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On Distributed Discrete-time Kalman Filtering in Large Linear Time-invariant Systems

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***Abstract*—The paper is concerned with the problem of distributed Kalman Filtering for discrete-time linear large-scale systems with decentralized sensors. Using the standard approach to the centralized Kalman filtering, the problem of distributed filtering is introduced, given the incidence of additive recurrence to realize such problem. The obtained solutions support the residual signal generation using Kalman filter innovations in the model-based fault detection design. The results, offering structures for fault detection filter realization, are illustrated with a numerical example to note the effectiveness of the approach.**

***Keywords*— linear systems, Kalman filtering, fault detection, residual generators, asymptotic stability.**

I. Introduction

The critical aspect for designing a fault-tolerant control (FTC) system is the conception of diagnostics operations that solve the fault detection and isolation (FDI) problems. These procedures most commonly use residual signals, generated by model-based fault detection filters (FDF), followed by their evaluation within decision functions. The main objective is to create residuals that are as a rule approximately zero in the fault free case, sensitive to faults, as well as robust to noises and disturbances. Occurred faults are detected by setting a threshold on the residual signal which is, in general, superposed minimally on measurement noise. Research in FDI has attracted many researches, and is now the subject of wide range of publications (see, e.g., [2], [4], [6], [8] and the reference therein).

The full decoupling of faults and noises cannot be realized completely, and so residual fault sensitivity to noises has to be minimized. One of the most commonly way is to apply Kalman as well as H_∞ filtering. Kalman filtering is an optimal state estimation process applied to a dynamic system that involves random noises, giving a linear, unbiased, and minimum error variance recursive algorithm to optimally estimate the unknown state of a dynamic system from noisy data taken from sensors at discrete real-time instants [5]. The state estimation obtained by the Kalman filter prediction-correction equations can be solved near-optimal but faster applying a distributed approach in that sense that the correction error reach the optimal values is decaying exponentially with time [10], [13]. Presented variant of this topic includes a distributed method that yields the filtering equations for each sensor [11]. Another applications can be find in [12], [14].

The outline of this paper is as follows. Section I. delineates the problem introduction and, in the following, Section II. draws the basic preliminaries. Dealing with the discrete-time systems description, the equations describing Kalman filters for correlated and uncorrelated measurement and system noises are trace out in Section III. Following this, the delineate mixed, as well as decentralized approaches in Kalman filter design are derived in Section IV and Section V, respectively, while the residual generation based on distributed Kalman filtering is rough in Section VI. Section VII. gives a numerical example, illustrating the properties of the proposed method, and Section VIII. presents some concluding remarks.

Throughout the paper, the notations is narrowly standard in such way that $\boldsymbol{x}^T$, $\boldsymbol{X}^T$ denotes the transpose of the vector $\boldsymbol{x}$ and matrix $\boldsymbol{X}$, respectively, $diag[\,\cdot\,]$ denotes a block diagonal matrix, for a square matrix $\boldsymbol{X} > 0$ means that $\boldsymbol{X}$ is a symmetric positive definite matrix, the symbol $\boldsymbol{I}_n$ indicates the n-th order unit matrix, $I\!R$ denotes the set of real numbers and $I\!R^n$, $I\!R^{n\times r}$ refer to the set of all n-dimensional real vectors and $n \times r$ real matrices, respectively.

II. Basic Preliminaries

In explaining how the Kalman filtering algorithm is given and how well it performs, it is necessary to use some formulas and inequalities from the matrix algebra. The basic ones are presented in the following lemmas.

Lemma 1: [8] (Sherman – Morrison – Woodbury formula) Given square invertible matrices $\boldsymbol{A} \in I\!R^{n\times n}$, $\boldsymbol{D} \in I\!R^{m\times m}$ and a matrix $\boldsymbol{B} \in I\!R^{n\times m}$ such that $(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{D}\boldsymbol{B}^T)$ is invertible, then

$$\begin{aligned}(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{D}\boldsymbol{B}^T)^{-1} = \\ = \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{D}^{-1} + \boldsymbol{B}^T\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{B}^T\boldsymbol{A}^{-1}\end{aligned} \tag{1}$$

Lemma 2: [5] (Schur complement) Given a partitioned matrix of the form

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} \tag{2}$$

where $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ are of compatible dimensions, then

a) if $\boldsymbol{A}^{-1}$ exists, a Schur complement of the matrix $\boldsymbol{M}$ is $\boldsymbol{D} - \boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B}$,

b) if $\boldsymbol{D}^{-1}$ exists, a Schur complement of the matrix $\boldsymbol{M}$ is $\boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C}$.

III. DISCRETE-TIME KALMAN FILTERS

To explain the basic properties, the discrete-time linear MIMO systems with the system and output noises are considered, described in the state-space form by the set of equations

$$\boldsymbol{q}(i+1) = \boldsymbol{F}\boldsymbol{q}(i) + \boldsymbol{G}\boldsymbol{u}(i) + \boldsymbol{v}(i) \tag{3}$$

$$\boldsymbol{y}(i) = \boldsymbol{C}\boldsymbol{q}(i) + \boldsymbol{o}(i) \tag{4}$$

where $\boldsymbol{q}(i) \in I\!R^n$, $\boldsymbol{u}(i) \in I\!R^r$, $\boldsymbol{y}(i) \in I\!R^m$ are vectors of the system state, inputs and outputs variables, respectively, $\boldsymbol{v}(i) \in I\!R^n$, $\boldsymbol{o}(i) \in I\!R^m$ are vectors of the system and measurement noise and matrices $\boldsymbol{F} \in I\!R^{n\times n}$, $\boldsymbol{G} \in I\!R^{n\times r}$, $\boldsymbol{C} \in I\!R^{m\times n}$ are real matrices. It supposed that the noises satisfy the properties

$$E\left\{\begin{bmatrix}\boldsymbol{v}(i)\\ \boldsymbol{o}(i)\end{bmatrix}\right\} = \begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{0}\end{bmatrix} \tag{5}$$

$$E\left\{\begin{bmatrix}\boldsymbol{v}(i)\\ \boldsymbol{o}(i)\end{bmatrix}\begin{bmatrix}\boldsymbol{v}^T(k) & \boldsymbol{o}^T(k)\end{bmatrix}\right\} = \begin{bmatrix}\boldsymbol{Q} & \boldsymbol{S}\\ \boldsymbol{S}^T & \boldsymbol{R}\end{bmatrix}\delta_{ik} \tag{6}$$

where $E\{\cdot\}$ is the mean value operator and

$$\delta_{ik} = \begin{cases} 1 & i = k\ , \\ 0 & i \neq k\ , \end{cases} \tag{7}$$

is the Kronecker delta-function.

The system and measurement noises are uncorrelated with the system initial state $\boldsymbol{q}(0)$, where it can be considered

$$E\left\{\boldsymbol{q}(0)\right\} = \boldsymbol{q}_0 \tag{8}$$

$$E\left\{(\boldsymbol{q}(0) - \boldsymbol{q}_0)(\boldsymbol{q}(0) - \boldsymbol{q}_0)^T\right\} = \boldsymbol{Q}^\bullet \tag{9}$$

The covariance matrices $\boldsymbol{Q}, \boldsymbol{Q}^\bullet \in I\!R^{n\times n}$, $\boldsymbol{R} \in I\!R^{m\times m}$ are symmetric positive definite matrices while, in general,

$$\boldsymbol{Q} - \boldsymbol{S}\boldsymbol{R}^{-1}\boldsymbol{S}^T > \boldsymbol{0} \tag{10}$$

In the following, the notation $\boldsymbol{q}_e(i|i{-}1)$ denotes the predicted estimation of the system state vector $\boldsymbol{q}(i)$ at the time instant i in the dependency on all noisy output measurement vector sequence $\{\boldsymbol{y}(j),\ j = 0, 1, \ldots, i{-}1\}$ up to time instant $i-1$ and $\boldsymbol{q}_e(i|i)$ means the corrected estimation of the system state vector $\boldsymbol{q}(i)$ at the time instant i in the dependency on all noisy output measurement vector sequence $\{\boldsymbol{y}(j),\ j = 0, 1, \ldots, i\}$ up to time instant i.

Lemma 3: If the Kalman filter, associated with the system (3), (4), is defined by the set of equations [5]

$$\begin{aligned}\boldsymbol{q}_e(i|i{-}1) = \boldsymbol{F}\boldsymbol{q}_e(i{-}1|i{-}1) + \boldsymbol{G}\boldsymbol{u}(i{-}1) +\\ +\boldsymbol{J}_S(\boldsymbol{y}(i{-}1) - \boldsymbol{y}_e(i{-}1|i{-}1))\end{aligned} \tag{11}$$

$$\boldsymbol{q}_e(i|i) = \boldsymbol{q}_e(i|i{-}1) + \boldsymbol{J}_F(i)(\boldsymbol{y}(i) - \boldsymbol{y}_e(i|i{-}1)) \tag{12}$$

$$\boldsymbol{y}_e(i|i{-}1) = \boldsymbol{C}\boldsymbol{q}_e(i|i{-}1), \quad \boldsymbol{y}_e(i|i) = \boldsymbol{C}\boldsymbol{q}_e(i|i) \tag{13}$$

then it yields, with $\boldsymbol{q}_e(0|0) = \boldsymbol{q}_0$, $\boldsymbol{P}(0|0) = \boldsymbol{Q}^\bullet$,

$$\boldsymbol{J}_S = \boldsymbol{S}\boldsymbol{R}^{-1} \tag{14}$$

$$\boldsymbol{J}_F(i) = \boldsymbol{P}(i|i{-}1)\boldsymbol{C}^T(\boldsymbol{R} + \boldsymbol{C}\boldsymbol{P}(i|i{-}1)\boldsymbol{C}^T)^{-1} \tag{15}$$

$$\boldsymbol{P}(i|i{-}1) = (\boldsymbol{F}{-}\boldsymbol{J}_S\boldsymbol{C})\boldsymbol{P}(i{-}1|i{-}1)(\boldsymbol{F}{-}\boldsymbol{J}_S\boldsymbol{C})^T + \boldsymbol{Q} - \boldsymbol{J}_S\boldsymbol{S}^T \tag{16}$$

$$\boldsymbol{P}(i|i) = (\boldsymbol{I} - \boldsymbol{J}_F(i)\boldsymbol{C})\boldsymbol{P}(i|i-1) \tag{17}$$

Proof: (for more details see [5]) The requirements is an unbiased filter with the estimates of minimum error variances.

Analyzing the prediction error $\boldsymbol{e}(i|i-1)$ at the sequence index i then

$$\begin{aligned}\boldsymbol{e}(i|i-1) = \boldsymbol{q}(i) - \boldsymbol{q}_e(i|i{-}1) =\\ = \boldsymbol{F}\boldsymbol{q}(i{-}1) + \boldsymbol{G}\boldsymbol{u}(i{-}1) + \boldsymbol{v}(i{-}1) -\\ -\boldsymbol{F}\boldsymbol{q}_e(i{-}1|i{-}1) - \boldsymbol{G}\boldsymbol{u}(i{-}1) -\\ -\boldsymbol{J}_S(\boldsymbol{C}\boldsymbol{q}(i{-}1) + \boldsymbol{o}(i{-}1) - \boldsymbol{C}\boldsymbol{q}_e(i{-}1|i{-}1)) =\\ = (\boldsymbol{F} - \boldsymbol{J}_S\boldsymbol{C})\boldsymbol{e}(i{-}1|i{-}1) + \boldsymbol{v}(i{-}1) - \boldsymbol{J}_S\boldsymbol{o}(i{-}1)\end{aligned} \tag{18}$$

where

$$\boldsymbol{e}(i{-}1|i{-}1) = \boldsymbol{q}(i{-}1) - \boldsymbol{q}_e(i{-}1|i{-}1) \tag{19}$$

Since it yields

$$\begin{aligned}E\{\boldsymbol{e}(i|i{-}1)\} = (\boldsymbol{F} - \boldsymbol{J}_S\boldsymbol{C})E\{\boldsymbol{e}(i{-}1|i{-}1)\} +\\ +E\{\boldsymbol{v}(i{-}1)\} - \boldsymbol{J}_S E\{\boldsymbol{o}(i{-}1)\} =\\ = (\boldsymbol{F} - \boldsymbol{J}_S\boldsymbol{C})E\{\boldsymbol{e}(i{-}1|i{-}1)\}\end{aligned} \tag{20}$$

it is evident that (20) can be satisfied only if $E\{\boldsymbol{e}(i|i{-}1)\} = E\{\boldsymbol{e}(i{-}1|i{-}1)\} = \boldsymbol{0}$. With this condition the covariance matrix of the prediction error satisfied the relation

$$\begin{aligned}\boldsymbol{P}(i|i{-}1) = E\{\boldsymbol{e}(i|i{-}1)\boldsymbol{e}^T(i|i{-}1)\} =\\ E\{((\boldsymbol{F}{-}\boldsymbol{J}_S\boldsymbol{C})\boldsymbol{e}(i{-}1|i{-}1) + \boldsymbol{v}(i{-}1) - \boldsymbol{J}_S\boldsymbol{o}(i{-}1))\boldsymbol{e}^T(i|i{-}1)\} =\\ = (\boldsymbol{F} - \boldsymbol{J}_S\boldsymbol{C})\boldsymbol{P}(i{-}1|i{-}1)(\boldsymbol{F} - \boldsymbol{J}_S\boldsymbol{C})^T +\\ +\boldsymbol{Q} + \boldsymbol{J}_S\boldsymbol{R}\boldsymbol{J}_S^T - \boldsymbol{J}_S\boldsymbol{S}^T - \boldsymbol{S}\boldsymbol{J}_S^T\end{aligned} \tag{21}$$

where the correction error covariance matrix is

$$\boldsymbol{P}(i{-}1|i{-}1) = E\{\boldsymbol{e}(i{-}1|i{-}1)\boldsymbol{e}^T(i{-}1|i{-}1)\} \tag{22}$$

Evidently, the minimal covariance matrix of the prediction error is obtained if

$$(\boldsymbol{J}_S\boldsymbol{R} - \boldsymbol{S})\boldsymbol{J}_S^T = \boldsymbol{0} \tag{23}$$

and, consequently, (21) implies (16) and (23) gives (14).

Analogously, the correction error $\boldsymbol{e}(i|i)$ at the sequence index i is given as

$$\begin{aligned}\boldsymbol{e}(i|i) = \boldsymbol{q}(i) - \boldsymbol{q}_e(i|i) =\\ = \boldsymbol{e}(i|i{-}1) - \boldsymbol{J}_F(i)(\boldsymbol{C}\boldsymbol{q}(i) + \boldsymbol{o}(i) - \boldsymbol{C}\boldsymbol{q}_e(i|i{-}1)) =\\ = (\boldsymbol{I} - \boldsymbol{J}_F(i)\boldsymbol{C})\boldsymbol{e}(i|i{-}1) - \boldsymbol{J}_F(i)\boldsymbol{o}(i)\end{aligned} \tag{24}$$

and, considering the above results, it yields

$$E\{\boldsymbol{e}(i|i)\} = (\boldsymbol{I} - \boldsymbol{J}_F(i)\boldsymbol{C})E\{\boldsymbol{e}(i|i{-}1)\} = \boldsymbol{0} \tag{25}$$

while the correction error covariance matrix is propagated as

$$\begin{aligned}\boldsymbol{P}(i|i) = E\{\boldsymbol{e}(i|i)\boldsymbol{e}^T(i|i)\} =\\ = E\{((\boldsymbol{I} - \boldsymbol{J}_F(i)\boldsymbol{C})\boldsymbol{e}(i|i{-}1) - \boldsymbol{J}_F(i)\boldsymbol{o}(i))\boldsymbol{e}^T(i|i)\} =\\ = (\boldsymbol{I} - \boldsymbol{J}_F(i)\boldsymbol{C})\boldsymbol{P}(i|i{-}1)(\boldsymbol{I} - \boldsymbol{J}_F(i)\boldsymbol{C})^T + \boldsymbol{J}_F(i)\boldsymbol{R}\boldsymbol{J}_F^T(i)\end{aligned} \tag{26}$$

Writing (26) as

$$\begin{aligned}\boldsymbol{P}(i|i) = (\boldsymbol{I} - \boldsymbol{J}_F(i)\boldsymbol{C})\boldsymbol{P}(i|i-1) +\\ +(\boldsymbol{J}_F(i)(\boldsymbol{R} + \boldsymbol{C}\boldsymbol{P}(i|i{-}1)\boldsymbol{C}^T) - \boldsymbol{P}(i|i{-}1)\boldsymbol{C}^T)\boldsymbol{J}_F^T(i)\end{aligned} \tag{27}$$

it is evident that with (15) then (27) implies (17). This concludes the proof. ■

Corollary 1: If the system and measurement noises are uncorrelated ($\boldsymbol{S} = \boldsymbol{0}$), the equations (11)–(17) are reduced to the set of recursive equtions

$$\boldsymbol{q}_e(i|i-1) = \boldsymbol{F}\boldsymbol{q}_e(i-1|i-1) + \boldsymbol{G}\boldsymbol{u}(i-1) \tag{28}$$

$$\boldsymbol{q}_e(i|i) = \boldsymbol{q}_e(i|i-1) + \boldsymbol{J}(i)(\boldsymbol{y}(i) - \boldsymbol{y}_e(i|i-1)) \tag{29}$$

$$\boldsymbol{y}_e(i|i-1) = \boldsymbol{C}\boldsymbol{q}_e(i|i-1), \quad \boldsymbol{y}_e(i|i) = \boldsymbol{C}\boldsymbol{q}_e(i|i) \tag{30}$$

$$\boldsymbol{J}(i) = \boldsymbol{P}(i|i-1)\boldsymbol{C}^T(\boldsymbol{R} + \boldsymbol{C}\boldsymbol{P}(i|i-1)\boldsymbol{C}^T)^{-1} \tag{31}$$

$$\boldsymbol{P}(i|i-1) = \boldsymbol{F}\boldsymbol{P}(i-1|i-1)\boldsymbol{F}^T + \boldsymbol{Q} \tag{32}$$

$$\boldsymbol{P}(i|i) = (\boldsymbol{I} - \boldsymbol{J}(i)\boldsymbol{C})\boldsymbol{P}(i|i-1) \tag{33}$$

where index F is omitted because there is only one filter gain matrix.

The discrete-time Kalman filter equations can be algebraically manipulated into a variety of forms [1], [9]. If the system and measurement noises are uncorrelated then for the updating the Kalman filter gain and the error covariances can be used the following lemma.

Lemma 4: If the system and measurement noises are uncorrelated then it yields for the Kalman filter gain and error covariance matrices

$$\boldsymbol{J}(i) = \boldsymbol{P}(i|i)\boldsymbol{C}^T\boldsymbol{R}^{-1} \tag{34}$$

$$\boldsymbol{P}^{-1}(i|i) = \boldsymbol{P}^{-1}(i|i-1) + \boldsymbol{C}^T\boldsymbol{R}^{-1}\boldsymbol{C} \tag{35}$$

Proof: (compare, e.g., [3]) Considering that there is known $\boldsymbol{q}_e(i|i-1)$ and $\boldsymbol{q}_e(i|i)$ is the best estimate of $\boldsymbol{q}(i)$ that minimizes the cost criterion

$$\begin{aligned} T(i) = &(\boldsymbol{q}(i) - \boldsymbol{q}_e(i|i-1))^T\boldsymbol{P}^{-1}(i|i-1)(\boldsymbol{q}(i) - \boldsymbol{q}_e(i|i-1)) + \\ &+(\boldsymbol{y}(i) - \boldsymbol{C}\boldsymbol{q}(i))^T\boldsymbol{R}^{-1}(\boldsymbol{y}(i) - \boldsymbol{C}\boldsymbol{q}(i)) \end{aligned} \tag{36}$$

Then, evaluating (36) it ields with the optimal setting of a state vector estimate $\boldsymbol{q}(i) = \boldsymbol{q}(i|i)$ the minimum expected cost is given by

$$\begin{aligned} \frac{\mathrm{d}T(i)}{\mathrm{d}\boldsymbol{q}(i)^T} = &\boldsymbol{P}^{-1}(i|i-1)(\boldsymbol{q}(i|i) - \boldsymbol{q}_e(i|i-1)) - \\ &-\boldsymbol{C}^T\boldsymbol{R}^{-1}(\boldsymbol{y}(i) - \boldsymbol{C}\boldsymbol{q}(i|i)) = 0 \end{aligned} \tag{37}$$

which implies

$$\begin{aligned} &(\boldsymbol{P}^{-1}(i|i-1) + \boldsymbol{C}^T\boldsymbol{R}^{-1}\boldsymbol{C})\boldsymbol{q}_e(i|i) = \\ &= \boldsymbol{P}^{-1}(i|i-1)\boldsymbol{q}_e(i|i-1) + \boldsymbol{C}^T\boldsymbol{R}^{-1}\boldsymbol{y}(i) = \\ &= (\boldsymbol{P}^{-1}(i|i-1) + \boldsymbol{C}^T\boldsymbol{R}^{-1}\boldsymbol{C}))\boldsymbol{q}_e(i|i-1) + \\ &+\boldsymbol{C}^T\boldsymbol{R}^{-1}(\boldsymbol{y}(i) - \boldsymbol{C}\boldsymbol{q}_e(i|i-1)) \end{aligned} \tag{38}$$

Therefore, at the i-th stage, using the above the equations (38) gives

$$\begin{aligned} &\boldsymbol{q}_e(i|i) = \boldsymbol{q}_e(i|i-1) + \\ &+(\boldsymbol{P}^{-1}(i|i-1) + \boldsymbol{C}^T\boldsymbol{R}^{-1}\boldsymbol{C})^{-1}\boldsymbol{C}^T\boldsymbol{R}^{-1}(\boldsymbol{y}(i) - \boldsymbol{C}\boldsymbol{q}_e(i|i-1)) = \\ &= \boldsymbol{q}_e(i|i-1) + \boldsymbol{P}(i|i)\boldsymbol{C}^T\boldsymbol{R}^{-1}(\boldsymbol{y}(i) - \boldsymbol{C}\boldsymbol{q}_e(i|i-1)) \end{aligned} \tag{39}$$

Then, pre-multiplying the left side by $\boldsymbol{P}(i|i)$ and post-multiplying the right side by $\boldsymbol{P}(i|i-1)$ it follows from (35) that

$$\boldsymbol{P}(i|i-1) = \boldsymbol{P}(i|i) + \boldsymbol{P}(i|i)\boldsymbol{C}^T\boldsymbol{R}^{-1}\boldsymbol{C}\boldsymbol{P}(i|i-1) \tag{40}$$

which can be proved recursively as follows

$$\boldsymbol{P}(i|i) = (\boldsymbol{I}_n - \boldsymbol{P}(i|i)\boldsymbol{C}^T\boldsymbol{R}^{-1}\boldsymbol{C})\boldsymbol{P}(i|i-1) \tag{41}$$

Comparing (39) with the covariance matrix of the filtering error given by (29) it is evident that

$$\boldsymbol{J}(i) = \boldsymbol{P}(i|i)\boldsymbol{C}^T\boldsymbol{R}^{-1} \tag{42}$$

and (42) implies (34).

On the other side, substituting (31) into (33) it can write

$$\begin{aligned} &\boldsymbol{P}(i|i) = \boldsymbol{P}(i|i-1) - \\ &-\boldsymbol{P}(i|i-1)\boldsymbol{C}^T(\boldsymbol{R} + \boldsymbol{C}\boldsymbol{P}(i|i-1)\boldsymbol{C}^T)^{-1}\boldsymbol{C}\boldsymbol{P}(i|i-1) \end{aligned} \tag{43}$$

and using the Sherman-Morrison-Woodbury formula (1) it yields

$$\begin{aligned} &\boldsymbol{P}^{-1}(i|i) = \boldsymbol{P}^{-1}(i|i-1) - \\ &-\boldsymbol{C}^T(-\boldsymbol{R} - \boldsymbol{C}\boldsymbol{P}(i|i-1)\boldsymbol{C}^T + \boldsymbol{C}\boldsymbol{P}(i|i-1)\boldsymbol{C}^T)^{-1}\boldsymbol{C} \end{aligned} \tag{44}$$

and so, evidently, (44) implies (35).

Moreover, since $\boldsymbol{C}^T\boldsymbol{R}^{-1}\boldsymbol{C}$ is at least a positive semi-definite matrix, it is evident from (40) that $\boldsymbol{P}(i|i)$ is never larger than $\boldsymbol{P}(i|i-1)$. This concludes the proof. ■

IV. Mixed Approach

Considering that the system is square in the sense that $r_j = m_j$, $j = 1, 2 \ldots w$, the number of output and inputs blocks is w and $m = r = \sum_{j=1}^{w} m_j = \sum_{j=1}^{w} r_j$, as well as

$$\boldsymbol{y}(i) = \begin{bmatrix} \boldsymbol{y}_1(i) \\ \boldsymbol{y}_2(i) \\ \vdots \\ \boldsymbol{y}_w(i) \end{bmatrix}, \; \boldsymbol{o}(i) = \begin{bmatrix} \boldsymbol{o}_1(i) \\ \boldsymbol{o}_2(i) \\ \vdots \\ \boldsymbol{o}_w(i) \end{bmatrix}, \; \boldsymbol{C} = \begin{bmatrix} \boldsymbol{C}_1 \\ \boldsymbol{C}_2 \\ \vdots \\ \boldsymbol{C}_w \end{bmatrix} \tag{45}$$

$$\boldsymbol{R}(i) = E\{\boldsymbol{o}(i)\boldsymbol{o}^T(i)\} = diag\left[\boldsymbol{R}_1(i)\ \boldsymbol{R}_2(i)\ \cdots\ \boldsymbol{R}_w(i)\right] \tag{46}$$

then (28)-(32), (45), (46) imply

$$\boldsymbol{q}_e(i|i-1) = \boldsymbol{F}\boldsymbol{q}_e(i-1|i-1) + \sum_{j=1}^{w}\boldsymbol{G}_j\boldsymbol{u}_j(i-1) \tag{47}$$

$$\begin{aligned} &\boldsymbol{q}_e(i|i) = \boldsymbol{q}_e(i|i-1) + \\ &+\boldsymbol{P}(i|i)\sum_{j=1}^{w}\boldsymbol{C}_j^T\boldsymbol{R}_j^{-1}(i)(\boldsymbol{y}_j(i) - \boldsymbol{C}_j\boldsymbol{q}_e(i|i-1)) = \\ &= \boldsymbol{q}_e(i|i-1) + \sum_{j=1}^{w}\boldsymbol{J}_{cj}(i)(\boldsymbol{y}_j(i) - \boldsymbol{C}_j\boldsymbol{q}_e(i|i-1)) \end{aligned} \tag{48}$$

$$\boldsymbol{y}_{ej}(i|i-1) = \boldsymbol{C}_j\boldsymbol{q}_e(i|i-1), \; \boldsymbol{y}_{ej}(i|i) = \boldsymbol{C}_j\boldsymbol{q}_e(i|i) \tag{49}$$

$$\boldsymbol{J}_{cj}(i) = \boldsymbol{P}(i|i)\boldsymbol{C}_j^T\boldsymbol{R}_j^{-1} \tag{50}$$

and (35), (32) give

$$\boldsymbol{P}^{-1}(i|i) = \boldsymbol{P}^{-1}(i|i-1) + \sum_{j=1}^{w}\boldsymbol{C}_j^T\boldsymbol{R}_j^{-1}\boldsymbol{C}_j \tag{51}$$

$$\boldsymbol{P}(i|i-1) = \boldsymbol{F}\boldsymbol{P}(i-1|i-1)\boldsymbol{F}^T + \boldsymbol{Q} \tag{52}$$

It is evident from this formulation that the relation of (48) computes data obtained at all the sensor nodes.

Corollary 2: (data structure separability) Considering that the filtered state vector can be prescribed as

$$\boldsymbol{q}_e(i|i) = \boldsymbol{q}_{ed}(i|i) + \boldsymbol{q}_{ec}(i|i) \tag{53}$$

that is, there are the components of the filtered system state which are dependent on the control signal as well as ones which are independent on the control signals. Then, substituting (53) in (47), it is

$$\begin{aligned} \boldsymbol{q}_e(i|i-1) = \sum_{j=1}^{w} \boldsymbol{G}_j \boldsymbol{u}_j(i-1) + \\ + \boldsymbol{F}(\boldsymbol{q}_{ed}(i-1|i-1) + \boldsymbol{q}_{ec}(i-1|i-1)) \end{aligned} \tag{54}$$

and it can be set

$$\boldsymbol{q}_{ec}(i|i-1) = \boldsymbol{F}\boldsymbol{q}_{ec}(i-1|i-1) + \sum_{j=1}^{w} \boldsymbol{G}_j \boldsymbol{u}_j(i-1) \tag{55}$$

$$\boldsymbol{q}_{ed}(i|i-1) = \boldsymbol{F}\boldsymbol{q}_{ed}(i-1|i-1) \tag{56}$$

Since the correction stage does not depend on the control inputs, using (55), (56) the relation (48) could be rewritten as

$$\begin{aligned} \boldsymbol{q}_{ed}(i|i) = \boldsymbol{q}_{ed}(i|i-1) + \\ + \boldsymbol{P}(i|i) \sum_{j=1}^{w} \boldsymbol{C}_j^T \boldsymbol{R}_j^{-1}(i)(\boldsymbol{z}_j(i) - \boldsymbol{C}_j \boldsymbol{q}_{ed}(i|i-1)) \end{aligned} \tag{57}$$

$$\boldsymbol{q}_{ec}(i|i) = \boldsymbol{q}_{ec}(i|i-1) \tag{58}$$

where

$$\boldsymbol{z}_j(i) = \boldsymbol{y}_j(i) - \boldsymbol{C}_j \boldsymbol{q}_{ec}(i|i-1) \tag{59}$$

$$\boldsymbol{z}_{dj}(i|i-1) = \boldsymbol{C}_j \boldsymbol{q}_{ed}(i|i-1) \tag{60}$$

while the sequence of the filter gain matrices, as well as recurrences of the covariance matrices are given by (50)–(52).

V. Partly Decentralized Approach

Theorem 1: Let the output mode based state estimates $\boldsymbol{q}_{edj}(i|i)$, $j = 1, 2, \ldots, w$ are computed using full decentralized formulas of the form

$$\boldsymbol{q}_{edj}(i|i) = \boldsymbol{q}_{edj}(i|i-1) + \boldsymbol{J}_j(i)(\boldsymbol{z}_j(i) - \boldsymbol{z}_{dj}(i|i-1)) \tag{61}$$

where are applied (55), (56) and (59), (60) and where

$$\boldsymbol{q}_{edj}(i|i-1) = \boldsymbol{F}\boldsymbol{q}_{edj}(i-1|i-1) \tag{62}$$

$$\boldsymbol{z}_{dj}(i|i-1) = \boldsymbol{C}_j \boldsymbol{q}_{edj}(i|i-1) \tag{63}$$

$$\boldsymbol{J}_j(i) = \boldsymbol{P}_j(i|i) \boldsymbol{C}_j^T \boldsymbol{R}_j^{-1} \tag{64}$$

$$\boldsymbol{P}_j^{-1}(i|i) = \boldsymbol{P}_j^{-1}(i|i-1) + \boldsymbol{C}_j^T \boldsymbol{R}_j^{-1} \boldsymbol{C}_j \tag{65}$$

then the filtered system state at the time instant i is approximately covered by the equations

$$\begin{aligned} \boldsymbol{q}_{ed}(i|i) = \sum_{j=1}^{w} \boldsymbol{P}(i|i) \boldsymbol{P}_j^{-1}(i|i) \boldsymbol{q}_{edj}(i|i) - \\ - \sum_{j=1}^{w} \boldsymbol{P}(i|i) \boldsymbol{P}_j^{-1}(i|i-1) \boldsymbol{q}_{edj}(i|i-1)) + \\ + \boldsymbol{P}(i|i) \boldsymbol{P}^{-1}(i|i-1) \boldsymbol{F} \boldsymbol{q}_{ed}(i-1|i-1) \end{aligned} \tag{66}$$

$$\boldsymbol{q}_{ec}(i|i) = \boldsymbol{q}_{ec}(i|i-1) \tag{67}$$

Proof: Substituting (64), the local-mode filter equation (61) takes the form

$$\begin{aligned} \boldsymbol{q}_{edj}(i|i) = \\ = \boldsymbol{q}_{edj}(i|i-1) + \boldsymbol{P}_j(i|i) \boldsymbol{C}_j^T \boldsymbol{R}_j^{-1} (\boldsymbol{z}_j(i) - \boldsymbol{z}_{dj}(i|i-1)) \end{aligned} \tag{68}$$

and it yields

$$\begin{aligned} \boldsymbol{P}_j^{-1}(i)(\boldsymbol{q}_{edj}(i|i) - \boldsymbol{q}_{edj}(i|i-1)) = \\ = \boldsymbol{C}_j^T \boldsymbol{R}_j^{-1} (\boldsymbol{z}_j(i) - \boldsymbol{z}_{dj}(i|i-1)) \end{aligned} \tag{69}$$

$$\begin{aligned} \boldsymbol{C}_j^T \boldsymbol{R}_j^{-1} \boldsymbol{z}_j(i) = \boldsymbol{C}_j^T \boldsymbol{R}_j^{-1} \boldsymbol{C}_j \boldsymbol{q}_{edj}(i|i-1) + \\ + \boldsymbol{P}_j^{-1}(i|i)(\boldsymbol{q}_{edj}(i|i) - \boldsymbol{q}_{edj}(i|i-1)) \end{aligned} \tag{70}$$

respectively. Inserting (65) then (70) gives

$$\begin{aligned} \boldsymbol{C}_j^T \boldsymbol{R}_j^{-1} \boldsymbol{z}_j(i) = \boldsymbol{P}_j^{-1}(i|i)(\boldsymbol{q}_{edj}(i|i) - \boldsymbol{q}_{edj}(i|i-1)) + \\ + \boldsymbol{P}_j^{-1}(i|i) \boldsymbol{q}_{edj}(i|i-1) - \boldsymbol{P}_j^{-1}(i|i-1) \boldsymbol{q}_{edj}(i|i-1) = \\ = \boldsymbol{P}_j^{-1}(i|i) \boldsymbol{q}_{edj}(i|i) - \boldsymbol{P}_j^{-1}(i|i-1) \boldsymbol{q}_{edj}(i|i-1) \end{aligned} \tag{71}$$

Combining (56) and (57) results in

$$\begin{aligned} \boldsymbol{q}_{ed}(i|i) = \boldsymbol{F}\boldsymbol{q}_{ed}(i-1|i-1) + \\ + \boldsymbol{P}(i|i) \sum_{j=1}^{w} \boldsymbol{C}_j^T \boldsymbol{R}_j^{-1}(i)(\boldsymbol{z}_j(i) - \boldsymbol{C}_j \boldsymbol{F} \boldsymbol{q}_{ed}(i-1|i-1)) \end{aligned} \tag{72}$$

which can be written as

$$\begin{aligned} \boldsymbol{q}_{ed}(i|i) = \sum_{j=1}^{w} \boldsymbol{P}(i|i) \boldsymbol{C}_j^T \boldsymbol{R}_j^{-1}(i) \boldsymbol{z}_j(i) + \\ + (\boldsymbol{I}_n - \sum_{j=1}^{w} \boldsymbol{P}(i|i) \boldsymbol{C}_j^T \boldsymbol{R}_j^{-1}(i) \boldsymbol{C}_j) \boldsymbol{F} \boldsymbol{q}_{ed}(i-1|i-1)) \end{aligned} \tag{73}$$

Since, pre-multiplying the left side of (51) by $\boldsymbol{P}(i|i)$, leads to

$$\boldsymbol{I}_n - \sum_{j=1}^{w} \boldsymbol{P}(i|i) \boldsymbol{C}_j^T \boldsymbol{R}_j^{-1} \boldsymbol{C}_j = \boldsymbol{P}(i|i) \boldsymbol{P}^{-1}(i|i-1) \tag{74}$$

then, considering (74), the relation (73) takes the following form

$$\begin{aligned} \boldsymbol{q}_{ed}(i|i) = \sum_{j=1}^{w} \boldsymbol{P}(i|i) \boldsymbol{C}_j^T \boldsymbol{R}_j^{-1}(i) \boldsymbol{z}_j(i) + \\ + \boldsymbol{P}(i|i) \boldsymbol{P}^{-1}(i|i-1) \boldsymbol{F} \boldsymbol{q}_{ed}(i-1|i-1)) \end{aligned} \tag{75}$$

Thus, the substitution of (71) into (75) implies (66). This concludes the proof. ■

Remark 1: It is evident that (66) is not a fully decentralized approach and an interleaving has to be used to modify (66). An approximation, used in decentralized LQG control, is as following [7], [11]

$$\begin{aligned} \boldsymbol{q}_{ed}(i-1|i-1)) = \sum_{j=1}^{w} \boldsymbol{h}(i-1|i-1)) + \\ + \sum_{j=1}^{w} \boldsymbol{P}_j^{-1}(i|i-1) \boldsymbol{F}^{-1} \boldsymbol{q}_{edj}(i|i-1)) \end{aligned} \tag{76}$$

where (66) is rewritten into two sequences

$$\boldsymbol{q}_{ed}(i|i) = \sum_{j=1}^{w} (\boldsymbol{P}(i|i) \boldsymbol{P}_j^{-1}(i|i) \boldsymbol{q}_{edj}(i|i) + \boldsymbol{h}_j(i|i)) \tag{77}$$

$$\begin{aligned} \boldsymbol{h}_j(i|i) = \boldsymbol{P}(i|i) \boldsymbol{P}^{-1}(i|i-1) \boldsymbol{F} \boldsymbol{h}_j(i-1|i-1) + \\ + \boldsymbol{P}(i|i) \boldsymbol{P}^{-1}(i|i-1) \boldsymbol{F} \boldsymbol{P}_j^{-1}(i|i-1) \boldsymbol{F}^{-1} \boldsymbol{q}_{edj}(i|i-1)) - \\ - \boldsymbol{P}(i|i) \boldsymbol{P}_j^{-1}(i|i-1) \boldsymbol{q}_{edj}(i|i-1)) \end{aligned} \tag{78}$$

However, this does not define a recursion for $\boldsymbol{P}_j^{-1}(i|i-1)$.

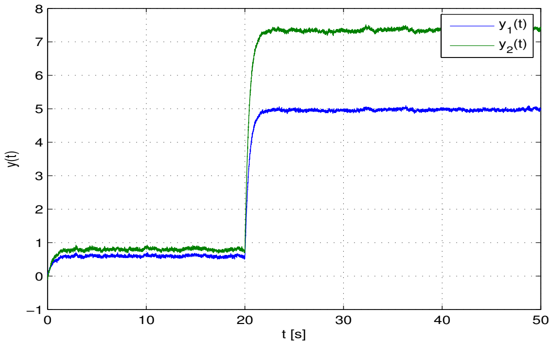

Fig. 1: Faulty system output responses - SAF_1

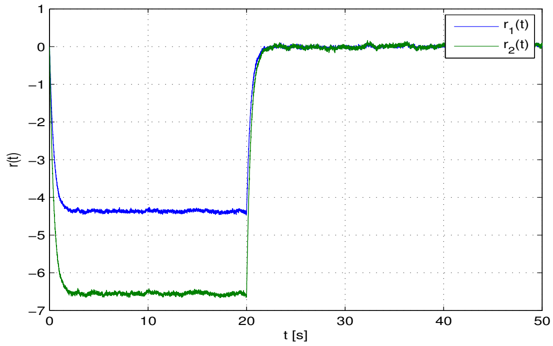

Fig. 2: Residual filter responses - SAF_1

VI. Residual Filters

Considering the distributed Kalman filtering principle applied to discrete-time linear systems the residual filter structure is appointed by the following theorem.

Theorem 2: Let the state estimates $\boldsymbol{q}_{edj}(i|i)$, $j = 1, 2, \ldots, w$ are computed using formulas of the form (61)-(64), (67), conditioned by (55), (56) and (59), (60). Then with

$$\boldsymbol{P}_j^{-1}(i|i) = \boldsymbol{P}^{-1}(i|i-1) + \boldsymbol{C}_j^T \boldsymbol{R}_j^{-1} \boldsymbol{C}_j \tag{79}$$

$$\boldsymbol{q}_{ed}(i|i) = \sum_{j=1}^{w} \boldsymbol{P}(i|i)\boldsymbol{P}_j^{-1}(i|i)\boldsymbol{q}_{edj}(i|i) + \\ +\boldsymbol{P}(i|i)\boldsymbol{P}^{-1}(i|i-1)\boldsymbol{F}(\boldsymbol{q}_{ed}(i-1|i-1) - \sum_{j=1}^{w} \boldsymbol{q}_{edj}(i-1|i-1)) \tag{80}$$

the residual filter takes the structure

$$\boldsymbol{r}_j(i) = \boldsymbol{z}_j(i), \quad j = 1, 2, \ldots w \tag{81}$$

Proof: Setting that $\boldsymbol{P}_j^{-1}(i|i-1) = \boldsymbol{P}^{-1}(i|i-1)$ and considering (62) then (65) gives (79) and (66) implies (80). This concludes proof. ■

To propose the inverse logic (the residual outputs are approximately equal zero if a single actuator fault occurs), then the residuals are evaluated using the formulas (81).

VII. Illustrative Example

The example is a simple demonstration of the Kalman filtering technique for the residual filter construction. The considered system can be put in the discrete-time state equation form (3)-(6) with the sampling period $t_s = 0.8s$, where $\boldsymbol{S} = \boldsymbol{0}$,

$$\boldsymbol{R} = \text{diag}\,[0.003\ 0.04], \quad \boldsymbol{Q} = 0.002\,\boldsymbol{I}_4$$

$$\boldsymbol{F} = \begin{bmatrix} 0.7650 & 0.6267 & 0.6058 & 0.0510 \\ 0.1048 & 0.1083 & 0.0813 & 0.0098 \\ 0.1484 & 0.1419 & 0.1171 & 0.0150 \\ 0.1709 & 0.2286 & 0.1603 & 0.1998 \end{bmatrix}$$

$$\boldsymbol{G} = \begin{bmatrix} 0.0241 & 0.0139 \\ 0.0151 & 0.0013 \\ 0.0109 & 0.0056 \\ 0.0142 & 0.0032 \end{bmatrix}, \ \boldsymbol{C} = \begin{bmatrix} 0.0001 & 0 & 1 & 0 \\ 0.0000 & 0 & 0 & 1 \end{bmatrix}$$

Since the system is stable, the feed-forward control, for simplicity, is used in simulations in such a way that

$$\boldsymbol{u}(i) = \boldsymbol{W}\boldsymbol{w}\,, \ \boldsymbol{w} = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}$$

$$\boldsymbol{W} = (\boldsymbol{C}(\boldsymbol{I}_4 - \boldsymbol{F})^{-1}\boldsymbol{G})^{-1} = \begin{bmatrix} -117.3841 & 79.3124 \\ 280.8078 & -187.1829 \end{bmatrix}$$

and the initial conditions are

$$\boldsymbol{q}^T(0) = \begin{bmatrix} 0.006 & 0 & 0 & 0 \end{bmatrix}, \ \boldsymbol{q}_e(0|0) = \boldsymbol{0}$$

$$\boldsymbol{P}(0|0) = \begin{bmatrix} 0.0028 & 0.0001 & 0.0043 & 0.0002 \\ 0.0001 & 0.0202 & 0.0000 & 0.0003 \\ 0.0043 & 0.0000 & 0.0189 & 0.0000 \\ 0.0002 & 0.0003 & 0.0000 & 0.0140 \end{bmatrix}$$

In simulations there were considered single actuator faults, modeled by the associated zero column in the matrix $\boldsymbol{G}$ and starting at any time instant in a system steady state. Thus, Fig. 1 presents the system outputs and Fig. 2 gives the fault residual filter responses (as the output of (81)), reflecting single fault of the first actuator (SAF_1) starting at the time instant $t = 20\ s$. Fig. 3 and Fig. 4 present the responses in analogous situations concerning the single second actuator fault, all starting at the time instant $t = 20\ s$. Evidently, this residual filter structure works in an inverse logic.

VIII. Concluding Remarks

Realization structures for distributed Kalman filtering, and their applications in fault detection residuals filters structure destined for noisy discrete-time systems, is derived in the paper. The main idea goes with introducing distributed sensor measurement noise corrector stage of a Kalman filter, applied in such a way to be locally uncorrelated with other sensor measurement. The problem accomplishes the manipulation in the manner giving guaranty of asymptotic stability of a local fault residual detection filter. Presented illustrative example confirms the effectiveness of the proposed filtration method.

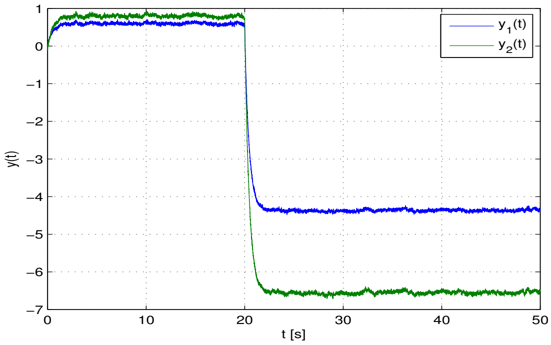

Fig. 3: Faulty system output responses - SAF_2

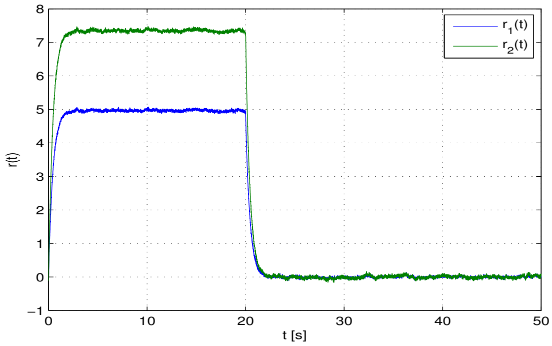

Fig. 4: Residual filter responses - SAF_2

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