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Stability Analysis of Wave Based Control: Practical Aspects

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***Abstract*—The wave-based control system has a potential to become effective method of vibration-damping controller design. The only design requirement of this method is to absorb the returning wave by the actuator. Stability or performance of the overall system is not included in the design specifications and the approach does not provide it in general. The advantage of this approach is that it does not need sensors along the entire length of system and it can simultaneously control position and damps vibration. On the other hand, this method is relatively young and there are many areas of research that have to be explored. This paper brings the stability analysis of wave based control for homogeneous chains. The paper also presents some remarks that extend obtained result for distributed systems.**

***Index Terms*—wave-based control, active damping of vibrations, cantilevered beam, control of distributed systems, identification of parameters of distributed systems**

I. Introduction

Although the vibration damping problem occurs in many applications, especially the development of robotics needs to master the control of flexible structures where vibrations are undesirable. There are many active and passive methods of suppressing the vibration [1],[2],[3],[4], but simple and effective approach how to design control system to suppress vibrations is still an open task. Approach designed by W.J. O'Connor and D. Lang in [5] has potential to become such an approach.

O'Connor and Lang designed new technique to control the lumped flexible structures. Their approach includes actuator that is able to send mechanical waves into system and simultaneously absorb waves coming back from the system. The approach is called Wave-Based Control (WBC). In order to decompose mechanical vibrations of the system to outgoing and returning waves, O'Connor and Lang presented the wave transfer function (WTF). Loop of wave transfer functions became a new way of modelling the flexible structures. [6].

This paper brings a brief description of the standard WBC approach in the section III. Section IV demonstrates convenient way to decompose the real structures. Smaller decomposed parts are used for modelling of the complex flexible structure. The way how to estimate required parameters of the controlled system is described in the section V. Alternative form of the wave based control is derived in section VI and it is used for analysis of the stability of the control loop. Several other approximations of real WTF are introduced and the approximations are compared. Section VII describes the way to control distributed systems. Necessary formulas are derived and analysed. Section VIII deals with application of the WBC into the real flexible distributed structure. Derived methods are used and results are presented.

II. Wave Based Control

Wave-based control uses the assumption that position of every point of the system can be described by superposition of two waves going in opposite directions and the system can be described by the loop of WTFs. Through the knowledge of this WTFs, the motion of the system can be decomposed to these two components, outgoing wave and returning wave, by using only one sensor. The controller is also designed to make the decomposition and force the actuator to do the move, that simultaneously absorb the returning wave and launch the outgoing wave, that moves the system to the desired position.

Main advantages of this method include the need of only one sensor, high speed of obtaining the desired values and it is proximity to time-optimal control, high robustness against external disturbances and parameter ignorance [7]. Nevertheless stability and performance of the control are not design requirements. Only design requirement is to absorb returning wave by actuator. The stability is not guaranteed in general and stability analysis has to be performed after the controller has been designed.

There are another scientific papers on this topic [8],[9],[10] however, most of them does not give the results in the real-world application but only in terms of the computer simulations.

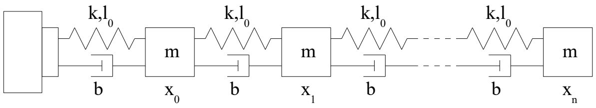

Fig. 1: Modelled system

III. Standard approach to WBC

Standard approach to WBC designed by O'Connor in [11] will be derived in this section. First of all, the WTF is derived on homogeneous infinite-infinite (fig. 2) mass-spring-damper string. The WBC can be used also for inhomogenous systems. However the WTF has to be derived using different approach. Let's consider the homogeneous system in this section.

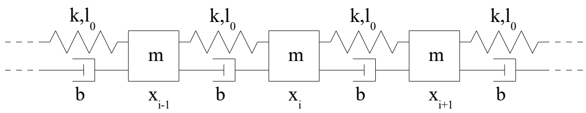

Fig. 2: Infinite-infinite mass-spring-damper system

Let's suppose that:

$$G = \frac{X_{i+1}}{X_i}, \tag{1}$$

where G is WTF between two masses whose position in s domain are X_{i+1}, respective X_i. s is the complex variable. Differential equation describing move of every tangible point is then:

$$\ddot{x}_i = \frac{k}{m}\,(x_{i+1} - 2x_i + x_{i-1}) + \frac{b}{m}(\dot{x}_{i+1} - 2\dot{x}_i + \dot{x}_{i-1}), \tag{2}$$

where x_i, $\dot{x}_i$, $\ddot{x}_i$ is position, velocity and acceleration of i-th mass, k is the stiffness coefficient of springs, b is the damping coefficient of dampers and m is the mass of tangible points.

After Laplace's transformation:

$$s^2\, X_i = \frac{k}{m}\,(G - 2 + G^{-1})X_i + \frac{b}{m}s\,(G - 2 + G^{-1}))X_i. \tag{3}$$

By solving eq. (3) we can see that G has two possible solutions:

$$G_a = 1 + \frac{1}{2}\,\frac{m}{k+bs}s^2 - \frac{1}{2}\sqrt{\frac{m}{k+bs}}\,s\,\sqrt{4 + \frac{2m}{k+bs}s^2}, \tag{4}$$

and

$$G_b = 1 + \frac{1}{2}\,\frac{m}{k+bs}s^2 + \frac{1}{2}\sqrt{\frac{m}{k+bs}}\,s\,\sqrt{4 + \frac{2m}{k+bs}s^2}. \tag{5}$$

Two possible solutions of (3) give us two possible directions of the spreading of waves. Nevertheless only solution (4) is causal and corresponds with outgoing wave. By application of boundary conditions we can create the model of the system as WTFs loops.

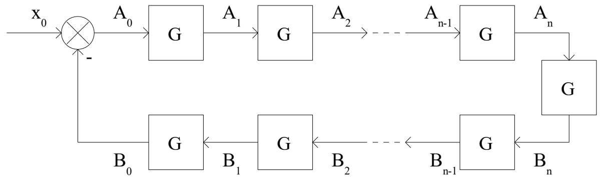

Fig. 3: Wave model of the system

Let's suppose that the displacement of every mass is given by the superposition of two waves A_i and B_i, where A_i is the outgoing wave and B_i is the returning wave.

$$X_i = A_i + B_i. \tag{6}$$

Considering that, it is a fact that

$$A_{i+1} = G_a\, A_i, \tag{7}$$

$$B_{i+1} = G_b\, B_i, \tag{8}$$

and

$$G_a = G_b^{-1}. \tag{9}$$

Then left boundary condition is

$$X_0 = A_0 + B_0,$$

thus

$$A_0 = X_0 - B_0 \tag{10}$$

and right boundary condition for the free end

$$X_n = X_{n+1}$$

$$A_n + B_n = A_{n+1} + B_{n+1}$$

$$A_n + B_n = G_a\, A_n + G_a^{-1}\, B_n$$

$$B_n = G_a\, A_n. \tag{11}$$

Left boundary condition can be considered as a negative feedback. Then we are getting the loop of transfer functions that models the examined system. Model of n-mass-spring-damper system is shown in the fig. 3.

Transfer function from x_0 to x_j derived from the wave model is the same one as the transfer function obtained by the standard way.

$$F_{j,0} = \frac{X_j}{X_0} = \frac{G^j\,(1 + G^{2(n-j)+1})}{1 + G^{2n+1}} \tag{12}$$

is the transfer function derived from the wave model and

$$\hat{F}_{j,0} = C\,(sI - A)^{-1}\,B \tag{13}$$

is the transfer function derived by standard way. A is dynamic matrix, B is the input matrix and C is output matrix of the system. Then

$$F_{j,0} - \hat{F}_{j,0} = 0 \tag{14}$$

Moreover this model suggests how to design the wave decomposer. WTF is transcendent and it is necessary to find its approximation $\hat{G}$. Decomposer is in the fig. 4.

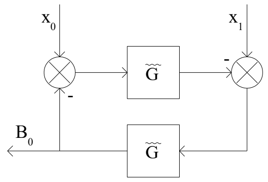

Fig. 4: Wave decomposer

Considering the fact that the static gain of each WTF in loop is equal to one, returning wave must have the same amplitude as the launching wave. This feature can be used to achieve the desired state. If the launching wave has half the amplitude of desired state we can add the returning wave to input values and get the other half of desired state together with the vibration damping.

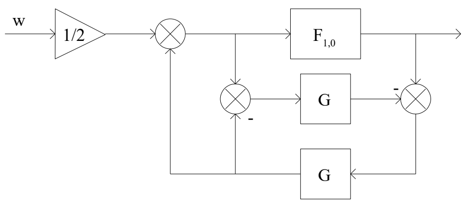

Fig. 5: Wave based control

IV. Modelling and control of real systems

Wave based control uses only one sensor located in the first mass point in the lumped system case. If the system is distributed, the sensor has to be colocated within the actuator-system interface. But that is not always possible.

It can be solved by using the so-called virtual spring. Force of a real spring is given as $k_0(x_1 - x_0)$. Actuator can be used to realise this force by using the position of actuator as x_1 and desired position as x_0. We can call the stiffness of the virtual spring k_0 and it can be set as necessary.

Nevertheless, WBC requires some amount of knowledge about the controlled system. A model of real systems that is not fully rigid, it is not always easy to obtain, because the system can contain some unknown parameters and his structure can be very complex. One approach to this problem is to decompose the system to smaller subsystems that can be modelled more easily. Naturally some systems that consist of more rigid and less rigid parts, can by simplified to masses and springs. [12] But generally, distributed systems like cantilevered beam can be also modelled this way.

The beam, with length L_0, mass M, stiffness K and damping B can be modelled like a string of tangible points linked by intangible springs and dampers. The system can be decomposed to n parts that have lengths $l_0 = \frac{L_0}{n}$, masses $m = \frac{M}{n}$, stiffnesses $k = Kn$ and dampings $b = Bn$. The parts can be simplified to tangible points with masses m, linked by springs with stiffnesses k, natural lengths l_0, and dampers with dampings b. In case that n has infinite values, we will obtain the exact description of the beam. The approximation of the beam is shown in fig. 6.

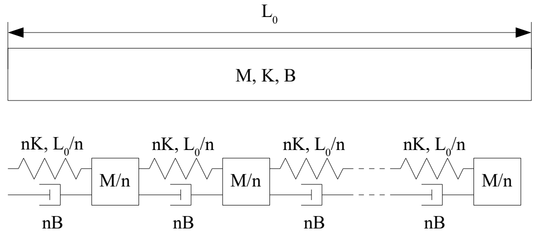

Fig. 6: Aproximation of beam

Although, there are many, less or more accurate, approaches to modelling of flexible structures [13] [14]. The one described above is sufficient and as we shall see later useful for the WBC purposes.

V. Parameter identification

Setting the wave based controller in every form depends on knowledge of the exact WTF or the parameters of the system respectively. Mass, stiffness and damping especially. Generally we do not know all these parameters. For simplicity we consider homogeneous systems. It is necessary to identify them. It depends on the type of system. We distinguish:

A. Parameter identification of lumped system

In case of we have to cope with unknown system that consists of n masses linked by springs with stiffnesses k and dampers with dampings b, there is a possibility to find parameter $\omega_n = \sqrt{\frac{k}{m+bs}}$ from the amplitude frequency response between x_0 and x_1. To get an easy example, let's assume that $b = 0$.

The frequency ω_n can be identified directly from the amplitude frequency response. The frequency ω_n is equal to the frequency of the $(\frac{n}{3} + \frac{2}{3}) - th$ resonance mode of the real system. ω_n corresponds to the resonance mode only if the $n \; mod \; 3 = 1$. Otherwise the $(\frac{n}{3} + \frac{2}{3})$ is not an integer. Thus the frequency ω_n is located between two resonance modes and the exact location can not be found. Then there is a possibility to use the fact that the last n-th resonance mode of the system is located on the frequency that is equal or less than $2\omega_n$. For higher n is the frequency of the last resonance mode closer to $2\omega_n$. With sufficient precision of the control purposes we can use $\omega_n = \frac{\omega(n)}{2}$ for the $n > 4$, where $\omega(n)$ is the frequency of n-th resonance mode. The fig. 7 shows the amplitude frequency response of the 10-mass-spring system with $\omega_n = 2rad/s$

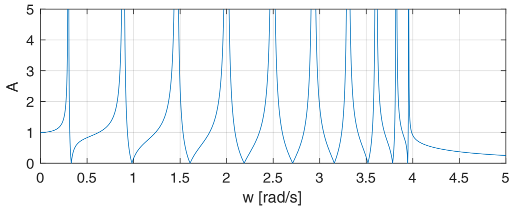

Fig. 7: Frequence responose of 10-mass-springs system with ω_n = 2

B. Parameter identification of distributed system

The last resonance mode can not be determined if the system is distributed, because the distributed system can contain infinite amount of resonance modes. Moreover if the system is damped, it can be very complicated to differentiate individual modes, but the first few modes are significant and distinguishable. Let's suppose that the system is approximated by n tangible points linked by springs and dampers. For higher n the first few modes of approximation will be closer to first few modes of real system. [15]

To determine the parameters of real distributed system Sys_{real} we need to measure the amplitude frequency response of the system between the required position and measured position of actuator. Also we have to choose the structure of the system Sys_{approx}, that approximates the real system Sys_{real}. For example the approximation of real system by the system with n masses without dampings can be described as

$$\dot{x} = Ax + B_m u;\ y = Cx, \tag{15}$$

where

$$A = \begin{bmatrix} A_{11}^{n\times n} & A_{12}^{n\times n} \\ A_{21}^{n\times n} & A_{22}^{n\times n} \end{bmatrix};\ B_m^{2n\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{2n} \end{bmatrix};$$

$$C^{1\times 2n} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix};$$

$$A_{21}^{n\times n} = \omega_n^2 \begin{bmatrix} -(1+\kappa) & 1 & 0 & \cdots & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & -2 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix};$$

$$b_i = 0\,\forall i \neq n+1;\ b_{n+1} = \omega_n^2 \kappa;$$

$A_{11}^{n\times n}$ and $A_{22}^{n\times n}$ are zero matrices, $A_{21}^{n\times n}$ is identity matrix, $\omega_n = \sqrt{\frac{k}{m}}$ and $\kappa = \frac{k_0}{k}$ are variable parameters of the Sys_{approx}.

By setting the ω_n and κ we are changing the frequency response of the Sys_{approx} and it can reach the colocation of the first few modes of the Sys_{approx} and Sys_{real}. Usually we choose a low n to avoid the complexity of the system and calculations. In case of low n colocation only the first mode is sufficient. If we can measure mass of the real system M, then we can calculate K. Parameter B determines the steepness of the amplitude frequency response. Assuming the $B = 0$ in control law has no significant effect on control. The comparison of frequency responses of the real system and the Sys_{approx} is shown in the fig. 8.

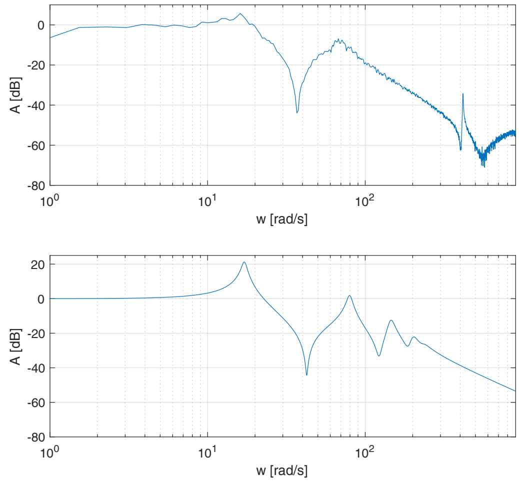

Fig. 8: Frequence response comparation of real and comparative system

VI. Analysis of Stability and Alternative Feedback Form of WBC

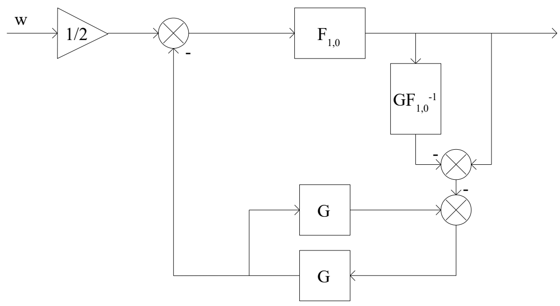

Fig. 9: Alternative form of WBC

System in the fig. 5 can be redrawn like in the fig. 9. In this form it is easy to obtain transfer function of the opened loop L and closed loop CL.

$$L = -\frac{\hat{G}}{1-\hat{G}^2}(F_{1,0} - \hat{G}), \tag{16}$$

and

$$CL = -\frac{F_{1,0}(1-\hat{G}^2)}{1-F_{1,0}\hat{G}}. \tag{17}$$

The feedback in the fig. 9 might be considered as a feedback controller C_f that is given as

$$C_f = -\frac{\hat{G}}{1-\hat{G}^2}\,(1 - F_{1,0}^{-1}\hat{G}), \tag{18}$$

If the $\hat{G}$ is given as a second order transfer function the relative order of the C_f equals two. However C_f is limited by the system dynamics. Moreover absolute order of C_f is too high. Controller in this form is not applicable even if we know all parameters of $\hat{G}$ because of the high order.

However the structure in the fig. 9 is favourable for stability analysis of the control loop. The transfer function of opened loop allows us to make Nyquist stability analysis. It gives us first tool for tuning parameters of the $\hat{G}$. We can tune one or more parameters to obtain desired shape of frequency response of the opened loop.

For example let the controlled system have parameters $M = 10$, $K = 0.3$, $B = 0.01$. By comparison of the frequency responses, we can choose the best approximation of WTF of the system (4). $\hat{G}_1$ is derived from continued fractions, $\hat{G}_2$ is the approximation of WTF designed by O'Connor in [11] and $\hat{G}_3$ is derived from Pade approximation of (4). Then

$$\hat{G}_1 = \frac{bs+k}{(b+m)s^2 + (2b+k)s + 2k}, \tag{19}$$

$$\hat{G}_2 = \frac{\frac{k}{m}}{s^2 + \sqrt{\frac{k}{m}}s + \frac{k}{m}}, \tag{20}$$

$$\hat{G}_3 = \frac{4k}{(2m - 2b\sqrt{\frac{m}{k}})s^2 + 4k\sqrt{\frac{m}{k}}s + 4k}. \tag{21}$$

Comparison of systems controlled within these approximations of (4) is shown in the fig. 10. By this way, we can tune unknown parameters of arbitrary structure of $\hat{G}$.

To analyse pole and zero placement the transfer function of closed system can be used. Nevertheless the pole placement method designed for $\hat{G}$ is not suitable. $\hat{G}$ reaches orders too high.

VII. Control of Distributed Systems

To obtain the best approximation of the controlled system, we need to decompose it to the infinite amount of tangible points. Then it is necessary to cope with the limit terms. The transfer function $\bar{F}_{1,0}$ of the distributed system is

$$\bar{F}_{1,0} = \lim_{n\to\infty} F_{1,0} = 1. \tag{22}$$

Transfer function $\bar{\hat{G}}$, if we substitute $\hat{G}$ by any approximation above or the exact WTF is

$$\bar{\hat{G}} = \lim_{n\to\infty} \hat{G} = 1. \tag{23}$$

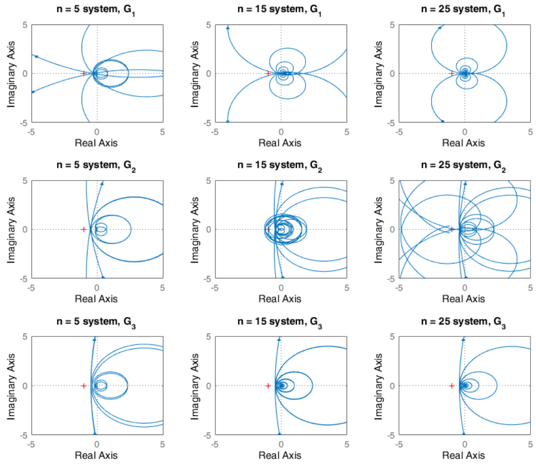

Fig. 10: Comparison of details of the Nyquist plots for n-mass-spring-damper systems cotrolled by controller with different approximations of WTF

Nevertheless $\bar{C}_f$ is different for various $\hat{G}$. Using C_f feedback for control purposes, it is possible to consider exact WTF without the loss of generality, because we do not implement $\hat{G}$ individually but we implement complex C_f. Then C_f has form

$$C_f = -\frac{G^{2n}}{1+G^{2n+1}} \tag{24}$$

and for desired limit form is given as

$$\bar{C}_f = \lim_{n\to\infty} C_f = -\frac{e^{-2s\sqrt{\frac{M}{Bs+K}}}}{1+e^{-2s\sqrt{\frac{M}{Bs+K}}}}. \tag{25}$$

After factoring $e^{-2s\sqrt{\frac{M}{K}}}$ out of the (25) and making the approximation of the residue we can consider this controller to produce the transport delay in series with lowpass filter. $\bar{C}_f$ the whole can be approximated as well.

Let's consider system with parameters $M = 10$, $K = 0.3$, $B = 0.01$ and apply the $\hat{\bar{C}}_f$ approximation of $\bar{C}_f$ to this system.

$$\hat{\bar{C}}_f = \frac{-0.8152s - 0.2848}{18.27s^3 + 9.522s^2 + 1.659s + 0.5697} \tag{26}$$

It is a Pade approximation of (25). We can analyse the stability of control loop after using the (26) controller on system approximated by n-tangible points. Results are in the fig. 11.

It is obvious that $\hat{\bar{C}}_f$ in the (26) form is valid for distributed systems and the stability of loop rises with more exact approximation of the controlled system. On the other hand, the lower is the n value in the loop, the more unstable it might be. Moreover other and more precise approximation

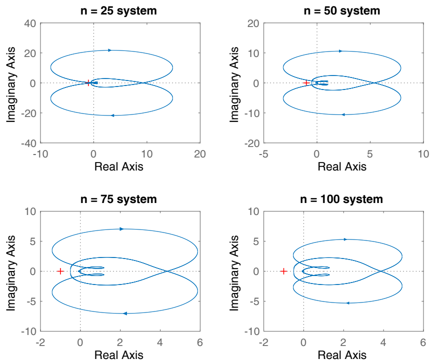

Fig. 11: Nyquist plots of n-mass-spring-damper approximation of real system controlled by Cf

of $\bar{C}_f$ that can be also stable for approximations of controlled systems with lower n values might be found.

VIII. Results of application of WBC to cantilevered beam

The method of identification of the parameters of distributed systems and wave-based feedback control procedure presented above in sections V and VII were applied to a small cantilevered beam. The frequency responses were measured first. The model with 5 masses linked by springs and dampers designed to be the best approximation of the model of cantilevered beam. Comparison of the frequency responses of the real model and comparising model is in the fig. 8.

Estimated parameters M, K and B were used to set controller $\hat{\bar{C}}_f$. The response of the loop with a $4\pi\ rad/s$ sloped ramp and final value 2π as the input controlled by $\hat{\bar{C}}_f$ is in the fig. 12.

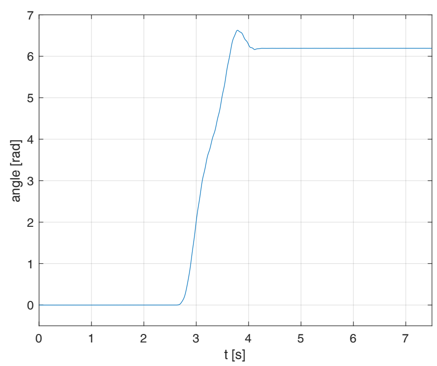

Fig. 12: Declination response of cantilevered beam to ramp with slope $4\pi\ rad/s$

IX. Conclusion

The WBC is an established stable and robust method for vibration control of chains consisting of lumped subsystems. However, if properly modified, the method is also applicable to distributed systems, i.e. systems descripted by partial differential equations (PDE). To design WBC of such systems, the model in the form of a chain consisting of lumped subsystems is neccesary. Such model is obtained by method of discretization of PDE. For aproximative model the WBC is then designed and resulted control low is converted with the limit process for the case that n goes to infinity. This paper aims on the application of WBC to model of distributed system. Parameters of the system are generally unknown. Necessary methods were developed and then applied. Structural parameters identification method referred in chapter V was used for the estimation of small cantilevered beam parameters. Controller $\hat{\bar{C}}_f$ was used for control of the beam. Results are presented in ch. VIII.

Acknowledgement

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