

Distributed adaptive consensus protocol with Laplacian eigenvalues estimation

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Abstract—This paper addresses distributed consensus problem for multi-agent systems with general linear time-invariant dynamics and undirected connected communication graphs. A distributed adaptive consensus protocol is found to solve problems of existing adaptive consensus protocols related to different, generally large and possibly unbounded coupling gains. This protocol guarantees ultimate boundedness under all conditions, however for an asymptotic stability, a proper estimation of reference values for coupling gains is required. Here, we propose an algorithm for the estimation of the coupling gain reference. The algorithm is based on a distributed estimation of the Laplacian eigenvalues. In comparison to the previously proposed algorithm based on the interval halving method, this algorithm offers robustness to change of the network topology. In addition, it decouples the estimation from the consensus protocol, hence it does not influence stability properties of the adaptive consensus protocol.

Keywords—consensus, adaptive control, Laplacian spectrum, decentralized estimation, multi-agent systems.

I. INTRODUCTION

In last two decades, a great effort has been made in distributed control and estimation in formations of mobile robots, satellites and vehicles. The inspiration came from the natural behaviour of swarms, flocks and schools. Connecting the graph theory, describing the topological structure of a network, and control theory, the basic consensus protocols for formation control in networked multi-agent systems are introduced in [1], [2], [3] and [4].

Previously developed theoretical results in control of single-agent system motivated the designs of recent distributed controllers and observers. For example the passivity-based design of cooperative controllers for cooperation and synchronization of multi-agent systems is described in [5]. A unified viewpoint on design of consensus regulator on directed graph topologies using the synchronizing region is introduced in [6]. The design of cooperative regulators and observers using state or output-feedback in continuous and discrete-time is considered in [7], [8] and [9].

The static consensus protocols presented in [6], [7] and [8] use a feedback coupling gain that satisfies a bound calculated from the smallest non-zero real part of Laplacian eigenvalues.

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The graph structure has to be known to calculate this bound. Therefore centralized information is required by each agent.

Distributed adaptive consensus protocols propose a solution to this problem on undirected connected graphs [10] as well as on directed graphs having a spanning tree with leader as a root node [11]. These protocols do not rely on any centralized information, therefore they can be implemented by each agent separately without using any global information. The protocols guarantee cooperative stability, however the benefits from adaptability suffer from possibly large control effort and lack of robustness to noise.

To avoid these drawback, a novel distributed adaptive consensus protocol is developed. First, it was designed to solve the cooperative regulator problem on undirected connected communication graphs [12]. This work was later extended to directed graphs having a spanning tree [13]. The protocol is fully distributed, therefore it does not require any centralized information. The unavailability of centralized information is compensated by its estimation. The protocol runs an algorithm for estimation of reference values for coupling gains. If the reference values are estimated properly, the network of agents is asymptotically stable. On the other hand the solution of the network dynamics is ultimately bounded.

In this paper, we introduce an algorithm for proper estimation of coupling gains' references. The algorithm is based on distributed estimation of Laplacian eigenvalues presented in [14] and [15]. Each agent implements this algorithm and estimates the smallest non-zero eigenvalue of the Laplacian matrix. This value is then used to calculate the coupling gain required for the asymptotic stability of the network. The algorithm offers better robustness to change of the network topology than the previously introduced estimation algorithm based on the interval halving method. Moreover, it decouples the estimation of coupling gains' references from the adaptive consensus protocol, thereby maintains the stability property of the adaptive consensus protocol.

This paper is organized as follows. Section II introduces the basic notation and graph preliminaries used throughout the paper. Section III introduces the problems addressed by the adaptive consensus protocol with the novel estimation algorithm. The adaptive consensus protocol is presented in Section IV. Section V introduces the novel algorithm estimating the eigenvalues of Laplacian matrix. Numerical simulation

are given in Section. VI. Section VII concludes the paper.

II. PRELIMINARIES

Through this paper the following notations and definitions are used. $\mathbb{R}^{m \times n}$ denotes the set of $m \times n$ real matrices. Denote $\mathbf{1}_N$ as a column vector with N entries, all equal to one. A matrix $M = \text{diag}(v)$ for $v \in \mathbb{R}^n$ denotes $\mathbb{R}^{n \times n}$ diagonal matrix with elements of v on the diagonal. Ordering eigenvalues of a matrix M in ascending order, its i -th eigenvalue is denoted by $\lambda_i(M)$ and the smallest and the largest eigenvalues are denoted by $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$, respectively. Positive (semi)-definite symmetric matrix is denoted by $M \succ (\succeq) 0$. The sum over all agents is denoted by \sum_i for $i = 1, \dots, N$ when it is not stated directly.

An undirected graph is given by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v_1, \dots, v_N\}$ is a non-empty finite set of vertices and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is a set of edges. An edge is a pair of nodes (v_i, v_j) , $v_i \neq v_j$, representing that agents i and j can exchange information between them. In sequel, the graph \mathcal{G} is assumed to be undirected, connected and simple.

The adjacency matrix $E = [e_{ij}] \in \mathbb{R}^{N \times N}$ associated with the graph \mathcal{G} is defined by $e_{ij} = e_{ji} > 0$ if $(v_i, v_j) \in \mathcal{E}$, otherwise $e_{ij} = e_{ji} = 0$. Define the vector of node degrees as $d = E\mathbf{1}_N$, and the degree matrix as $D = \text{diag}(d)$. Then the graph Laplacian is defined by $L = D - E$.

III. PROBLEM STATEMENT AND MOTIVATION

Consider a group of N identical agents. Each agent is described by a general LTI dynamics

$$\dot{x}_i = Ax_i + Bu_i, \quad i = 1, \dots, N, \quad (1)$$

where $x_i \in \mathbb{R}^n$ is the agents state, $u_i \in \mathbb{R}^m$ is the agents input, and $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are constant matrices. The matrix A does not need to be stable but the pair of matrices (A, B) is assumed stabilizable. The communication topology of the network of agents is given by an undirected graph \mathcal{G} , that is assumed connected.

Our goal is to synchronize the states of agents in the sense of $\lim_{t \rightarrow \infty} \|x_j - x_i\| = 0, \forall i, j$ without requiring any centralized information. There has been developed many adaptive consensus protocols that can reach this goal. However, all these adaptive consensus protocols suffer from several drawbacks:

- different final coupling gain values,
- high final coupling gain values,
- lack of robustness to noise.

Our recent work [12], [13] present a novel adaptive consensus protocol that avoids these drawback and solves the cooperative regulator problem on undirected connected communication graphs [12], later extended to directed strongly connected communication graphs [13].

In this paper, we are going to extend the results on undirected connected graphs [12] by proposing a new method for estimation of coupling gain values.

IV. DISTRIBUTED ADAPTIVE CONSENSUS PROTOCOL

Let each agent implements an adaptive control law [12], [13] given by a control input and a coupling gain dynamics

$$u_i = c_i K \sum_j e_{ij} (x_j - x_i), \quad i = 1, \dots, N, \quad (2)$$

$$\dot{c}_i = \sum_j e_{ij} (x_j - x_i)^T \Gamma (x_j - x_i) + \sum_j e_{ij} (c_j - c_i) - \ell (c_i - \kappa_i), \quad (3)$$

where $\ell > 0$ is a constant, $c_i > 0$ is the coupling gain associated with the i -th agent and $\kappa_i \geq 0$ is a coupling gain reference estimated by the i -th agent.

The gain matrices K and Γ are designed by LQR method. Let $Q = Q^T \in \mathbb{R}^{n \times n}$ and $R = R^T \in \mathbb{R}^{m \times m}$ be positive definite symmetric matrices, then

$$K = R^{-1} B^T P, \quad (4)$$

$$\Gamma = \Gamma^T = P B K = P B R^{-1} B^T P, \quad (5)$$

where matrix $P \succ 0$ is the unique solution of the algebraic Riccati equation

$$A^T P + P A - P B R^{-1} B^T P = -Q. \quad (6)$$

The task of the coupling gain dynamics (3) is to adapt the coupling gains c_i . It consists of three main terms. The first term on the right hand side pushes the coupling gains to higher values until the states of agents get synchronized. The second term on the right hand side synchronizes the coupling gains. The third term on the right hand side pushes the coupling gain c_i to its reference κ_i . The value of the reference κ_i is estimated by an estimation algorithm. The strength of the third term is determined by the positive constant ℓ .

The estimation algorithm determines the stability of the network of agents implementing the adaptive control law (2, 3). Each agent runs this algorithm to estimate its own coupling gain reference κ_i . Since the coupling gain c_i is pushed to its reference κ_i by the coupling gain dynamics (3), each agent estimates its own coupling gain. For small values of κ_i below some bound $\kappa_i < \beta$ the network dynamics is ultimately bounded and the agent's trajectories oscillate. For large values of κ_i bigger than this bound $\kappa_i \geq \beta$, the network dynamics is asymptotically stable and the agents reach consensus. Note that the ultimate boundedness is the worst case scenario that guaranties stability albeit bounded. This property provides sufficient time for the estimation of κ_i .

With increasing κ_i increases also c_i and thereby the control effort. Hence, the aim of the estimation algorithm is to estimate proper κ_i , that is sufficiently high to reach asymptotic stability of the network but at the same time as low as possible to minimize the control effort.

In the next section we present an estimation algorithm for estimation of κ_i . We discuss its benefits and drawback, and provide a comparison with the previously proposed algorithm.

V. EIGENVALUE ESTIMATION ALGORITHM

The main contribution of this paper is an estimation algorithm for estimation of the coupling gain reference κ_i . The algorithm is based on the estimation of Laplacian eigenvalues proposed in [14], [15]. Each agent estimates the Laplacian eigenvalues by performing an algorithm with the following updating rule

$$\begin{aligned} \dot{p}_i(t) &= -\sum_j e_{ij}(t)(q_i(t) - q_j(t)), \\ \dot{q}_i(t) &= \sum_j e_{ij}(t)(p_i(t) - p_j(t)), \end{aligned} \quad (7)$$

where $p_i, q_i \in \mathbb{R}$ are artificial states of i -th agent.

The network implementing the updating rule can be described by a time-varying autonomous linear system

$$\begin{bmatrix} \dot{p}(t) \\ \dot{q}(t) \end{bmatrix} = \mathcal{A}(t) \begin{bmatrix} p(t) \\ q(t) \end{bmatrix}, \quad (8)$$

where

$$\mathcal{A}(t) = \begin{bmatrix} \mathbf{0}_N & -L(t) \\ L(t) & \mathbf{0}_N \end{bmatrix}, \quad (9)$$

and $\mathbf{0}_N \in \mathbb{R}^{N \times N}$ is a matrix of zeros.

Since \mathcal{A} is a skew symmetric matrix, all its eigenvalues are on the imaginary axis. Moreover, the eigenvalues of \mathcal{A} can be derived from the eigenvalues of the Laplacian matrix L .

Lemma 1 ([14], Thm. 1): Consider an undirected graph \mathcal{G} given by a Laplacian matrix L . Let \mathcal{A} be given as in (9). Then for every eigenvalue of the Laplacian matrix L there is a complex pair of eigenvalues of matrix \mathcal{A}

$$\lambda_i(\mathcal{A}), \bar{\lambda}_i(\mathcal{A}) = \pm j\lambda_i(L). \quad (10)$$

From Lemma 1 follows, that each state p_i and q_i follow an oscillating trajectory, that is a linear combinations of sinusoid oscillating at frequencies given by the Laplacian eigenvalues. Note, that by change of the network topology the trajectories remain continuous only the phase and the module of the signal change. Using a Fast Fourier Transformation (FFT), each agent can independently estimate the eigenvalues of the Laplacian matrix. The smallest non-zero estimated eigenvalues is then used to calculate the new κ_i . The formula for calculation of κ_i was adopted from the stability condition of a static consensus protocol.

Consider a control law with the control input (2) and one static common coupling gain c , i.e. the coupling gain dynamic (3) is neglected and the coupling gains c_i are replaced by only one static coupling gain c . A static consensus protocol presented in [7] and [8] is obtained. Following from [7, Thm. 1], a static consensus protocol is globally asymptotically stable if

$$c \geq \frac{1}{2\lambda_{\min>0}(L)}, \quad (11)$$

where $\lambda_{\min>0}(L)$ is the smallest non-zero eigenvalue of the Laplacian matrix L .

One could expect, that having one coupling gain separately for each agents does not change the conclusion on stability. If each c_i satisfy (11), then the network of agents should

be asymptotically stable. The estimation algorithm uses this conclusion to determine the new value of κ_i as

$$\kappa_i = \frac{1}{2\lambda_{\min>0}(L)}. \quad (12)$$

If the agents correctly estimate the smallest non-zero eigenvalue of the Laplacian matrix the network of agents should be globally asymptotically stable. Note, that this eigenvalue might be unobservable for some agents or some agents might catch some other bigger eigenvalue instead of the smallest non-zero one. In this case the adaptive control protocol (2, 3) still guarantees ultimate boundedness of the solution. The network of agents will be stable but the agents' trajectories might oscillate.

To estimate κ_i , each agent performs the estimation algorithm consisting of following steps:

- 1) At the beginning $t = 0$, generate the initial conditions $p_i, q_i \in \{-1, 1\}$.
- 2) Perform the state updating rule (7).
- 3) In a time window Δt , estimate the frequencies of sinusoids of its artificial state p_i or q_i . The values of the estimated frequencies correspond to the eigenvalues of the Laplacian matrix.
- 4) Use the smallest non-zero estimated eigenvalue to calculate the new κ_i from (12).

Previously proposed estimation algorithm based on the interval halving method represents a low-pass filter that can handle only fast disturbances in the network. Change of the network topology, like for instance adding an agent, generates an abrupt changes in the local neighbourhood errors, what results in increase of the values of κ_i and thereby also c_i . This estimation algorithm is therefore not robust to change of the network topology.

Moreover, the interconnection of the estimation algorithm and the adaptive consensus protocol creates an additional artificial feedback in network dynamics. This interconnection appears to be stable, however to prove analytical stability is not an easy task. The approach we propose in this paper decouples the control and estimation dynamics. Hence the adaptive consensus protocol and the Laplacian eigenvalue estimator can be designed separately and their interconnection does not create any artificial feedback in the network.

VI. SIMULATION RESULTS

The adaptive control protocol (2, 3) has been simulated on a graph \mathcal{G} consisting of agents described by linear double integrator dynamics

$$\dot{x}_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_i + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i, \quad x_i = \begin{bmatrix} x_{i1} \\ x_{i2} \end{bmatrix}, \quad \forall i. \quad (13)$$

For comparison, both estimation algorithm are implemented to estimate κ_i . The interval-halving estimation algorithm uses the time window $\Delta t_1 = 5s$ and the sampling frequency $f_{s1} = 10Hz$. The eigenvalue estimation algorithm uses the sampling

frequency $f_{s2} = 1\text{Hz}$. The positive constant ℓ was set to 1 in both cases. Initial conditions of the agents are

$$x_{i_1}(0) \in \langle -10, 10 \rangle, \quad x_{i_2}(0) = 0, \quad c_i(0) = 0, \quad \forall i. \quad (14)$$

The simulations of the control protocol (2, 3) with the previously proposed interval-halving estimation algorithm and with the current eigenvalue estimation algorithm are situated on the top and bottom of the figures, respectively.

Each simulation of an algorithm consists of two charts located in one row of a figure. To visualize the evolution of agents' states and show the cooperative stability of a network of agents, the chart on the left shows the agents' position errors with respect to some average value $\delta_i = x_i - x^*$, where $x^* = 1/N \sum_i x_i$ is an average of agents' states. The chart on the right shows the evolution of coupling gains.

The simulations on the circular graph consisting of 50 agents are shown in Figure 1. The proposed protocol implemented with the interval-halving estimation algorithm reaches lower coupling gains with preserving stability. The distributed eigenvalue estimation algorithm estimates the eigenvalues in the first 100 seconds. During this period $\kappa_i = 0, \forall i$. The smallest estimated non-zero eigenvalues is then used for calculation of κ_i . After 100 second the consensus is reached. The higher oscillating coupling gains come from the poor accuracy in the eigenvalue estimation. To increase accuracy longer estimation time period is required.

Assuming small noise acting on states, the responses of 10 agents in circular topology are shown in Figure 2. The distributed eigenvalue-estimation approach uses estimation period of 20 seconds. From the figure it follows that the proposed protocol implemented with both approaches is robust to noise acting on states.

Figure 3 shows the response to the change in the network topology. At the time instance of 30 seconds the graph topology was switched from the circular graph of 4 synchronized agents to the path graph and 5-th agent was connected to the end of the path. The distance of 5-th agent from the rest of the network was chosen to be 10. The graphs can be seen in Figure 4. The protocol implemented with interval-halving estimation algorithm detected slight increase of the coupling gains. This happened because of an abrupt change in the network triggered by insertion of an agent. The eigenvalue estimation algorithm adapted to the change in the network by estimated the new smallest non-zero Laplacian eigenvalue. Therefore, it was found robust to change in the network topology.

VII. CONCLUSION

In this paper we extend the results on distributed adaptive consensus protocol [12] by proposing a novel algorithm for estimation of the reference value κ_i for the coupling gain c_i . The coupling gains' references are then used by the adaptive consensus protocol to reach asymptotic stability of the network.

The estimation algorithm is based on the distributed estimation of Laplacian eigenvalues presented in [14], [15]. It

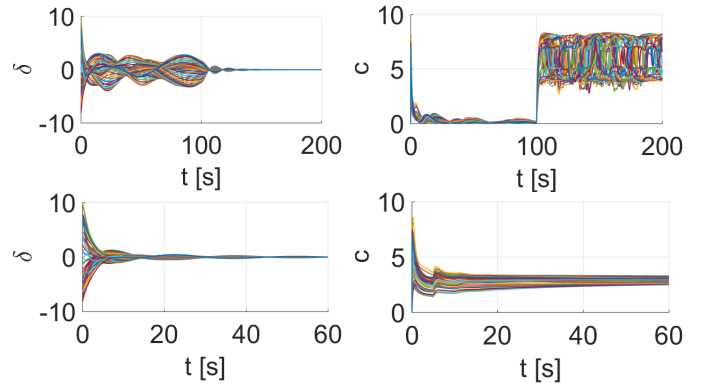


Fig. 1: Simulation of 50 agents on a circular topology.

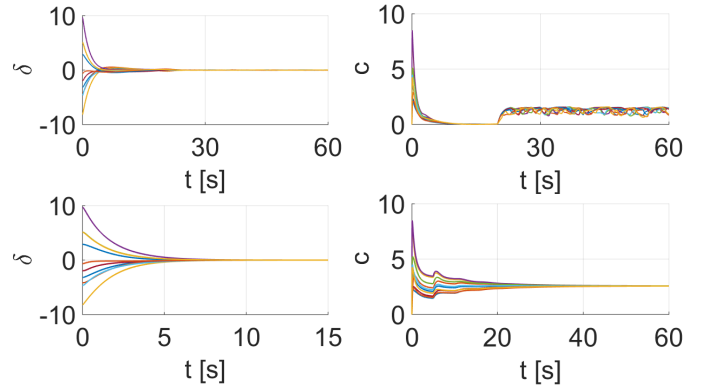


Fig. 2: Simulation of 10 agents on a circular topology assuming noise in state measurements.

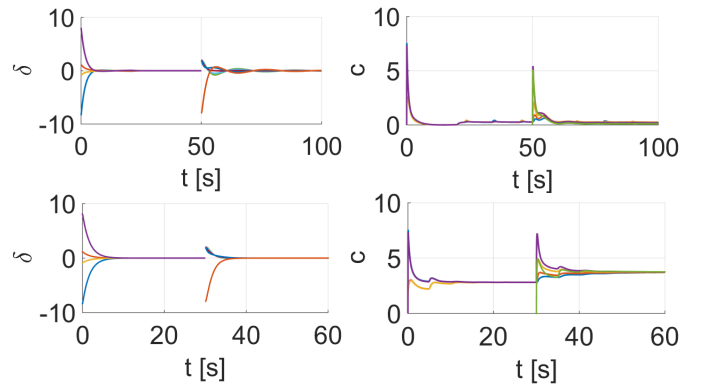


Fig. 3: Simulation of 5 agents by change in a graph topology.

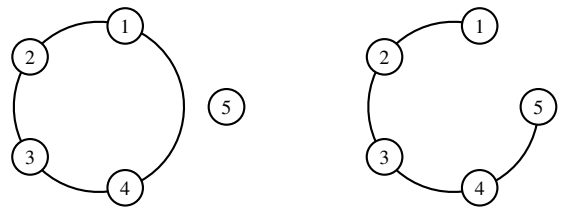


Fig. 4: Two graphs used to simulate a change in the network topology. At the time instance of 30 seconds, the network topology was switched from the left graph to the right graph.

contains a local updating rule to generate a signal oscillating at frequencies corresponding to eigenvalues of the Laplacian matrix. The FFT is then applied at this oscillating signal to obtain the estimates of Laplacian eigenvalues. The smallest non-zero estimated eigenvalue is then used to calculate the coupling gain reference k_i .

An advantage of this estimation algorithm is, that it can adjust the coupling gains to the network topology. This allows the adaptive consensus protocol to be used on switching networks. Additionally, the algorithm decouples the estimation of the coupling gains' references from the control law, thus it does not influence the conclusions on the stability of the adaptive consensus protocol.

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