2017 21st International Conference on Process Control (PC)
June 6–9, 2017, Štrbské Pleso, Slovakia

Robust control for a discrete-time uncertain system novel LMI based approach

Danica Rosinová, Peter Balko
Institute of Automotive Mechatronics
Slovak University of Technology in Bratislava
Bratislava, Slovakia
danica.rosinova@stuba.sk, peter.balko@stuba.sk

***Abstract*—In the paper, a novel robust controller design method is developed for discrete-time parameter varying system using matrix inequalities. Auxiliary matrices are used to reduce conservatism of the proposed robust stability conditions. The resulting design method is illustrated on examples.**

***Keywords*—robust control, matrix inequalities, discrete-time system, uncertain system**

I. INTRODUCTION

Robust control provides an attractive controller design approach, applicable on real world systems control, since it is resistant to a wide range of uncertainties and imperfections. Various approaches and methods for robust control design have been developed in time and frequency domain. In state space, quadratic, or polynomially dependent Lyapunov function is often used to receive stabilizing control, e.g. [1], [2], [3], formulated as matrix inequalities. There is still an effort to simplify the respective robust stability conditions or to reduce their conservatism (difference between "sufficient" and "necessary and sufficient" conditions). One possible way to relax the conservatism is to include auxiliary matrices, which means lifting of problem into higher dimension space.

In this paper we propose the discrete-time counterpart to the recently presented robust stability condition for uncertain continuous-time system, [3]. Important feature of the developed novel robust stability condition is the fact, that product of input matrix and control gain matrix is dilated owing to extra degree of freedom introduced by additional matrices, which opens the way to use the proposed approach to the gain scheduling control. The proposed robust controller design scheme is illustrated on two examples (nonlinear boiler-turbine system and magnetic levitation system).

II. ROBUST CONTROL PROBLEM AND PRELIMINARIES

Consider the discrete-time uncertain polytopic system

$$\begin{aligned} x(k+1) &= A(\alpha)x(k) + B(\alpha)u(k) \\ y(k) &= Cx(k) \end{aligned} \tag{1}$$

where $x(k) \in R^n$, $u(k) \in R^m$, $y(k) \in R^l$ are state, control and output vectors respectively;

$$\big(A(\alpha), B(\alpha)\big) \in \left\{ \left(\sum_{i=1}^{N} \alpha_i A_i, \sum_{i=1}^{N} \alpha_i B_i \right), \quad \sum_{i=1}^{N} \alpha_i = 1,\ \alpha_i \geq 0 \right\}, \tag{2}$$

A_i, B_i and C are known constant matrices of the respective dimensions. Parameter $\alpha = \alpha(k)$ varies in time, dependence on k is omitted for better readability.

The static output feedback control is considered

$$u(k) = FCx(k) \tag{3}$$

where F is a controller gain matrix conforming to the structure of B and C.

The respective uncertain closed-loop system is then

$$x(k+1) = A_C(\alpha)x(k) \tag{4}$$

where

$$A_C(\alpha) = A(\alpha) + B(\alpha)FC\,. \tag{5}$$

To assess the performance quality, a quadratic cost function known from LQ theory is used.

$$J_d = \sum_{k=0}^{\infty} J(k) = \sum_{k=0}^{\infty} [x(k)^T Qx(k) + u(k)^T Ru(k)] \tag{6}$$

where $Q \in R^{n \times n}, R \in R^{m \times m}$ are symmetric positive definite matrices.

The concept of guaranteed cost control is used in a standard way. Let there exist a feedback gain matrix F_0 and a constant J_0 such that

$$J_d \leq J_0 \tag{7}$$

holds for the closed loop system (4). Then the respective control (3) is called the guaranteed cost control and the value of J_0 is the guaranteed cost.

The main aim is to develop a static output feedback (SOF) control algorithm that stabilizes the uncertain system (1), with *guaranteed cost* with respect to the cost function (6).

The development of main result is based on Lyapunov stability approach, the following lemma known from LQ theory is recalled for a reader's convenience.

Lemma 1

Consider the discrete time system (1) with control algorithm (3). Control algorithm (3) is the guaranteed cost control law for uncertain system (1) if and only if there exists a Lyapunov function $V(k)>0$ such that the following condition holds

$$\Delta V(k)+J(k)\le 0\,. \tag{8}$$

We use Parameter dependent Lyapunov function (PDLF) in the form, [1]

$$\begin{gathered}V(k)=x(k)^T P(\alpha)x(k)\\ P(\alpha)=\sum_{i=1}^{N}\alpha_i P_i\,.\end{gathered} \tag{9}$$

We further assume that the change of PDLF matrix $P(\alpha)$ within one sampling period is given by a limited parameter α change

$$\alpha(k+1)=\alpha(k)+\Delta\alpha;\ |\Delta\alpha|\le\Delta\alpha_{\max}\,. \tag{10}$$

From (10) and (9) we have an upper bound on matrix $P(\alpha)$ change

$$\begin{gathered}P(\alpha(k+1))=P(\alpha(k)+\Delta\alpha)\le P(\alpha(k))+P_{\Delta\alpha}\\ P_{\Delta\alpha}\le\Delta\alpha_{\max}\sum_{i=1}^{N}P_i\,.\end{gathered} \tag{11}$$

In the following we use denotation $P(\alpha+\Delta\alpha)$ for $P(\alpha(k+1))$ and $P(\alpha)$ for $P(\alpha(k))$.

Robust stability is considered in the sense of the following definition.

Definition 1 ([2])

System (4) is robustly stable in the convex uncertainty domain (2) with parameter-dependent Lyapunov function (9) if and only if there exists a matrix $P(\alpha)=P(\alpha)^T>0$ such that

$$A_C^T(\alpha)P(\alpha)A_C(\alpha)-P(\alpha)<0 \tag{12}$$

for all α such that $A_C(\alpha)$ is given by (5).

III. ROBUST CONTROLLER DESIGN

In this section, sufficient conditions for robust SOF controller design are derived in the form of matrix inequalities.

The following developments provide a discrete-time counterpart of recently published result for continuous time systems, [3].

To achieve robust stability of the closed loop with guaranteed cost, we consider condition (8), which is rewritten for PDLF (9) and $J(k)$ from (6) as

$$\begin{gathered}x(k+1)^T P(\alpha+\Delta\alpha)x(k+1)-x(k)^T P(\alpha)x(k)\\ +x(k)^T Qx(k)+u(k)^T Ru(k)\le 0\end{gathered} \tag{13}$$

Inequality (13) can be rewritten in a matrix form as

$$\begin{bmatrix}x(k+1)\\ x(k)\\ u(k)\end{bmatrix}^T\begin{bmatrix}P(\alpha+\Delta\alpha) & 0 & 0\\ 0 & -P(\alpha)+Q & 0\\ 0 & 0 & R\end{bmatrix}\begin{bmatrix}x(k+1)\\ x(k)\\ u(k)\end{bmatrix}\le 0\,. \tag{14}$$

We use auxiliary matrices to include constraints (1) and (3) for $x(k+1)$ and $u(k)$ as well as to lift the parameter space. For this reason we consider equalities

$$\begin{gathered}[x(k+1)-A(\alpha)x(k)-B(\alpha)u(k)]^T[N_1x(k+1)+N_2x(k)\\ +N_3u(k)]+*=0\end{gathered} \tag{15}$$

$$[u(k)-FCx(k)]^T[N_4x(k+1)+N_5x(k)+N_6u(k)]+*=0 \tag{16}$$

where $*$ denotes a transpose term to the previous one.

Rewrite now equalities (15) and (16) into a matrix form

$$\begin{bmatrix}x(k+1)\\ x(k)\\ u(k)\end{bmatrix}^T W_i\begin{bmatrix}x(k+1)\\ x(k)\\ u(k)\end{bmatrix}=0\,,\quad i=1,2 \tag{17}$$

where for (15) we have

$$W_1=\begin{bmatrix}N_1^T+N_1 & * & *\\ N_2^T-A(\alpha)^T N_1 & -N_2^T A(\alpha)+A(\alpha)^T N_2 & *\\ N_3^T-B(\alpha)^T N_1 & -N_3^T A(\alpha)-B(\alpha)^T N_2 & -N_3^T B(\alpha)-B(\alpha)^T N_3\end{bmatrix}$$

and for (16)

$$W_2 = \begin{bmatrix} 0 & * & * \\ -C^T F N_4 & -N_5^T FC - C^T F N_5 & * \\ N_4 & -N_6^T FC + N_5 & N_6^T + N_6 \end{bmatrix}$$

$*$ denotes the transpose term for a symmetric element.

Summing (14) with (17) for W_1 and W_2, the final condition for stability with guaranteed cost is obtained as summarized in Theorem 1.

Theorem 1

Consider discrete-time system (1) with cost function (6). Output feedback control (3) stabilizes system (1) with guaranteed cost if there exist symmetric positive definite matrices $P_i \in R^{nxn}$ and matrices $F \in R^{mxp}$, $N_1, N_2 \in R^{nxn}$, $N_3 \in R^{nxm}$, $N_4, N_5 \in R^{mxn}$ and $N_6 \in R^{mxm}$ such that the following inequality holds for $i = 1,2,...,N$

$$\begin{bmatrix} w_{11i} & * & * \\ w_{21i} & w_{22i} & * \\ w_{31i} & w_{32i} & w_{33i} \end{bmatrix} \le 0 \tag{18}$$

where

$$w_{11i} = P_i + P_{\Delta\alpha} + N_1 + N_1^T$$
$$w_{21i} = N_2^T - A_i^T N_1 - C^T F^T N_4$$
$$w_{31i} = N_3^T - B_i^T N_1 + N_4$$
$$w_{22i} = -N_2^T A_i - A_i^T N_2 - N_5^T F\,C - C^T F^T N_5 - P_i + Q$$
$$w_{32i} = -N_3^T A_i - B_i^T N_2 - N_6^T F\,C + N_5$$
$$w_{33i} = -N_3^T B_i - B_i^T N_3 + N_6 + N_6^T + R.$$

Proof: The proof is based on the fact that matrices $A(\alpha), B(\alpha), P(\alpha)$ are convex combinations of A_i, B_i, P_i for $i = 1,2,...,N$ respectively.

Taking convex combination of (18) for $i = 1,2,...,N$ we arrive at

$$\begin{bmatrix} w_{11} & * & * \\ w_{21} & w_{22} & * \\ w_{31} & w_{32} & w_{33} \end{bmatrix} \le 0 \tag{19}$$

for

$$w_{11} = P(\alpha) + P_{\Delta\alpha} + N_1 + N_1^T$$
$$w_{21} = N_2^T - A(\alpha)^T N_1 - C^T F^T N_4$$
$$w_{31} = N_3^T - B(\alpha)^T N_1 + N_4$$
$$w_{22} = -N_2^T A(\alpha) - A(\alpha)^T N_2 - N_5^T FC - C^T F^T N_5 - P(\alpha) + Q$$
$$w_{32} = -N_3^T A(\alpha) - B(\alpha)^T N_2 - N_6^T FC + N_5$$
$$w_{33} = -N_3^T B(\alpha) - B(\alpha)^T N_3 + N_6 + N_6^T + R.$$

Considering the arguments above Theorem 1, it can be easily shown that (19) implies (13), therefore it is sufficient stability condition with guaranteed cost. This ends the proof.

Condition (18) provides a stability condition in the form of Linear matrix inequality (LMI) for known control gain matrix *F*; for a robust controller design it is in the form of Bilinear matrix inequality (BMI).

Remark 1

It is important to note that in the developed condition (18), controller gain matrix *F* does not appear in product with parameter dependent matrix $B(\alpha)$, which enables to use this condition directly for a gain scheduling controller design with parameter dependent gain matrix $F(\alpha)$. Details of robust controller design for a discrete time uncertain system with gain scheduling control are under research. Other possibility to use (18) is to consider parameter dependent output matrix C.

In the next section, robust controller design using (18) is illustrated on two examples: one of them is unstable SISO system, the other is two input - two output system with additional disturbance input, decentralized control is designed.

IV. EXAMPLES

In this section, results for robust PID controller design are shown for:

- magnetic levitation system [4], [8] – Example 1
- nonlinear boiler-turbine system, [5], [6] - Example 2

Example 1 Magnetic levitation

In this example we consider state space linearized model of magnetic levitation laboratory system, [8], where the aim is to control a ball position within the air space by voltage input controlling the corresponding current in a coil; details about model can be found in [4]. Note that the system is highly nonlinear and unstable.

The state space model is obtained for sampling period T=0.001s and is augmented by additional 2 states and 2 outputs to include PID controller dynamics (augmentation procedure can be found e.g. in [7]).

We consider two working points respective to two ball positions

$$A_1 = \begin{bmatrix} 1.003 & 0.001 & 0 & 0 & 0 \\ 6.074 & 1.003 & -0.0382 & 0 & 0 \\ -71.3 & -0.038 & 0.704 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1.003 & 0.001 & 0 & 0 & 0 \\ 6.900 & 1.003 & -0.0310 & 0 & 0 \\ -113.11 & -0.061 & 0.601 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 \\ -0.018 \\ 0.746 \\ 0 \\ 0 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 0 \\ -0.0216 \\ 1.006 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Solving (18) we obtain controller gain matrix

$$F = \begin{bmatrix} 4129.8 & 3526.6 & -3059.7 \end{bmatrix}$$

The corresponding PID controller transfer function is

$$G_C(z) = K_P + \frac{K_I}{1 - z^{-1}} + K_D(1 - z^{-1})$$

where

$K_P = 136.316$

$K_I = 466.87$

$K_D = 3526.6$

The respective step responses for nonlinear simulation model are shown in Fig. 1. The obtained controller stabilizes the system in the considered region, with relatively quick response.

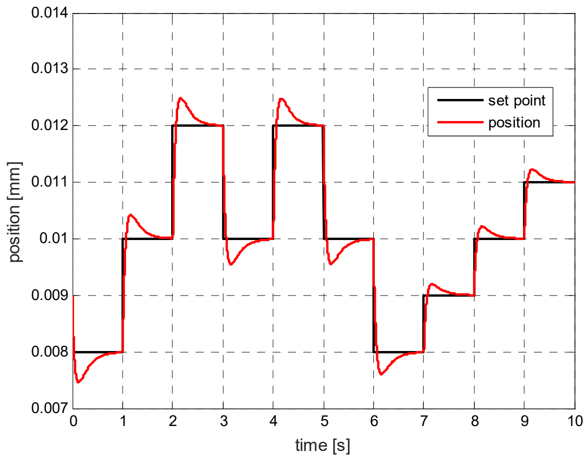

Fig. 1. Step responses for Maglev ball position – closed loop with the designed PID controller.

Example 2 Boiler-turbine system, [5], [6]

In this case, we consider nonlinear boiler-turbine system with two control inputs (feed water flow and fuel flow) and two controlled outputs (drum pressure and water level). The third input – load (changing steam demand) is considered as a disturbance. Decentralized control structure is considered, with 2 PI controllers, where drum pressure is controlled by fuel flow and water level by feed water flow.

The state space model is augmented by additional 2 states and 2 outputs to include dynamics of 2 PI controllers for the corresponding subsystem loops.

The augmented state space models for 3 working points discretized for sampling period T=1s are given by (1) for

$$A_1 = \begin{bmatrix} 0.964 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.946 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.973 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.994 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0.089 & 0 & 0 & -0.031 & 0 & 1 & 0 \\ 0 & 0.228 & -0.0013 & 0 & 0.084 & 0 & 1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0.128 & 0 \\ 0.243 & 0 \\ 0.123 & 0 \\ 0 & 0.0312 \\ 0 & 0.1250 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0.089 & 0 & 0 & -0.031 & 0 & 0 & 0 \\ 0 & 0.228 & -0.0013 & 0 & 0.0840 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$
A_2 = \begin{bmatrix} 0.984 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.961 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.980 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.995 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0.116 & 0 & 0 & -0.0272 & 0 & 1 & 0 \\ 0 & 0.165 & -0.0002 & 0 & 0.0824 & 0 & 1 \end{bmatrix}
$$

$$
B_2 = \begin{bmatrix} 0.062 & 0 \\ 0.245 & 0 \\ 0.123 & 0 \\ 0 & 0.0312 \\ 0 & 0.125 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}
$$

$$
C_2 = \begin{bmatrix} 0.116 & 0 & 0 & -0.0272 & 0 & 0 & 0 \\ 0 & 0.165 & -0.0002 & 0 & 0.0.324 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
A_3 = \begin{bmatrix} 0.963 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.952 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.976 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.996 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0.0965 & 0 & 0 & -0.0192 & 0 & 1 & 0 \\ 0 & 0.177 & -0.0022 & 0 & 0.0816 & 0 & 1 \end{bmatrix}
$$

$$
B_3 = \begin{bmatrix} 0.123 & 0 \\ 0.244 & 0 \\ 0.123 & 0 \\ 0 & 0.0312 \\ 0 & 0.125 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}
$$

$$
C_3 = \begin{bmatrix} 0.0965 & 0 & 0 & -0.0192 & 0 & 0 & 0 \\ 0 & 0.177 & -0.0022 & 0 & 0.0816 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

Solving (18) we obtain controller gain matrix for decentralized control

$$
F = \begin{bmatrix} 2.153 & 0 & 0.038 & 0 \\ 0 & 1.12 & 0 & 0.039 \end{bmatrix},
$$

the corresponding PID controller transfer functions for individual loops are

$$
G_{R1}(z) = 2.116 + \frac{0.0368}{1-z^{-1}}; \quad G_{R2}(z) = 1.08 + \frac{0.0394}{1-z^{-1}}
$$

We compare this result with controllers presented in [5]

$$
G_{R1m}(z) = 2.5 + \frac{0.05}{1-z^{-1}}; \qquad G_{R2m}(z) = 1.25 + \frac{0.025}{1-z^{-1}}
$$

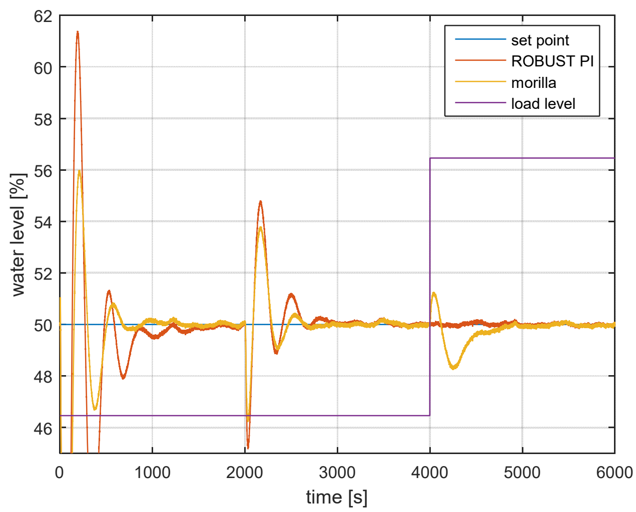

Fig. 2. Step response of water level: comparison of the designed controller (red line) and the one from literature (yellow line), disturbance (load change) appears in 4000s time.

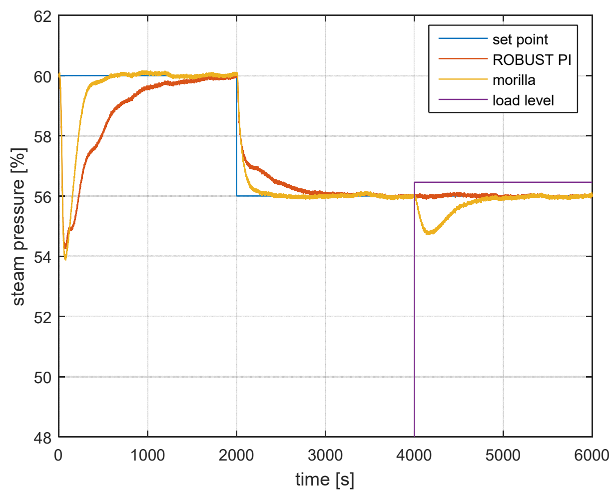

Fig. 3. Step response of steam pressure: comparison of the designed controller (red line) and the one from literature (yellow line), disturbance (load change) appears in 4000s time.

We can observe on output responses in Fig.2 and Fig.3 that the proposed robust controller has slightly slower response than the reference one, however its robustness competes the other one when a disturbance appears. Note that in 2000s time, where steam pressure setpoint is changed, the couplings between system variables cause the oscillations of water level.

V. Conclusion

The novel robust stability condition for a discrete time polytopic system with output feedback control was developed, which does not include a product of controller gain matrix and other unknown number, therefore it enables to consider parameter dependent controller gain matrix. Moreover, the developed condition includes extra degrees of freedom, thus can be assumed as less conservative. The result in the form of matrix inequality can be used for robust stability analysis (as LMI) or robust control design (as BMI). Two examples illustrate the applicability of the proposed condition for a robust controller design also for unstable and MIMO systems.

Acknowledgment

The work has been supported by the SRDA grant No APVV 0772-12 and Slovak Scientific Grant Agency, grant No 1/0733/16.

References

[1] J. Daafouz, J. Bernussou, "Parameter dependent Lyapunov functions for discrete time systems with time varying parametric uncertainties", Systems & Control Letters 43, 2001, pp. 355–359.

[2] M.C. de Oliveira, J. Bernussou, J.C. Geromel, "A new discrete-time robust stability condition", Systems & Control Letters 37, 1999, pp. 261–265.

[3] V. Veselý," A new method to robust controller design", in 17th International Carpathian Control Conference (ICCC), May 29-June 1 2016

[4] P. Balko, D. Rosinová, "Modeling of magnetic levitation system", submitted to Process Control 2017

[5] F. Morilla, "Benchmark for PID control based on the Boiler Control Problem". In: Proceedings from IFAC Conference on Advances in PID Control PID'12, Brescia, March 2012, Italy, available on: http://www.dia.uned.es/~fmorilla/benchmarkPID2012/

[6] P. Balko, D. Rosinová, "Robust decentralized control of nonlinear drum boiler" In IFAC-PapersOnLine: 8th IFAC Symposium on Robust Control Design ROCOND 2015. Bratislava, Slovakia. 8-11 July 2015. Vol. 48, No. 14, 2015, p. 431-436.

[7] V. Veselý, D. Rosinová, "Robust PID-PSD Controller Design: BMI Approach", Asian Journal of Control, Vol. 15, Iss. 2, 2013, p. 469-478.

[8] Magnetic Levitation System 2EM - User's Manual (Inteco Ltd, Krakow, Poland, 2008).