \forall i, j$; the *target invariant subspace*. Due to linearity $x = 0 \in \mathcal{X}$ is a trivial invariant subspace of (9). Assume a nontrivial invariant subspace of (9) spanned by columns of matrix Π, $\operatorname{spancol}\Pi$. Then Π satisfies the invariance condition,

$$A^{*}\Pi = \Pi X, \quad (11)$$

and the y-consensus condition

$$C^{*}\Pi = \overline{R}_1, \quad (12)$$

where $\overline{R}_1 = \underline{1}_N \otimes R_1$, for some matrix R_1, [17]. The shorthand starred symbols in (11), (12) stand for

$$A^{*} = diag(A_1 ... A_N) - diag(B_1 K_1 ... B_N K_N)(L \otimes I_q) D^{*}, \quad (13)$$
$$C^{*} = diag(C_1 ... C_N), D^{*} = diag(D_1 ... D_N).$$

The system matrix X in (11), describing dynamics on the target invariant subspace, $\operatorname{spancol}\Pi$, does not have any stable poles for those imply trivial consensus, [17]. Take the dimension of the target invariant subspace (11) to be $0 < l \le \min_i n_i$, whence X is an $l \times l$ matrix. Expanding (11) with respect to the block structure of A^{*}, (13), one has

$$\begin{bmatrix} A_1 - l_{11}B_1K_1D_1 & -l_{12}B_1K_1D_2 & \cdots & -l_{1N}B_1K_1D_N \\ -l_{21}B_2K_2D_1 & A_2 - l_{22}B_2K_2D_2 & \cdots & -l_{2N}B_2K_2D_N \\ \vdots & & \ddots & \\ -l_{N1}B_NK_ND_1 & \cdots & \cdots & A_N - l_{NN}B_NK_ND_N \end{bmatrix} \begin{bmatrix} \Pi_1 \\ \Pi_2 \\ \vdots \\ \Pi_N \end{bmatrix} = \begin{bmatrix} \Pi_1 \\ \Pi_2 \\ \vdots \\ \Pi_N \end{bmatrix} X, \quad (14)$$

where l_{ij} are components of the graph or augmented graph Laplacian matrix, yielding coupled Francis equations, [26],

$$(A_i - l_{ii}B_iK_iD_i)\Pi_i + \sum_j e_{ij}B_iK_iD_j\Pi_j = \Pi_i X, \quad (15)$$
$$C_i\Pi_i = R_1. \quad (16)$$

Assumption 1. Distributed control (8) vanishes on the target invariant subspace in $\mathcal{X}$, $\operatorname{spancol}\Pi$,

$$K_i \sum_{j=1}^{N} e_{ij}(D_j\Pi_j - D_i\Pi_i) = 0, \ \forall i. \quad (17)$$

If the distributed control did not vanish on the target invariant subspace it would necessarily take part in determining that set. Then the target invariant subspace would depend on the exact communication topology, hence would not be robust to changes of topology. Assumption 1 is robustly satisfied, for any choice of K_is, *via* z-consensus, but that is not necessary for the following developments.

Definition 1. Given a Cartesian product of N identical sets $\mathcal{S}$, $\mathcal{S}^N$, the *diagonal* has the form $(x, x, ..., x)$ for all $x \in \mathcal{S}$. Diagonal thus defined is homeomorphic to a single set $\mathcal{S}$.

The following result reduces the invariant subspace structure of total system (9) to that of single-agent systems.

Lemma 1. Under Assumption 1 there exist invariant subspaces for single-agent drift dynamics A_i,

$$A_i\Pi_i = \Pi_i X, \quad (18)$$
$$C_i\Pi_i = R_1, \quad (19)$$

and the target invariant subspace, $\operatorname{spancol}\Pi$, is a diagonal in the Cartesian product of all these single-agent drift dynamics invariant subspaces.

Proof: Under Assumption 1 (14) reduces to $diag(A_1 ... A_N)\Pi = \Pi X$. Each agent hence has an invariant subspace in its state-space $\mathbb{R}^{n_i}$, spanned by columns of Π_i; $x_i = \Pi_i\xi_i$, $\xi_i \in \mathbb{R}^l$, where Π_i satisfies the invariance and y-consensus conditions (18), (19). The target invariant subspace given as $x_i = \Pi_i\xi, \forall i$, corresponds then to $\xi_i = \xi, \forall i$. ■

Specific structure imposed on A_i matrices, such as explicitly incorporating an m-copy model of X by a hierarchical construction, leads to specially structured solutions Π_i, [27][29][30]. Single-agent passivity, as used in [36], leads to a similar structure, albeit necessarily with a stable X. Here we do not make any such special assumptions.

Remark 1. If Assumption 1 is satisfied by z-consensus; $z_i = z_j, \forall (i, j)$ if $x_i = \Pi_i\xi$, $\forall i$,

$$\sum_{j=1}^{N} e_{ij}(D_j\Pi_j - D_i\Pi_i) = 0, \ \forall i, \quad (20)$$

then one has similarly to (12) also $D^{*}\Pi = \overline{R}_2$, where $\overline{R}_2 = \underline{1}_N \otimes R_2$. The spancol Π_i in that case also satisfies

$$D_i\Pi_i = R_2. \quad (21)$$

Remark 2. Under (18), (19) and (21) the invariant subspaces (18) of single-agents are all parameterized by the same number of coordinates, and on all those subspaces the drift dynamics is described by X. Each single-agent state-space thus embeds a copy of the same, shared, invariant subspace. This is called system intersection in [31], but its structure is not revealed there. By Lemma 1 the target invariant subspace is also an embedding of the shared invariant subspace in $\mathcal{X}$.

Proposition 1. For each single-agent there are l linearly independent scalar outputs that reach consensus on the target invariant subspace.

Proof: The relation $C_i\Pi_i = R_1$ is a nonhomogeneous linear equation for C_i. Π_i has full column rank, *i.e.* l linearly independent columns, so a particular solution exists,

$$C_i = R_1(\Pi_i^T\Pi_i)^{-1}\Pi_i^T. \tag{22}$$

Since R_1 has l columns one can write up to l linearly independent rows, and each such row in (22) gives one scalar output, *i.e.* one row of C_i. With l linearly independent rows R_1 is nonsingular, which by regularity of $\Pi_i^T\Pi_i$ implies the rows of C_i thus constructed are linearly independent. ■

The choice of outputs in Proposition 1 is of course not unique. Moreover, not all l linearly independent scalar outputs need to be communicated for cooperative control; the measured outputs should suffice.

V. Sufficient Condition for Output Synchronization

This section brings the main results of the paper. Assuming a solution of (15) and (16) exists and satisfies Assumption 1, a sufficient condition for asymptotic stability of the target invariant subspace is given. A coordinate transformation is used that clearly reveals how the distributed z-output signal affects and interconnects the single-agent systems.

Each single-agent drift dynamics A_i, under Assumption 1, (17), necessarily has an invariant subspace, $\operatorname{spancol}\Pi_i$, where Π_i satisfies (18), (19) and (21). Supplement the columns of Π_i by those of Ψ_i to form a basis of the single-agent state-space $\mathbb{R}^{n_i}$. Then in such basis, one has the transformed state, $x_i \mapsto (\xi_i^T, \vartheta_i^T)^T$,

$$x_i = \begin{bmatrix}\Pi_i & \Psi_i\end{bmatrix}\begin{bmatrix}\xi_i \\ \vartheta_i\end{bmatrix}, \tag{23}$$

with (18) generally leading to

$$A_i\begin{bmatrix}\Pi_i & \Psi_i\end{bmatrix} = \begin{bmatrix}\Pi_i & \Psi_i\end{bmatrix}\begin{bmatrix}X & F_i \\ 0 & \tilde{A}_i\end{bmatrix} \tag{24}$$

Therefore, in ξ_i, ϑ_i coordinates the drift dynamics is

$$\frac{d}{dt}\begin{bmatrix}\xi \\ \vartheta\end{bmatrix}_i = \begin{bmatrix}X & F_i \\ 0 & \tilde{A}_i\end{bmatrix}\begin{bmatrix}\xi \\ \vartheta\end{bmatrix}_i. \tag{25}$$

The roles of transformed coordinates ξ, ϑ are evident from (25); the ξ_i s parameterize the subspace $\operatorname{spancol}\Pi_i$, and ϑ_i s are the remaining coordinates on $\mathbb{R}^{n_i}$. Namely $\vartheta_i = 0$ is the single-agent shared invariant subspace, $\operatorname{spancol}\Pi_i$; $\vartheta_i = 0 \Rightarrow \dot{\vartheta}_i = 0, \dot{\xi}_i = X(\xi_i)$. This corresponds to the system intersection in [31].

The closed-loop dynamics of (9) in the coordinates (ξ_i, ϑ_i) then equals

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix}\xi \\ \vartheta\end{bmatrix}_i &= \begin{bmatrix}X & F_i \\ 0 & \tilde{A}_i\end{bmatrix}\begin{bmatrix}\xi \\ \vartheta\end{bmatrix}_i \\ &+ \begin{bmatrix}\tilde{B}_1 \\ \tilde{B}_2\end{bmatrix}_i K_i \sum\nolimits_{j=1}^{N} e_{ij}\left(\begin{bmatrix}\tilde{D}_1 & \tilde{D}_2\end{bmatrix}_j \begin{bmatrix}\xi \\ \vartheta\end{bmatrix}_j - \begin{bmatrix}\tilde{D}_1 & \tilde{D}_2\end{bmatrix}_i \begin{bmatrix}\xi \\ \vartheta\end{bmatrix}_i\right),\end{aligned} \tag{26}$$

where

$$\begin{bmatrix}\tilde{B}_1 \\ \tilde{B}_2\end{bmatrix}_i := \begin{bmatrix}\Pi_i & \Psi_i\end{bmatrix}^{-1} B_i,\ \begin{bmatrix}\tilde{D}_1 & \tilde{D}_2\end{bmatrix}_i := D_i\begin{bmatrix}\Pi_i & \Psi_i\end{bmatrix}, \tag{27}$$

are given by the transformation (23). Assuming (21), $\tilde{D}_{1i} = D_i\Pi_i = R_2, \forall i$, one writes

$$\begin{aligned}\dot{\xi}_i &= X\xi_i + \tilde{B}_{1i}K_iR_2\sum\nolimits_{j=1}^{N} e_{ij}(\xi_j - \xi_i) + F_i\vartheta_i \\ &+ \tilde{B}_{1i}K_i\sum\nolimits_{j=1}^{N} e_{ij}(\tilde{D}_{2j}\vartheta_j - \tilde{D}_{2i}\vartheta_i),\end{aligned} \tag{28}$$

$$\begin{aligned}\dot{\vartheta}_i &= \tilde{A}_i\vartheta_i + \tilde{B}_{2i}K_i\sum\nolimits_{j=1}^{N} e_{ij}(\tilde{D}_{2j}\vartheta_j - \tilde{D}_{2i}\vartheta_i) \\ &+ \tilde{B}_{2i}K_iR_2\sum\nolimits_{j=1}^{N} e_{ij}(\xi_j - \xi_i).\end{aligned} \tag{29}$$

Lemma 2. Let in (28) and (29) $\|\xi_i - \xi_j\| \to 0, \forall i, j$ as $t \to \infty$, and $\vartheta_i \to 0, \forall i$ as $t \to \infty$. Then *y*-consensus is asymptotically achieved.

Proof: It follows from the set-up of the system that $\xi_i = \xi_j, \forall i, j$, $\vartheta_i = 0, \forall i$ imply $x_i = \Pi_i\xi_i$, hence $y_i = C_i\Pi_i\xi_i = R_1\xi_i = R_1\xi_j = C_j\Pi_j\xi_j = y_j$. Transform (23) used to obtain (28) preserves asymptotic partial stability, [23], hence convergence of transformed states to the transformed target set is therefore equivalent to convergence of original state to the original target set. ■

Remark 3. (23) reveals how a general measured output distributed feedback affects and interconnects different subsystems of each agent. Condition (21) brings the distributed ξ_i dynamics in (28) close to the familiar form of identical agent cooperative state synchronization, [3][15]. If (21) were not assumed the ξ_i dynamics in (28) would resemble that of heterogeneous agents, *i.e.* coordinate transformation would reduce the original problem to a version thereof. The form of dynamics in (28) and (29), motivates the following.

Theorem 1. Let the graph contain a spanning tree. Let the columns of Ψ_i span the stable invariant subspace of A_i. Let K_i s cooperatively stabilize the system

$$\dot{\xi}_i = X\xi_i + \tilde{B}_{1i}K_iR_2\sum\nolimits_{j=1}^{N} e_{ij}(\xi_j - \xi_i). \tag{30}$$

Let all $\tilde{A}_i$ be asymptotically stable. Then for $\tilde{D}_{2i} = D_i\Psi_i$ sufficiently small and convergence rate for $\tilde{A}_i$ dynamics sufficiently fast, *y*-consensus is asymptotically achieved.

Proof: The transformed system (28) and (29) appears as an interconnected system with nominal systems taken as

$$\dot{\xi}_i = X\xi_i + \tilde{B}_{1i}K_iR_2\sum_{j=1}^{N} e_{ij}(\xi_j - \xi_i), \tag{31}$$

$$\dot{\vartheta}_i = \tilde{A}_i\vartheta_i + \tilde{B}_{2i}K_i\sum_{j=1}^{N} e_{ij}(\tilde{D}_{2j}\vartheta_j - \tilde{D}_{2i}\vartheta_i). \tag{32}$$

Under cooperative stability of (30), *via* a spanning tree, there exists a Lyapunov function

$$V_\xi(\delta_\xi) = \frac{1}{2}\delta_\xi^T P_\xi \delta_\xi > 0$$

whose time-derivative satisfies

$$\dot{V}_\xi(\delta_\xi) \le -\delta_\xi^T Q_\xi \delta_\xi < 0,$$

where $\delta_\xi = \xi - \overline{\xi}^* \in \mathbb{R}^{Nl}$ is the synchronization error for an irreducible graph with

$$\xi^* = (\sum_{j=1}^{N} p_j)^{-1}\sum_{j=1}^{N} p_j\xi_j,$$

where $p_i > 0$ are elements of the left zero eigenvector of the graph Laplacian [37], or $\delta_\xi = \xi - \overline{\xi}_0$ with ξ_0 a state of an isolated leader, contained in the graph. Under asymptotic stability of all $\tilde{A}_i$ s there exist Lyapunov functions

$$V_i(\vartheta_i) = \frac{1}{2}\vartheta_i^T P_i\vartheta_i > 0$$

satisfying

$$\dot{V}_i(\vartheta_i) = \frac{1}{2}\vartheta_i^T(\tilde{A}_i^T P_i + P_i\tilde{A}_i^T)\vartheta_i = -\vartheta_i^T Q_i\vartheta_i < 0.$$

Then for the interconnected nominal system (32) one constructs a composite Lyapunov function

$$V_\vartheta(\vartheta) = \sum_{i=1}^{N} d_iV_i(\vartheta_i),\ d_1,...,d_N > 0,$$

whose time-derivative is

$$\begin{aligned}
&\dot{V}_\vartheta(\vartheta)\\
&= \sum_{i=1}^{N}\frac{1}{2}d_i\vartheta_i^T(A_i^TP_i + P_iA_i)\vartheta_i + d_i\vartheta_i^TP_i\tilde{B}_{2i}K_i\sum_{j=1}^{N} e_{ij}(\tilde{D}_{2j}\vartheta_j - \tilde{D}_{2i}\vartheta_i)\\
&= \sum_{i=1}^{N} d_i\vartheta_i^T(-Q_i - h_iP_i\tilde{B}_{2i}K_i\tilde{D}_{2i})\vartheta_i + d_i\vartheta_i^TP_i\tilde{B}_{2i}K_i\sum_{j=1}^{N} e_{ij}\tilde{D}_{2j}\vartheta_j.
\end{aligned} \tag{33}$$

Expression (33) leads to an inequality

$$\begin{aligned}
&\dot{V}_\vartheta(\vartheta) \le \sum_i^N -d_i(\underline{\sigma}(Q_i) - h_i\overline{\sigma}(P_i\tilde{B}_{2i}K_i\tilde{D}_{2i}))\|\vartheta_i\|^2\\
&+\sum_{i,j=1}^{N} d_ie_{ij}\overline{\sigma}(P_i\tilde{B}_{2i}K_i\tilde{D}_{2j})\|\vartheta_i\|\|\vartheta_j\| = -\frac{1}{2}[\|\vartheta_1\|...\|\vartheta_N\|](DS+S^TD)[\|\vartheta_1\|...\|\vartheta_N\|]^T
\end{aligned}$$

where $S_{ii} = \underline{\sigma}(Q_i) - h_i\overline{\sigma}(P_i\tilde{B}_{2i}K_i\tilde{D}_{2i}),\ S_{ij} = -e_{ij}\overline{\sigma}(P_i\tilde{B}_{2i}K_i\tilde{D}_{2j})$. S is a nonsingular *M*-matrix for sufficiently small $\tilde{D}_{2i}$ s. Then a diagonal matrix $D := diag(d_1...d_N) > 0$ can be found such that $DS + S^TD > 0$, [24]. Hence for sufficiently small $\tilde{D}_{2i}$ s the origin of (32) is asymptotically stable. The total interconnected system (28) and (29) can now be written as

$$\frac{d}{dt}\begin{bmatrix}\delta_\xi\\ \vartheta\end{bmatrix} = \begin{bmatrix}A_\xi & B_{\xi\vartheta}\\ B_{\vartheta\xi} & A_\vartheta\end{bmatrix}\begin{bmatrix}\delta_\xi\\ \vartheta\end{bmatrix}, \tag{34}$$

where the matrices $A_\xi = I_N \otimes X - diag(B_{1i}K_i)(L\otimes R_2)$, $A_\vartheta = diag(\tilde{A}_i) - diag(\tilde{B}_{2i}K_i)(L\otimes I_q)diag(\tilde{D}_{2i})$ are cooperatively and asymptotically stable by assumptions of the Theorem and (33) respectively. In (34) the off-diagonal blocks are

$$B_{\xi\vartheta} = diag(F_i) - diag(\tilde{B}_{1i}K_i)(L\otimes I_q)diag(\tilde{D}_{2i}),$$

$$B_{\vartheta\xi} = -diag(\tilde{B}_{2i}K_i)(L\otimes R_2).$$

Construct a new composite Lyapunov function for (34) as

$$V(\delta_\xi, \vartheta) = c_1V_\xi(\delta_\xi) + c_2V_\vartheta(\vartheta).$$

with some $c_1 > 0, c_2 > 0$. Its time-derivative with (28), (29) is

$$\begin{aligned}
&\dot{V}(\delta_\xi,\vartheta) = c_1\dot{V}_\xi(\delta_\xi) + c_2\dot{V}_\vartheta(\vartheta)\\
&\le -c_1\delta_\xi^TQ_\xi\delta_\xi + c_1\delta_\xi^TP_\xi(diag(F_i) - diag(\tilde{B}_{1i}K_i)(L\otimes I_q)diag(\tilde{D}_{2i}))\vartheta\\
&\quad -c_2\frac{1}{2}[\|\vartheta_1\|...\|\vartheta_N\|](DS+S^TD)[\|\vartheta_1\|...\|\vartheta_N\|]^T\\
&\quad +c_2\sum_{i=1}^{N} d_i\vartheta_i^TP_i\tilde{B}_{2i}K_iR_2\sum_{j=1}^{N} e_{ij}(\xi_j - \xi_i)
\end{aligned}$$

The Ψ_i spanning the stable invariant subspace gives $F_i = 0, \forall i$. Such choice is always possible as X has no strictly stable poles. Then one has

$$\begin{aligned}
&\dot{V}(\delta_\xi,\vartheta) \le -c_1\underline{\sigma}(Q_\xi)\|\delta_\xi\|^2 - c_1\delta_\xi^TP_\xi diag(\tilde{B}_{1i}K_i)(L\otimes I_q)diag(\tilde{D}_{2i})\vartheta\\
&-c_2\frac{1}{2}[\|\vartheta_1\|...\|\vartheta_N\|](DS+S^TD)[\|\vartheta_1\|...\|\vartheta_N\|]^T - c_2\vartheta^Tdiag(d_iP_i\tilde{B}_{2i}K_i)\underbrace{(L\otimes R_2)\xi}_{=(L\otimes R_2)\delta_\xi}\\
&\le -c_1\underline{\sigma}(Q_\xi)\|\delta_\xi\|^2 + c_1\overline{\sigma}(P_\xi diag(\tilde{B}_{1i}K_i)(L\otimes I_q)diag(\tilde{D}_{2i}))\|\delta_\xi\|\|\vartheta\|\\
&-\frac{c_2}{2}\underline{\sigma}(DS+S^TD)\|\vartheta\|^2 + c_2\overline{\sigma}(diag(d_iP_i\tilde{B}_{2i}K_i)(L\otimes R_2))\|\vartheta\|\|\delta_\xi\|\\
&= -\frac{1}{2}[\|\delta_\xi\|\|\vartheta\|][diag(c_1,c_2)S' + S'^T diag(c_1,c_2)][\|\delta_\xi\|\|\vartheta\|]^T
\end{aligned}$$

where

$$S' = \begin{bmatrix}\underline{\sigma}(Q_\xi) & -\overline{\sigma}(P_\xi diag(\tilde{B}_{1i}K_i)(L\otimes I_q)diag(\tilde{D}_{2i}))\\ -\overline{\sigma}(diag(d_iP_i\tilde{B}_{2i}K_i)L\otimes R_2) & ½\underline{\sigma}(DS+S^TD)\end{bmatrix}$$

is again a nonsingular *M*-matrix for sufficiently small $\tilde{D}_{2i}, \forall i$, and sufficiently large diagonal elements, related to convergence rates of nominal systems (31) and (32). This implies existence of $c_1, c_2 > 0$ guaranteeing $\delta_\xi \to 0, \vartheta \to 0$, [24], hence *y*-consensus, by Lemma 3. ■

Theorem 1 does not require *a priori* special single-agents, *e.g.* observers or dynamic compensators, [17][33]; rather it relies on general measured output static distributed feedback and applies to general single-agents. Moreover, it expresses the main design requirements on the K_i s. Conditions of Theorem 1 on single-agents' structure can be guaranteed by single-agent feedback *via* eigenvalue-eigenvector placement.

Remark 4. The goal in (31) and (32) is cooperative consensus in ξ_i, while keeping ϑ_i asymptotically stable. In order not to reduce (32) to the original problem, the stability of (32) is guaranteed on single-agent basis, [24]. Hence the distributed control signal takes the cooperatively stabilizing role in (31) for ξ-synchronization, while it is taken as detrimental in (32), [24]. The synchronization of identical drift dynamics agents (31) is familiar from [15][19]. Different $\tilde{B}_{1i}$s are treated by multi-player games [15], while $\tilde{B}_{1i} = B_1, \forall i$ is treated by conventional results [3][16][19]. Hence, designs for identical agent state synchronization, satisfying requirements of Theorem 1, [15][34][37], yield heterogeneous agent output synchronization controls. Observability of (X, R_2) is then necessary [3]. This is guaranteed by detectability of (A_i, D_i). No additional assumptions on graphs, *e.g.* irreducibility or no-loop condition [27][31] are needed as those are generally not required for identical agent state synchronization. Detectability of (A_i, D_i) reveals relations between single-agent invariant subspaces and partitions of single-agent state-spaces into slices of indistinguishable points with respect to D_i, [32].

Remark 5. For detectable (A_i, D_i) $W_i = \bigcap_{k=0}^{n_i} \ker D_i A_i^k$, must not contain any nontrivial subspace of $\operatorname{spancol} \Pi_i$, thus $\operatorname{spancol} \Pi_i \cap W_i = \{0\}$. One generally has $W_i \subseteq \operatorname{spancol} \Psi_i$. A special case thereof is afforded by the coincidence $\operatorname{spancol} \Psi_i \equiv W_i$; $\tilde{D}_{2j} = D_j \Psi_j = 0, \forall j$. The coupling between (28) and (29) is then unidirectional. A ξ-cooperative regulator or tracker, (28), synchronizes separately from the ϑ-dynamics and the ϑ-system ultimately converges to zero, similarly to [17][25][29]. Theorem 1, in contrast, allows for $D_i \Psi_i \neq 0$ small enough, [24]. This goes beyond [17][25], which construct hierarchical augmented agents.

Remark 6. K_is appear in nominal systems both as controls in (31) and as ϑ-interconnections in (32), but also as interconnections $B_{\xi\vartheta}, B_{\vartheta\xi}$ in (34). Synchronization of (31) requires the controls to exceed a threshold, implying a lower bound on K_is' magnitude, [18]. But stability of interconnected systems implies an upper bound on the magnitude of interactions. The distributed controls need to be strong enough to synchronize (31), but also weak enough not to destroy the nominal stability of invariant subspaces for uncontrolled single-agents $\tilde{A}_i$, (32). This is a consequence of treating ϑ-interconnections as detrimental and is reminiscent of small gain theorems. The result of this paper ultimately relies on single-agents' property that their shared dynamics can be synchronized using relatively weak controls.

VI. CONCLUSIONS

This paper considers consensus of regulated outputs by communicating measured outputs. Emphasis is placed on the structure of single-agents. Coordinate transformations clearly reveal how the distributed feedback affects heterogeneous agents. The resulting design implies a lower and an upper bound on feedback gains, but requires no restrictions on the graph topology.

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