A Robustness Study of Process Control Loops Designed Using Exact Linearization

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Abstract—Exact linearization is often applied to nonlinear processes. This method requires not only the knowledge of the model structure but also the accurate parameter values. If the real parameter values of the controlled process are different from the nominal values used for the exact linearization, the resulting system may not be linear or may have different gains and time constants as expected. A simple procedure is suggested that allows first the parameter grid based characterisation of the composition of the uncertain nonlinear system and its linearizing feedback for the nominal parameters and then the design of a H_{∞} controller. The approach is illustrated by simple examples including 1st order systems and the nonlinear Van der Pol oscillator.

I. INTRODUCTION

One of the available options to address the control of nonlinear processes is to transform them into linear systems and then to design a feedback using techniques available from the linear control theory (e.g. pole placement for the error dynamics). The set of transformations available includes linearization in the neighborhood of a setpoint (usually an equilibrium) and exact linearization. Linearization around the setpoint has limited validity. Exact linearization [1]–[3] involves an (eventually) dynamical state feedback so that the resulting closedloop dynamical system is not a linear approximation but an exact linear system. Exact linearization has been successfully applied for process models [4] and it is known that the second order models of the Van der Pol oscillator and the continuousstirred tank reactors may be also linearized by feedback [5].

The linearizing feedback mentioned above may involve parameters of the model which are known with only a limited accuracy so that their nominal values are used in the linearizing feedback law. As the real parameter values may be different from their nominal values due to uncertainties, the resulting system may not be exactly linear or may not be the same linear system as for the nominal parameter values. It follows that the robustness of the closed-loop performance and stability against uncertainty have to be addressed.

The H_{∞} , LMI¹, and similar design techniques [6], [7] for uncertain linear systems allow to represent parametric or frequency domain uncertainties and to take them into account during controller synthesis. Based on this observation, it is natural to study if the set of systems resulting from the application of inaccurate linearizing feedbacks can be represented as a set of linear uncertain systems. This idea is

¹Linear Matrix Inequality

already appeared in the literature, an example for a similar approach was used by Li et al. [8] to ensure the robust control of the longitudinal dynamics of a vehicle platoon.

This paper describes a systematic four steps procedure to cope with uncertainties resulting from the inaccurate knowledge of parameter values used for exact linearization. The procedure considers a finite grid of parameter values. Taking the closed-loop systems for the combination of the nonlinear system for each possible parameter value over the grid and the linearizing feedback law endowed with the nominal parameter values, a linear system is fitted. This results in a set of linear systems which is then covered using a standard structure with multiplicative uncertainty. Using the multiplicative description of the uncertainty, a weighted mixed sensitivity problem can be formulated so that an H_{∞} controller can be designed with the help of Matlab. The limitation of the procedure is the "size" of the uncertainty system set since a small variation in the parameter values may result in a large variation of the closed-loop systems.

The input of the procedure is the nonlinear plant with the linearizing feedback, and the output is a robust controller that is able to cope with parametric uncertainty and to ensure desired tracking performance. Recall however that the design of the weighting transfer matrices for the H_{∞} synthesis of the controller for the uncertain linear system depends on the application at hand.

The paper is structured as follows. The next section describes the methodology suggested and presents each step in details. Section III presents two application examples to illustrate the methodology. Based on the examples, the next section reports a simple generalization of the procedure for 1st order systems. The last example illustrates the useability of the method for a second order dynamics, namely the Van der Pol oscillator in Section V. The last section presents the conclusions.

II. METHODOLOGY

The methodology is presented here first in a general setup. Consider a nonlinear system (NL) with a finite dimensional state space so that its description depends on the elements of a parameter vector p. The uncertainty of each parameter is bounded. The vector of their nominal values is denoted by p_0 . It is understood that a grid of parameter values can cover the



Fig. 1. The nonlinear system and its inverse to result in a combination that is linear



Fig. 2. Multiplicative structure to represent parameter uncertainty

parameter space with sufficient resolution. The vertices of the grid are denoted by p_i for $i \in [1, ..., P]$ where P is finite.

Suppose now that a linearizing feedback is designed and it is denoted by NL^{-1} so that the closed-loop system becomes exactly a linear system for the nominal parameter values p_0 . The nominal values p_0 are supposed to be known or specified as well as the dynamics of the linear system obtained by applying NL^{-1} to NL. To be able to apply the H_{∞} synthesis methods, this linear transfer is expected to contain no integrator. The feedback NL^{-1} may represent a complex dynamic feedback law [9]. Let us enumerate the four steps of the suggested procedure.

- 1) Calculation or identification of the nominal linear plant.
- 2) Identification or linearization for all parameters p_i , $i \in [1, ..., P]$.
- 3) Pulling out the Δ (representing uncertainties) and determination of its weighting function to cover the resulting set of linear systems using a multiplicative uncertainty structure.
- 4) H_{∞} controller synthesis.

Figures 1 and 2 help to illustrate the objective of steps 1-3 for the SISO case (in the MIMO case, weighting functions may be needed on both sides of the uncertainty block Δ and their calculation is also supported by the Robust Control Toolbox of Matlab). Although the uncertainties are taken into account by using a multiplicative structure here, other approaches are equally possible. The first step is to determine (by calculation or by identification) the linear plant G(s) for the nominal parameter values where the inversion feedback produces exactly a linear system. The second step requires to identify *P* linear models, one for each parameter in the grid that covers the bounded uncertainty range. The identified model, denoted by $G(s, p_i)$, is a linear approximation of

$$\mathrm{NL}(p_i) \circ \mathrm{NL}^{-1}(p_0) \tag{1}$$

(see also Figure 1). The identification of linear models for Step 2 can be carried out by standard techniques (usually supported

by Simulink or by the Identification toolbox of Matlab).

The third step requires to find a single W(s) so that for all $G(s, p_i)$ (i = 1, ..., P), the equation

$$G(s, p_i) = G(s)(1 + W(s)\Delta)$$
⁽²⁾

holds for some $\|\Delta\|_{\infty} \leq 1$. In the SISO case, it is easy to verify that (2) can be rearranged so that the inequality constraint $\|\Delta\|_{\infty} \leq 1$ is equivalent to

$$\frac{\|G(j\omega,p_i) - G(j\omega)\|}{\|G(j\omega)\|} \le \|W(j\omega)\| \quad \forall \omega > 0 \quad i = 1, \dots, P \quad (3)$$

where we used that $s = j\omega$. Recall that this inequality can be solved readily with a tool such as Matlab. For the details, the reader may refer to a Gu et al. [10].

The last step is the solution of a standard weighted mixed sensitivity H_{∞} controller synthesis problem, illustrated in Figure 3. Recall that M(s) specifies the desired closed-loop behavior and the weighting transfer functions $W_u(s)$ and $W_e(s)$ express performance requirements and constraints on the closed-loop and actuator bandwidths. The transfer matrix P(s) reads

$$P(s) = \begin{bmatrix} 0 & 0 & W(s)G(s) \\ W_e(s) & -W_e(s)M(s) & W_e(s)G(s) \\ 0 & 0 & W_u(s) \\ -1 & 1 & -G(s) \end{bmatrix}$$
(4)

The controller K(s) can be obtained by using again Matlab (e.g. using the Robust Control Toolbox) applying ARE² or LMI based synthesis techniques. Controller structures different from Figure 3 can be also applied and further performance specifications for noise and disturbance rejection can be also added. We denote in the sequel by u_k the input of the controller and by y_k its output.



Fig. 3. Illustration of the standard H_{∞} weighted mixed sensitivity problem

²Algebraic Ricatti Equation



Fig. 4. Static nonlinearity at the input

III. FIRST ORDER EXAMPLE SYSTEMS

Two examples are presented in this section. The first includes a static nonlinear inversion and the second a nonlinear feedback to transform the system into a specific linear dynamics. Each step in the proposed procedure is implemented using Matlab, Simulink and the Robust Control Toolbox.

A. Static nonlinear inversion

The simplest scenario fitting into the scheme specified in Figure 1 is the static inversion of a nonlinearity (without feedback but still resulting in a linear closed-loop behavior) at the input or output of the nonlinear system. Such a scheme is given in Figure 4. The two uncertain parameters are $p_1 \in [2,3]$ and $p_2 \in [6,8]$ with mid-range nominal values $p_{10} = 2.5$ and $p_{20} = 7$. A grid of p_i is obtained by dividing the bounded interval of both uncertain parameters into four segments. The identification of the nominal linear plant G(s) and the calculation of a second order W(s) according to (3) resulted in

$$G(s) = \frac{0.143}{s + 0.143} \quad W(s) = \frac{0.43s^2 + 0.52s + 0.045}{s^2 + 1.22s + 0.182}, \quad (5)$$

which concludes steps 1-3 of our proposed algorithm. For the H_{∞} synthesis, we used the following model and performance weighting functions, M(s), $W_u(s)$ and $W_e(s)$, respectively:

$$M(s) = \frac{1}{2s+1} \tag{6}$$

$$W_u(s) = \frac{0.05(0.02s+1)^2}{(0.002s+1)^2} \tag{7}$$

$$W_e(s) = \frac{100(0.02s+1)}{2s+1} \tag{8}$$

The closed-loop behavior is depicted in Figure 5. Each curve shows the plant input and plant output transients for a given value p_i covering the full uncertainty range such that the reference is a step function at t = 1 s. Both the performance and stability are satisfactory.

B. Nonlinear inversion with feedback

For the second example, the exact transformation of the nonlinear system into a linear one involves feedback. We have again two uncertain parameters p_1 and p_2 with bounds and nominal values identical to the previous example. The scheme is shown in Figure 6.

Applying the feedback law

$$u = f(v, y, p_0) = p_{10} \tan(v \cdot p_{20} - y \cdot p_{20} + \arctan y)$$
(9)



Fig. 5. Plant inputs and outputs in closed-loop for different values of p_i .



Fig. 6. Linearization using feedback

for the system, it is easy to verify that the closed-loop transfer function for the nominal parameter values is

$$G(s) = \frac{1}{s+1} \tag{10}$$

The desired closed-loop model M(s) and performance weighting functions W_e and W_u are all identical to the previous example. For the sake of completeness, Figure 7 shows the left hand side of the inequality (3) as a function of the angular frequency and for all p_i of the grid. The weighting function W(s) for the multiplicative uncertainty representation has to have a larger magnitude at every frequency than all transfer functions traced in Figure 7. Such a transfer function reads

$$W(s) = \frac{0.51s^2 + 8.7s + 0.8}{s^2 + 20.4s + 26.5}.$$
 (11)

The transients for a step reference are shown in Figure 8.



Fig. 7. Left hand side of the inequality (3)



Fig. 8. Closed-loop input and output of the plant for all p_i in the grid covering the parameter uncertainty range

IV. GENERALIZATIONS FOR FIRST ORDER SYSTEMS

Based on the examples above, an easy generalization can be proposed for first order nonlinear systems. Consider a first order dynamical system in the form

$$\dot{\mathbf{y}} = f(\mathbf{y}, p) + g(u, p) \tag{12}$$

where it is assumed that functions f and g satisfy conditions for existence and uniqueness of the solution. Such a system can be graphically represented by a block diagram as in Figure 9. Suppose that the linear system to be obtained after nonlinear inversion is specified in the form



Fig. 9. 1st order nonlinear system

$$T\dot{y} = -y + v \tag{13}$$

for some T > 0. It can be easily verified that this linear system can be obtained by applying the feedback

$$u = g_{p_0}^{-1} \left(\frac{1}{T} \left(v - y \right) - f(y, p_0) \right)$$
(14)

where $g_{p_0}^{-1}$ denotes the inverse of the mapping g for the nominal parameter set $p = p_0$. The procedure suggested in Section II can be then applied.

Another generalization may be obtained if one allows dynamic feedback. Consider now a first order dynamics of the form

$$\dot{\mathbf{y}} = f(\mathbf{y}, \mathbf{u}, p) \tag{15}$$

In order to have an affine system w.r.t. the input, one may introduce a new state (part of the feedback) so that $\zeta = u$ and $\dot{\zeta} = w$. The goal of the exact linearization is to obtain a double integrator $\ddot{y} = v$. After some straightforward calculations, one can get the NL⁻¹ system which is now a dynamic feedback of the form

$$w = \frac{v - f'_{y}(y, \zeta, p_{0})f(y, \zeta, p_{0})}{f'_{\zeta = u}(y, \zeta, p_{0})}$$
(16)

$$\dot{\boldsymbol{\zeta}} = \boldsymbol{w} \tag{17}$$

where ζ is the state (and output) of the feedback (and the input of the nonlinear plant) and f'_y denotes the derivative of f w.r.t. *y*. Again, the procedure described in Section II may then be applied.

V. VAN DER POL MODEL OSCILLATOR

The state space equation of the controlled Van Der Pol oscillator reads [3].

$$\dot{x} = \begin{bmatrix} x_2 \\ 2\omega\zeta(1-\mu x_1^2)x_2 - \omega^2 x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad (18)$$

Suppose the output function is chosen as

$$y = x_1. \tag{19}$$

The uncertainty range of the parameters are $\omega \in [1,2]$, $\zeta \in [2,3]$ and $\mu \in [0.01, 0.09]$ with mid-range nominal values $\omega_{10} = 1.5$, $\zeta_{20} = 2.5$ and $\mu_{30} = 0.05$.

We would like to obtain a second order under-damped system as the result of the exact linearization with the parameters $\omega_m = 10$ and $\xi = 0.8$, so its transfer function reads

$$G(s) = \frac{\omega_m^2}{s^2 + 2\xi\omega_n + 1}.$$
(20)

The underlying linearizing feedback law reads

$$u = \phi(\omega_m, \xi_m, \omega_{10}, \zeta_{20}, \mu_{30})$$

= $\omega_m^2 v - 2\omega_{10}\zeta_{20}(1 - \mu_{30}x_1^2)x_2 + \omega_{10}^2x_1 - 2\xi_m\omega_mx_2 - \omega_m^2x_1.$ (21)

The suggested procedure can be followed. Figure 10 shows the magnitude plots for the linear systems which were fitted for the closed-loop system obtained applying the feedback (21) with the nominal parameters to the dynamics (18) with a parameter set from the grid. The multiplier of the uncertainty block Δ



Fig. 10. Frequency response of the fitted linear systems obtained for the feedback linearized Van der Pol oscillator using different parameter values



Fig. 11. Plant input (Van der Pol example)

reads

$$W(s) = \frac{0.0723s^2 + 4.423s + 2.1}{s^2 + 11.25s + 97.62}.$$
 (22)

For the weighted mixed sensitivity problem, we used the following model and performance weighting functions, M(s), $W_u(s)$ and $W_e(s)$, respectively:

$$M(s) = \frac{\omega_m^2}{s^2 + 2\xi_m \omega_m s + \omega_m^2} = G(s)$$
(23)

$$W_u(s) = \frac{20(s+10\omega_m)^2}{(s+200\omega_m)^2}$$
(24)

$$W_e(s) = \frac{1400\omega_m}{s + 2\omega_m} \tag{25}$$

The closed-loop behavior with the controller obtained as the solution of the associated weighted mixed sensitivity problem is shown on Figures 11 and 12 for a step reference signal.

VI. CONCLUSION

A procedure to deal with parameter uncertainty in the case of exact linearization of nonlinear systems is presented. The procedure aims to take into consideration the parameter uncertainties in a multiplicative structure and to ensure robust stability and performance by H_{∞} controller synthesis. The



Fig. 12. Plant output (Van der Pol example)

procedure has been demonstrated on first and second order systems.

The method involving only static inversion can be easily generalized for higher order systems if the nonlinearity involves only the input and/or the output.

Nonlinear inversion may generate quite large uncertainties for the linear model set which may impede the synthesis of a single H_{∞} controller with satisfactory performance. Hence the switching between different controllers designed for subsets of the linear uncertain models may be also possible similar to the multiple-model switching (MMS) method which is reported by Gao and al. [11].

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