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Linear analysis of lateral vehicle dynamics

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***Abstract*—Systematic analysis of lateral dynamics of a ground vehicle (e.g. passenger car) is presented in this paper. The results are based on the simplest possible single-track model. Effects of variations in the physical parameters - mass, the moment of inertia, tire priorities, vehicle geometry - on the response times, damping ratios, natural frequencies and other dynamical characteristics are presented and confronted with intuitive and "common sense" expectations and with real-life experience of race car drivers and constructors. We believe that such a report is quite unique and useful by itself: we are not aware of any similar existing report which would provide this systems-and-controls viewpoint on the vehicle dynamics phenomena. In addition, future plans are to apply modern systematic model-based control design approaches to come up with active dynamics modifications solutions - using for instance torque vectoring - surpassing current approaches based mainly on mechanical re-designs and, to some extent, simplest possible local feedback controllers.**

I. Introduction

From a historical point of view lot of knowledge of the dynamic behavior of the vehicle is known for mechanical engineers. However, this phenomenon was not covered from the control engineer point of view. Frequency characteristics are well known for mechanical engineers, but for example roots of the characteristic polynomial of the state space vehicle model and their location depending on the vehicle parameters were not well described.

Latest technological improvement of vehicles with electric drivetrain brings new opportunities for the vehicle control and stabilization. Understanding how the parameters affect the vehicle handling from the systems-and-controls viewpoint can lead to an improved control system of the vehicle. This work can also bring interesting information for newly interested in the vehicle dynamics systems.

This paper is organized as follows. Selection of existing tire models is introduced and described first in section II. Then the nonlinear single track vehicle model is derived in section III and the model is linearized afterward. Section IV is focused on linear analysis and dynamic properties of the developed linear vehicle model. Finally, in section V conclusion and future development and research plans are provided.

II. Tire models

Mathematical description of the interaction between the vehicle tires and the road surface is the biggest challenge of models and simulations describing vehicle behavior. Such models can evaluate longitudinal and lateral tire forces using vehicle states as input. Advanced tire characteristics and behavior can be found in [1] or [4].

The forces transferred by the tire in longitudinal and lateral direction are commonly expressed by slip curve. The example of slip curve is presented in Fig. 1. Only lateral tire forces are considered in this paper, therefore only sideslip angle to lateral force slip characteristic is considered from now on.

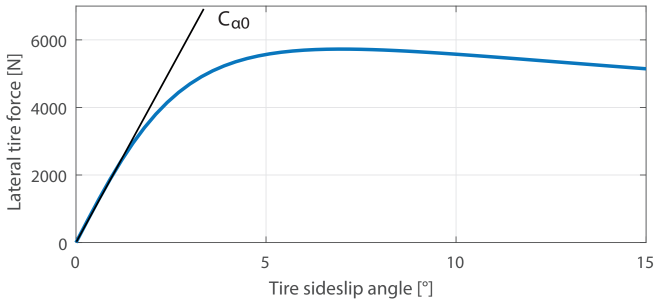

Fig. 1. Typical slip curve for lateral motion.

The initial slope at zero sideslip angle of the characteristics is called nominal cornering stiffness $C_{\alpha 0}$.

The sideslip angle of the tire is defined as

$$\alpha = \arctan\left(\frac{v_y}{v_x}\right), \tag{1}$$

where α is the sideslip angle, v_x and v_y are the velocities of the tire center in x and y direction of the tire coordination system.

For small sideslip angle α the lateral tire characteristics is linear and side force F_y is equal to sideslip angle multiplied by the nominal cornering stiffness. This characteristic is used in the linear tire model described later on. However, as the sideslip angle grows, the tire starts to be overloaded to the point where the slip curve reaches the maximum of the friction coefficient μ_{max}. With further increase of the slip angle, the tire is not able to transfer bigger forces F_y.

The lateral slip curve depends not only on tire characteristics but also on different conditions such as inflation of the tire, surface conditions (eg. dry/wet tarmac, snow, ice), the tire and the road temperatures. It also differs with varying normal load F_z. As the normal force grows, the maximal transferred side force F_y. It is assumed that normal forces depend only on the weight distribution of the car and that these forces are constant during the vehicle movement. In other words, neither

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longitudinal nor lateral load transfer is considered within this paper. The aerodynamic downforce of the vehicle is also neglected for the same purposes.

A. Pacejka 'Magic' formula

The widely used empirical formula of Hans Bastiaan Pacejka [3] was simplified into 4 main parameters (B,C,D and E) based on empirical measurements of the tire behavior. Longitudinal and lateral tire forces are computed independently for simpler implementation and representation of the formula.

The general simplified form of Pacejka's 'Magic' formula is:

$$F_y(\alpha) = D\cdot\sin\left(C \cdot \arctan\left(B\alpha - E(\,B\alpha - \arctan\left(B\alpha\right)\,)\right)\right), \tag{2}$$

where parameters B,C,D and E give the shape of the tire characteristics, F_y is lateral tire force and α is sideslip angle of the tire.

The same formula (with different empirical parameters) can be used for estimating the longitudinal tire force F_x if the sideslip angle is replaced with the slip of the tire λ and also for the aligning moment of the tire.

B. Linear tire model

Linear model is the simplest model of the tire. It is defined as

$$F_x(\lambda) = C_{\lambda 0}\lambda, \qquad F_y(\alpha) = C_{\alpha 0}\alpha, \tag{3}$$

where F_x is longitudinal tire force, F_y is lateral tire force, λ is tire slip, α is sideslip of the tire, $C_{\lambda 0}$ is tire slip coefficient and $C_{\alpha 0}$ is tire sideslip coefficient.

This model is accurate only when sideslip angle not far from 0. The model does not include the non-linear behavior of the tire. However, this model can be used for vehicle model linearization. This linearized model is later used in used section IV.

III. Single track model

Simple kinematic vehicle model is required for simpler control of the vehicle dynamics during steady state cornering. Only a planar motion of the vehicle is considered in vehicle single track model described within this paper. The vehicle center of gravity is projected into the plane of the surface in order to neglect the load transfer during the vehicle motion. Thus only one rotatory and one translatory degree of freedom is required to sufficiently estimate the vehicle state.

The vehicle coordinate system has to be defined first. The x axis of the vehicle points from the center of gravity towards the front of the vehicle and the y axis towards the right side of the vehicle from the driver's perspective. Finally, the z axis points towards the ground to follow commonly used a right-handed coordinate system.

The single track vehicle model describing planar vehicle motion is introduced in figure 2. The vehicle is moving with velocity v. The angle between the x axis of the vehicle and

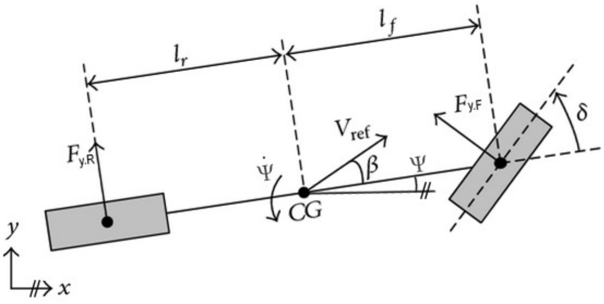

Fig. 2. Single track kinematic model of the vehicle.

the velocity vector is called vehicle sideslip angle β and is defined as

$$\beta = \arctan\left(\frac{v_y}{v_x}\right), \tag{4}$$

where v_x and v_y are the vehicle velocities in x and y direction of the vehicle coordinate frame respectively.

The differential equations of the vehicle model shown in figure 2 can be directly derived by creating equilibrium of all forces in the x (5) and y (6) vehicle direction and of all moments about the z axis (7) of the vehicle. It is assumed, that modeled vehicle is usual passenger car with front wheel steering only. As mentioned before, the aerodynamic forces are neglected.

$$\begin{aligned} &- m\dot{v}\cos(\beta) + mv(\dot\beta + \dot\psi)\sin(\beta) - F_{y,F}\sin(\delta) + \\ &\qquad + F_{x,F}\cos(\delta) + F_{x,R} = 0 \end{aligned} \tag{5}$$

$$\begin{aligned} &- m\dot{v}\sin(\beta) - mv(\dot\beta + \dot\psi)\cos(\beta) + F_{y,F}\cos(\delta) + \\ &\qquad + F_{x,F}\sin(\delta) + F_{y,R} = 0 \end{aligned} \tag{6}$$

$$-I_z\ddot\psi + F_{y,F}l_f\cos(\delta) - F_{y,R}l_r + F_{x,F}l_f\sin(\delta) = 0 \tag{7}$$

In differential equations of motion above m is the mass of the vehicle, v is the velocity of the vehicle, $\dot\psi$ is the yaw rate of the vehicle,I_z is moment of inertia about the z-axis, l_f and l_r are the distances of the center of the front and rear tire from the vehicle gravity center respectively, β is the sideslip angle of the vehicle, δ is the wheel steering angle, $F_{x,F}$ and $F_{x,R}$ are longitudinal forces of front and rear tire respectively, $F_{y,F}$ and $F_{y,R}$ are lateral forces of front and rear tire respectively. The change of the sideslip angle $\dot\beta$ is very small compared to the yaw rate $\dot\psi$, thus it can be neglected.

The tire forces $F_{y,F}$ and $F_{y,R}$ are defined within equations of selected tire model. The tire position and steering angle has significant impact on sideslip angles of tires, which are defined as

$$\alpha_F = \delta - \arctan\left(\frac{v\sin(\beta) + l_f\dot\psi}{v\cos(\beta)}\right), \tag{8}$$

$$\alpha_R = -\arctan\left(\frac{v\sin(\beta) - l_r\dot{\psi}}{v\cos(\beta)}\right), \tag{9}$$

where δ is front steering angle, v is vehicle velocity, $\dot{\psi}$ is the yaw rate, β is sideslip of the entire vehicle, l_f and l_r are the distances of the centre of the front and rear tire from the vehicle gravity centre respectively and α_F and α_R are sideslip angles of front and rear tire respectively.

The relations 8 and 9 can be rewritten for small steering angles as

$$\alpha_F = \delta - \beta - \frac{l_f\dot{\psi}}{v_x}, \tag{10}$$

$$\alpha_R = -\beta + \frac{l_r\dot{\psi}}{v_x}, \tag{11}$$

where δ is front steering angle, v is vehicle velocity, $\dot{\psi}$ is the yaw rate, β is sideslip of the entire vehicle and α_F and α_R are sideslip angles of front and rear tire respectively.

Assuming small steering angle δ and sideslip angles β the vehicle differential equations of motion 5, 6 and 7 can be then linearized as

$$-m\dot{v} + F_{x,F} + F_{x,R} = 0, \tag{12}$$

$$-mv(\dot{\beta} + \dot{\psi}) + F_{y,F} + F_{y,R} = 0, \tag{13}$$

$$-I_z\ddot{\psi} + F_{y,F}l_f - F_{y,R}l_r = 0, \tag{14}$$

where m is the vehicle mass, I_z is moment of inertia about the z-axis, v is vehicle velocity, δ is steering angle, β is sideslip angle, ψ is vehicle yaw rate, $F_{x,F}$ and $F_{y,F}$ are longitudinal and lateral forces of the front tire respectively and $F_{x,R}$ and $F_{y,R}$ are longitudinal and lateral forces of the rear tire respectively.

The sideslip angles of the front and rear tire are defined in equations 10 and 11 respectively. The lateral tire forces $F_{y,F}$ and $F_{y,R}$ can be estimated using selected tire model - for simplification the linear model is chosen.

It is assumed that acceleration of the vehicle $\dot{v}$ is equal to zero during the steady state cornering maneuver. The vehicle differential equations are after substitution of side forces $F_{y,F}$ and $F_{y,R}$ (see eq. 3) and using equations 10 and 11 transformed into following state space model

$$\begin{bmatrix}\dot{\beta}\\ \ddot{\psi}\end{bmatrix} = \mathbf{A}\begin{bmatrix}\beta\\ \dot{\psi}\end{bmatrix} + \begin{bmatrix}\frac{C_{\alpha0,F}}{mv}\\ \frac{C_{\alpha0,F}l_f}{Iz}\end{bmatrix}\delta \tag{15}$$

$$\mathbf{A} = \begin{bmatrix} -\frac{C_{\alpha0,F}+C_{\alpha0,R}}{mv} & -\left(1+\frac{C_{\alpha0,F}l_f - C_{\alpha0,R}l_r}{mv^2}\right)\\ -\frac{C_{\alpha0,F}l_f - C_{\alpha0,R}l_r}{I_z} & -\frac{C_{\alpha0,F}l_f^2 + C_{\alpha0,R}l_r^2}{I_z}\end{bmatrix} \tag{16}$$

where the vehicle state is represented by vehicle sideslip angle β and vehicle yaw rate $\dot{\psi}$.

IV. Linear analysis

In the previous section, the linear steady-state cornering model of the vehicle motion through the corner was derived. This simple model can be used designing the control systems for the vehicle which improves the vehicle stability or handling.

TABLE I
DEFAULT PARAMETRIZATION OF THE VEHICLE MODEL

Weight m	1500 kg
Vehicle speed v	15.5 m/s
Moment of inertia I_z	2000 kg $\cdot$ m^2
Vehicle length	3 m
Distance of front wheel and CG l_f	1.3 m
Distance of rear wheel and CG l_r	1.7 m
Nominal cornering stiffness of front tire $C_{\alpha0,F}$	100000 N/rad
Nominal cornering stiffness of rear tire $C_{\alpha0,R}$	120000 N/rad

Obtained linear model (eq. 15) is a simple second order linear state space model with one input (the steering angle of the front wheel δ) and two states (vehicle sideslip angle β and yaw rate $\dot{\phi}$). However, each of the physical parameters influences the static and dynamic characteristics of this model in a different way. It is, therefore, important for us as control designers to understand what are the impacts of variations in mass and geometric parameters of the vehicle to time constants, natural frequencies and damping ratios of the lateral model modes. This is the goal of this section, and one of the main contributions of the whole paper.

The parametrization of the vehicle model was selected to match a usual passenger car. One selected parameter is varied in each subsection. This simulation does not correspond with the reality, for example, if the weight of the vehicle is increased or the center of gravity is shifted forward it influences directly the cornering stiffness coefficient of the tire (via different normal load on each tire) and moment of inertia of entire vehicle.

The influence on the location of roots of the characteristic polynomial is shown and well described in each subsection. The time response of vehicle yaw rate to step change of the direction of the front wheels together with the Bode plot is shown. Each figure contains an arrow which shows the direction of change as the value of selected parameter increase.

A. Vehicle velocity

During the vehicle cornering, one of the main parameters which have a big influence on vehicle handling is the speed. The vehicle velocity v appears only in denominators of components of the system matrix A (eq. 16). Thus as the velocity increases, the poles of the system should be closer to zero.

The tendency mentioned before is shown in figure 3. As the vehicle velocity v increases, the poles of the system move towards zero (indicated by black arrow). It is possible to see, that at some point the poles become complex.

This behavior has another effect on vehicle handling which can be seen in time response of the system (fig. 5). As the vehicle velocity increases, the vehicle yaw rate $\dot{\phi}$ (steady state) also increases until a critical point. With further increase of the velocity, the yaw rate (steady state) decreases which results in an increase of radius of the corner.

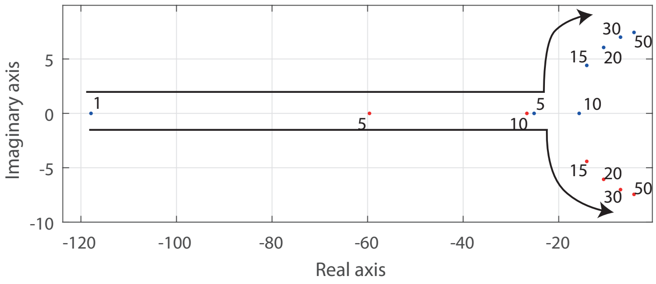

Fig. 3. Poles of linear vehicle model with increasing velocity.

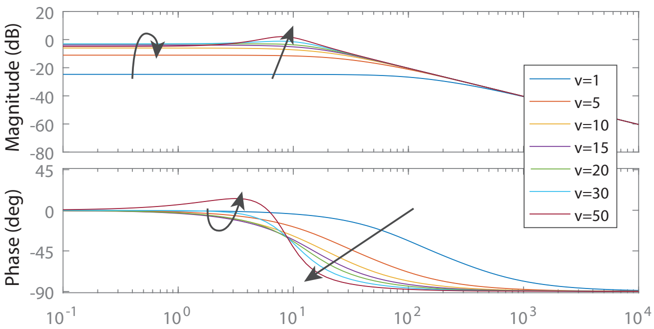

Fig. 4. Bode plot of linear vehicle model with increasing velocity.

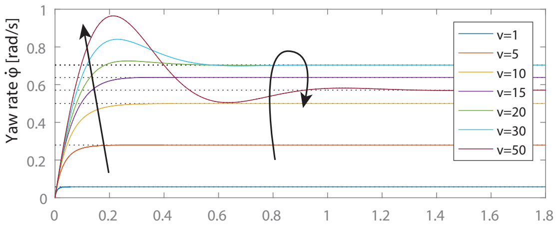

Fig. 5. Step response of linear vehicle model with increasing velocity.

B. Position of the centre of gravity

Another interesting parameter influencing the vehicle handling is the location of the center of gravity. In figures 6, 7 and 8 change of location of the centre of gravity is shown as increase of the distance between front wheel and centre of gravity lf. The length of the vehicle remains the same, thus the distance between rear wheel and center of gravity lr decreases. The vehicle velocity was set to $v = 50km/h$.

The common knowledge says that moving the center of gravity towards the front wheel forces the vehicle to have a quicker response and less rear wheel grip. Moving the center of gravity towards the rear axle does the opposite - less steering and more rear wheel grip.

This phenomenon can be seen in time response of the vehicle system in figure 11. The rising time decreases up to the moment, where the center of gravity is closer to the front wheel ($lf \ = 1.25m$). Then the rising time grows again.

The location and movement of poles are also different when

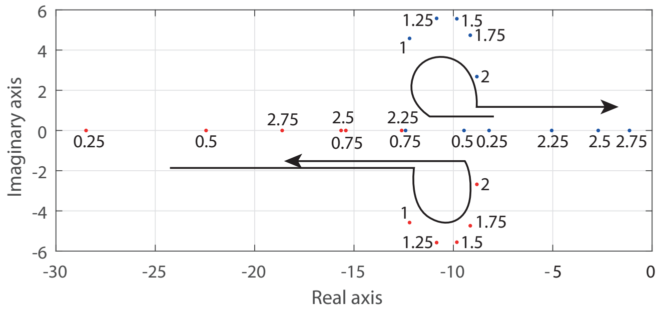

Fig. 6. Poles of linear vehicle model with increasing CG from front to rear wheel.

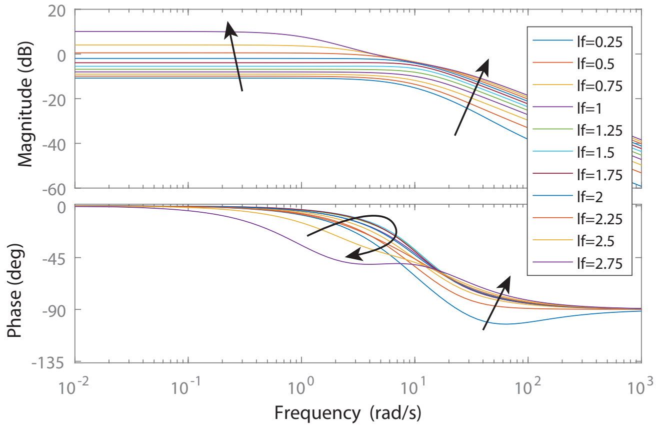

Fig. 7. Bode plot of linear vehicle model with increasing CG from front to rear wheel.

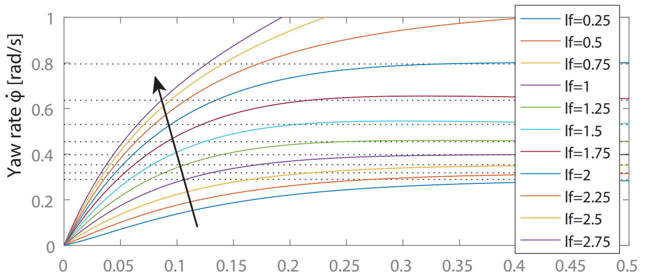

Fig. 8. Step response of linear vehicle model with moving CG from front to rear wheel.

the vehicle is moving with bigger velocity. If the center of gravity is close to the rear wheel the vehicle can become unstable in terms of the location of the poles. This behavior is shown in figures 9, 10 and 11.

C. Moment of inertia

The moment of inertia is varied in this subsection. This effect can be achieved by moving the vehicle engine and transmission from the center of the vehicle to the front and back. The vehicle center of gravity should remain in the same location.

The common practice of the sports vehicle design is to keep the moment of inertia of the entire vehicle as small as

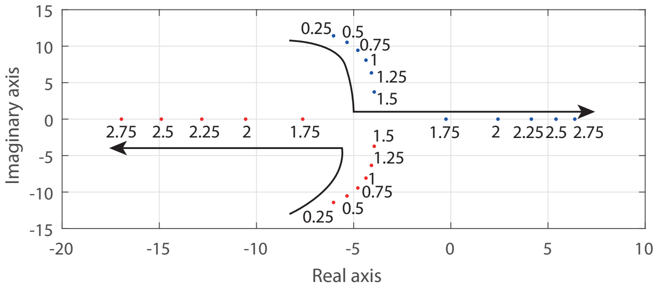

Fig. 9. Poles of linear vehicle model with increasing CG from front to rear wheel - higher velocity.

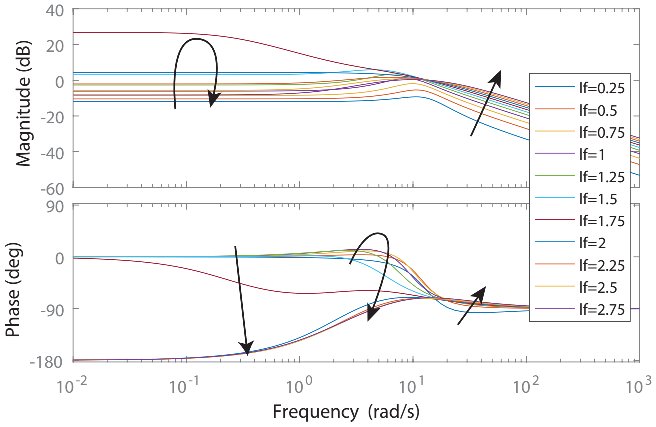

Fig. 10. Bode plot of linear vehicle model with increasing CG from front to rear wheel - higher velocity.

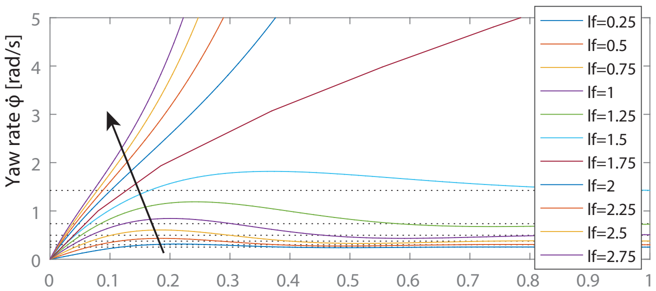

Fig. 11. Step response of linear vehicle model with moving CG from front to rear wheel - higher velocity.

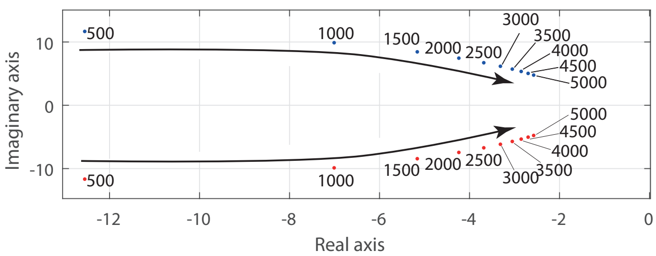

Fig. 12. Poles of linear vehicle model with different moment of inertia of the vehicle.

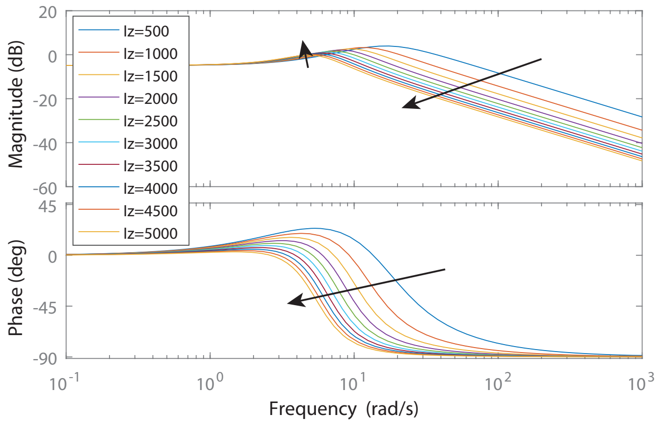

Fig. 13. Bode plot of linear vehicle model with different moment of inertia of the vehicle.

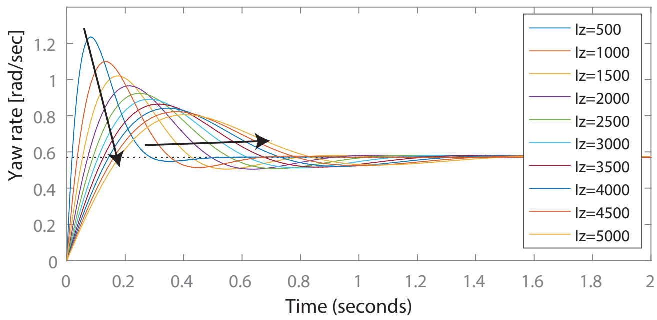

Fig. 14. Step response of linear vehicle model with different moment of inertia of the vehicle.

possible. The time response of the system (fig. 18) confirms this phenomenon. As the moment of inertia increases, the time response of the system output is slower.

D. Weight

Adding the weight usually change the moment of inertia of the vehicle. However, we assume in this subsection adding weight only to the vehicle center of gravity, thus the moment of inertia is not modified. This modification can be achieved by adding or removing some weight of the motor of a mid-engine vehicle since the motor is usually placed near the center of gravity of the mid-engine vehicle.

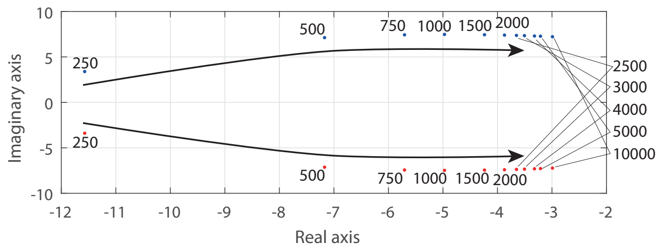

Fig. 15. Poles of linear vehicle model with increasing weight.

Additional vehicle weight added into the vehicle center of gravity results in the smaller yaw rate and less dumped yaw

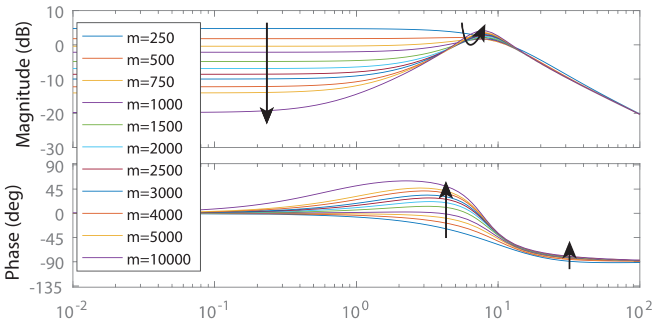

Fig. 16. Bode plot of linear vehicle model with increasing weight.

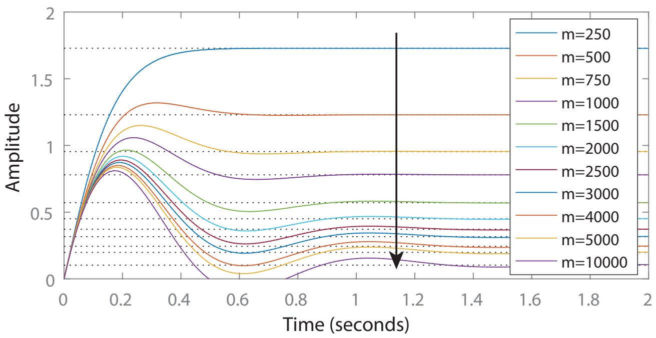

Fig. 17. Step response of linear vehicle model with increasing weight.

rate response of the vehicle. This phenomenon corresponds with common sense - as the vehicle gains weight it becomes less steerable. If we remove some weight, we can achieve bigger yaw rate and quicker vehicle response for the same steering angle.

E. Surface conditions

The change of the surface conditions has a big influence on lateral vehicle behavior. The difference in tire friction coefficient is studied in this subsection. Other parameters should remain the same.

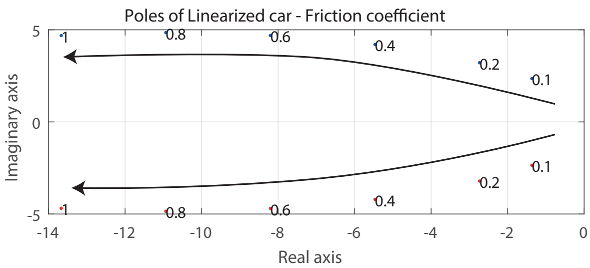

Fig. 18. Poles of linear vehicle model for different tire friction coefficient.

As the friction coefficient grows, the tires have more grip and are able to transfer bigger lateral forces and the vehicle has bigger yaw rate. Since the vehicle speed is set to the constant value, this increment of yaw rate leads to a reduce of the cornering radius.

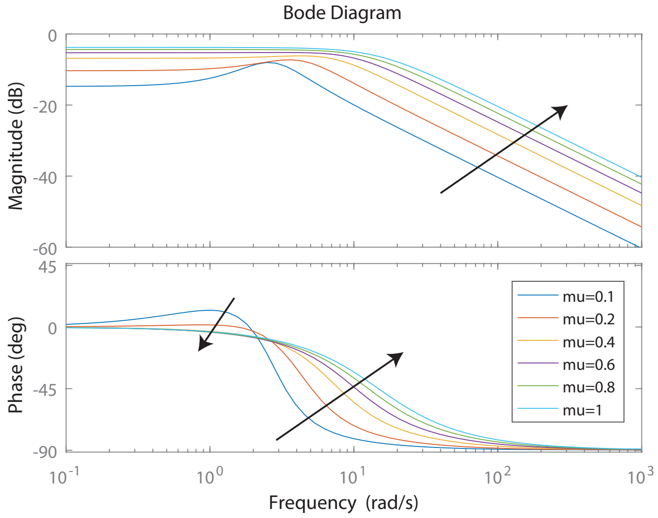

Fig. 19. Bode plot of linear vehicle model with increasing tire friction coefficient.

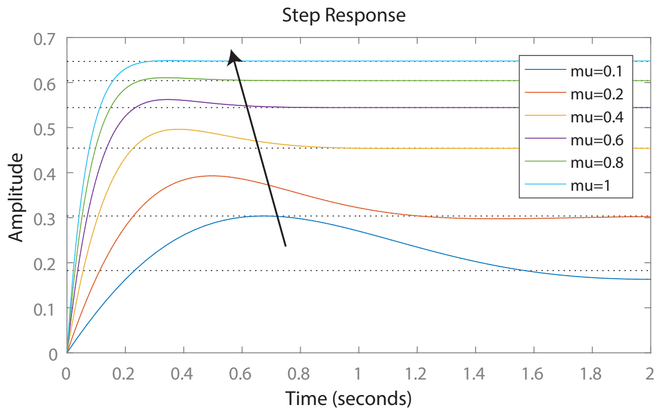

Fig. 20. Step response of linear vehicle model with increasing tire friction coefficient.

V. Conclusion

The main focus of this paper was to show some interesting behavior of the single track vehicle model from the system-and-control point of view. In section III the single track model of the vehicle was developed and linearized into steady-state cornering model. In section IV all main parameters of this linear steady-state cornering model were varied. The impact on the time response of vehicle yaw rate and roots of characteristic polynomial was analyzed and described.

This work can provide useful information to all people interested in vehicle dynamics, mainly the students of undergraduate control engineer courses. This work is also the starting project of the new group of vehicle dynamics and control at the Faculty of Electrical engineering, Czech technical university in Prague.

The paper will be further extended by analysis of variations of mutually dependent parameters of the linear vehicle model. Finally based on all results of linear analysis fully functional torque vectoring control system for the vehicle with electric drivetrain should be developed.

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