

# Modelling of Ball and Plate System Based on First Principle Model and Optimal Control

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**Abstract**—This paper presents modelling of ball and plate systems based on first principles by considering balance of forces and torques. A non-linear model is derived considering the dynamics of motors, gears, ball and plate. The non-linear model is linearized near the operating region to obtain a standard state space model. This linear model is used for discrete optimal control of the ball and plate system – the trajectory of the ball is controlled by control voltages to the motor.

**Keywords**—Ball and plate system; first principle model; optimal control; LQ control

## I. INTRODUCTION

Laboratory experiments are an integral part of control education. There are lots of educational platforms available e.g. inverted pendulum, magnetic levitation, ball and beam system etc. Ball and plate system is an upgraded version of ball and beam system where the position of the ball can be manipulated in two directions [1]. The educational model ball and plate, consists of a plate, pivoted at its center, such that the slope of the plate can be manipulated in two perpendicular directions. The ball and plate is a non-linear, multi-variable and open-loop unstable system. There are basically two control problems: point stabilization and trajectory tracking. In point stabilization, the aim is to carry the ball to a specific position and hold it there. In trajectory tracking control, the goal is to make the ball follows a predefined trajectory (linear, square, circle and Lissajous curve) [2-5].

The first step is finding out a mathematical model which describes the system. There are basically two modelling approaches for the ball and plate system in literature: the Lagrangian method and the Newton-Euler method. The derivation of dynamical equations of ball and plate system by Lagrangian method can be seen in [6]. The modelling based on Newton-Euler method is quite rare in the literature. Even though, Newton-Euler method is quite cumbersome, the variables and equations have physical meaning which is suitable for control educational purpose. The balance of forces and torques are considered in the Newton-Euler method to derive the mathematical model.

In this paper, a non-linear mathematical model of ball and plate system is derived by considering the dynamics of the ball and plate system, DC motors and gear system, based on balance of forces and torques. The model is linearized around equilibrium points to arrive at a linear state space model. Infinite

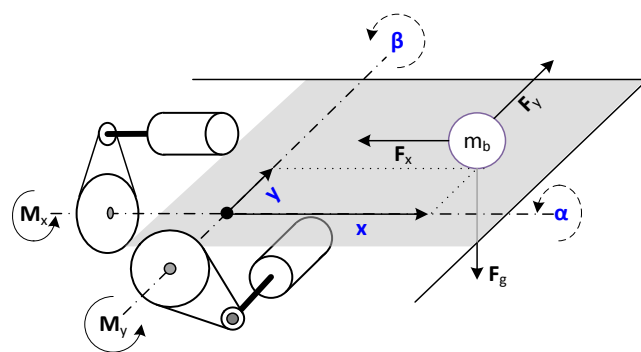


Fig. 1. Ball and plate system

horizon optimal linear quadratic (LQ) control is applied to the trajectory tracking problem by penalizing the state and control effort. Simulation results of model verification and trajectory tracking control are also provided.

## II. MATHEMATICAL MODELLING

Mathematical model of nonlinear dynamics takes account of the position of the ball on the plate depending on the voltage of the motors that control the tilt of the plate, in two perpendicular axes (see Fig. 1). A ball of mass  $m_b$ , moment of inertia  $J_b$  and radius  $r_b$  is located on a square surface (plane), tilting in the two perpendicular axes  $x$  and  $y$ . The origin of the axis is located at the intersection of coordinate axes. The moment of inertia of the plate is  $J_p$  (relative to each axis of rotation). On each axis, the torques  $M_x$  and  $M_y$  are operated. Moments are created by two DC geared motors via two cable systems with the same gear ratio of  $G$ , but with different moments of inertia  $J_{Gx}$  and  $J_{Gy}$ .

The mathematical model is based on the balance of forces and torques acting on the ball, and the dynamic model of DC series motor. The real behaviour is taken into account by including an approximation of linear mechanical losses, depending on the speed of the rotational motion. In the case of moving balls, the mechanical losses are proportional to the square of opposition translational speed of movement.

The model is built on the following assumptions:

- there is no loss of contact area with the ball
- the ball is hollow ball (ping-pong ball)
- an infinitely large area (not considering the rebound)
- connection of the motor to tilt the axis is perfectly rigid

### A. Balance of forces - ball

Since the ball is placed on a surface rotating in two axes, we need to consider, in addition to the inertia of the translational movement, the influence of the apparent forces (Euler, centrifugal and Coriolis). These forces are caused by the rotational movement i.e. all the forces in curvilinear motion consisting of the movement of translation and rotation. A general representation of the resultant forces acting upon the curvilinear motion on the mass point in vector form ( $\vec{v}$  is the translational velocity vector and  $\vec{\omega}$  is the angular velocity vector) is then described [7],

$$\vec{F}_{ext} = \underbrace{m \frac{d\vec{v}}{dt}}_{\text{Inertia force } \vec{F}_{acc}} + \underbrace{m \frac{d\vec{\omega}}{dt} \times \vec{r}}_{\text{Euler force } \vec{F}_{Eul}} + \underbrace{2m\vec{\omega} \times \frac{d\vec{r}}{dt}}_{\text{Coriolis force } \vec{F}_{Cor}} + \underbrace{m\vec{\omega} \times (\vec{\omega} \times \vec{r})}_{\text{Centrifugal force } \vec{F}_{cen}}. \quad (1)$$

All the apparent forces acting in a plane are perpendicular to the axis of rotation. Fig. 2 shows a situation where all the forces are acting in the plane of motion. The mass point is at a perpendicular distance  $r$  from the rotational axis, and the vector (line connecting the point and the axis of rotation) forms an angle  $\alpha$  from the selected  $x$  axis. The Euler force  $F_{Eul}$  (also apparent inertia force) acts only when there is a change in the speed of rotation  $\omega$  and its direction is perpendicular to the vector. Centrifugal force  $F_{cen}$  acts at non-zero rotation speed and its direction is in the direction of the vector. Coriolis force  $F_{Cor}$  acts when the velocity of motion is not perpendicular to the vector (i.e. there is a change in the size of the vector). Its direction is perpendicular to the direction of translational speed.

In the arrangement (rotational axis in the coordinate axes), the forces are decomposed in the directions of axes  $x$  and  $y$  directions. The translational velocity  $v_x, v_y$  and rotation speed  $\omega_x$  and  $\omega_y$  are either parallel or perpendicular to each other. Decomposition of the axes describes the movement in one direction only i.e. the apparent Euler and Coriolis force is applied only in the balance of moments. The situation of moving in the  $x$ -axis is shown in Fig. 3. Since it is not a fixed point we need to consider more ball spin (moment of inertia) with speeds of rotation  $\omega_x$  and  $\omega_y$ , loss due to rolling resistance and environment resistance. Rolling resistance is proportional to the rotational speed of the ball and resistance is proportional to the square of environment translational velocity. The overall balance of forces can be expressed as,

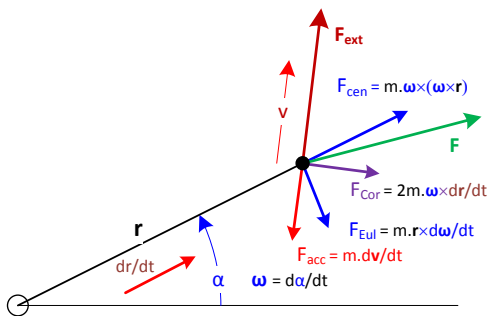


Fig. 2. Forces acting on ball and plate

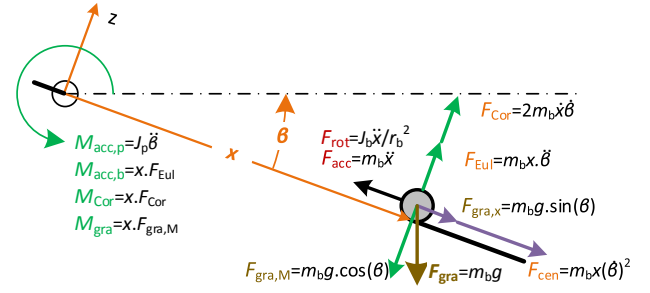


Fig. 3. Ball and plate – movement in the  $x$ -axis ( $x$ - $z$  plane) and rotation (torque) acting in  $y$ -axis

$$F_{acc} + F_{rot} + F_{los} + F_{Eul} + F_{Cor} + F_{cen} = F_{gra}(\alpha, \beta) \quad [kg \cdot m \cdot s^{-2}]$$

$$\begin{aligned} \text{Translational force} \quad F_{acc} &= m_b \frac{d^2x}{dt^2} \\ \text{Rotational force} \quad F_{rot} &= \frac{J_b}{r_b} \frac{d\omega_x}{dt} = \frac{J_b}{r_b^2} \frac{d^2x}{dt^2} \\ \text{Rolling resistance} \quad F_{los} &= k_b \omega_x = \frac{k_b}{r_b} \frac{dx}{dt} \\ \text{Environmental resistance} \quad F_{los} &= c_x \frac{1}{2} \rho_{air} S_b \frac{dx}{dt} \left| \frac{dx}{dt} \right| = k_c \frac{dx}{dt} \left| \frac{dx}{dt} \right| \\ \text{Centrifugal force} \quad F_{cen} &= m_b \omega_\beta^2 x = m_b x \left( \frac{d\beta}{dt} \right)^2 \\ \text{External force (gravity)} \quad F_{gra,x} &= -m_b g \cdot \sin(\beta) \end{aligned}$$

where,

$m_b$	Mass of the ball
$r_b$	Radius of the ball
$S_b$	Total area $S_b = \pi r_b^2$
$\Delta$	Thickness of the ball
$J_b$	Torque of ball $J_b = \frac{2}{5} m_b (2r_b - \Delta) \Delta$
$\omega_x, \omega_y$	Angular velocity $x$ -axis $\frac{dx}{dt} = r_b \omega_x$
$k_b$	Coefficient approximation rolling losses
$c_x=0.5$	Coefficient of aerodynamic resistance ball
$\rho_{air}=1.2$	Density of dry air at 20 ° C and a pressure of 101.4 kPa
$k_c = c_x \rho_{air} \pi r_b^2 / 2$	Coefficient of resistance of the environment
$g=9.81$	Gravitational constant

By considering balance of forces on  $x$  axis we get,

$$m_b \frac{d^2x}{dt^2} + \frac{J_b}{r_b^2} \frac{d^2x}{dt^2} + \frac{k_b}{r_b} \frac{dx}{dt} + k_c \frac{dx}{dt} \left| \frac{dx}{dt} \right| + m_b x \left( \frac{d\beta}{dt} \right)^2 = -m_b g \cdot \sin(\beta)$$

$$\left( 1 + \frac{J_b}{m_b r_b^2} \right) \frac{d^2x}{dt^2} + \frac{dx}{dt} \left( \frac{k_b}{m_b r_b} + \frac{k_{cx}}{m_b} \left| \frac{dx}{dt} \right| \right) + x \left( \frac{d\beta}{dt} \right)^2 = -g \cdot \sin(\beta) \quad (2)$$

Similarly, by considering balance of forces on  $y$  axis we get,

$$m_b \frac{d^2y}{dt^2} + \frac{J_b}{r_b^2} \frac{d^2y}{dt^2} + \frac{k_b}{r_b} \frac{dy}{dt} + k_c r_b^2 \frac{dy}{dt} \left| \frac{dy}{dt} \right| + m_b y \left( \frac{d\alpha}{dt} \right)^2 = -m_b g \cdot \sin(\alpha)$$

$$\left( 1 + \frac{J_b}{m_b r_b^2} \right) \frac{d^2y}{dt^2} + \frac{dy}{dt} \left( \frac{k_b}{m_b r_b} + \frac{k_{cy}}{m_b} \left| \frac{dy}{dt} \right| \right) + y \left( \frac{d\alpha}{dt} \right)^2 = -g \cdot \sin(\alpha) \quad (3)$$

### B. Balance of moments - plate with ball

Overall the balance of moments can be expressed as,

$$M_{acc,p} + M_{acc,b} + M_{Cor,b} + M_{los} = M_{mot} - M_{gra} \quad [kg \cdot m^2 \cdot s^{-2}]$$

Torque of plate	$M_{acc,p} = (J_p + J_{Gx}) \frac{d^2\alpha}{dt^2}$
Torque of ball	$M_{acc,b} = y \cdot F_{Eul} = m_b y^2 \frac{d^2\alpha}{dt^2}$
Coriolis moment	$M_{Cor,b} = y \cdot F_{Cor} = 2m_b y \frac{dy}{dt} \frac{d\alpha}{dt}$
Torque losses	$M_{los} = k_{px} \frac{d\alpha}{dt}$
Gravitational moment	$M_{gra} = -m_b y \cdot g \cos(\alpha)$

where,

$J_p$	Plate moment of inertia (MI) $J_p = \frac{1}{12} m_p a^2$
$m_p$	Mass of the plate
$a$	Pivot length, passes through the center axis of
$J_{Gx}, J_{Gy}$	Pivot length, passes through the center axis
$k_{px}, k_{py}$	Coefficient of approximation of rotational losses
$x, y$	Current position of the ball
$M_{Gx}, M_{Gy}$	Actual moments of the drives
$\alpha, \beta$	Current the angle of the platform according to the respective axes

By considering balance of forces on x axis we get,

$$(J_p + J_{Gx}) \frac{d^2\alpha}{dt^2} + m_b y^2 \frac{d^2\alpha}{dt^2} + 2m_b y \frac{dy}{dt} \frac{d\alpha}{dt} + k_{px} \frac{d\alpha}{dt} = M_{Gx} - m_b y \cdot g \cos(\alpha)$$

$$\left(\frac{J_p + J_{Gx}}{m_b} + y^2\right) \frac{d^2\alpha}{dt^2} + \left(2y \frac{dy}{dt} + \frac{k_{px}}{m_b}\right) \frac{d\alpha}{dt} = \frac{M_{Gx}}{m_b} - y \cdot g \cos(\alpha) \quad (4)$$

Similarly, by considering balance of torques on y axis we get,

$$(J_p + J_{Gy}) \frac{d^2\beta}{dt^2} + m_b x^2 \frac{d^2\beta}{dt^2} + 2m_b x \frac{dx}{dt} \frac{d\beta}{dt} + k_{py} \frac{d\beta}{dt} = M_{Gy} - m_b x \cdot g \cos(\beta)$$

$$\left(\frac{J_p + J_{Gy}}{m_b} + x^2\right) \frac{d^2\beta}{dt^2} + \left(2x \frac{dx}{dt} + \frac{k_{py}}{m_b}\right) \frac{d\beta}{dt} = \frac{M_{Gy}}{m_b} - x \cdot g \cos(\beta) \quad (5)$$

### C. Gear system

The gear box reduces the angular velocities of motors to output angular velocities with respect to the gear ratio. Similarly, the torques of motors are increased to output torques.

$$\alpha = \frac{1}{G} \varphi_x \quad \beta = \frac{1}{G} \varphi_y$$

$$M_{Gx} = G \cdot M_x \quad M_{Gy} = G \cdot M_y \quad (6)$$

where,

$G$	Gear transmission ratio of the drive
$\varphi_x, \varphi_y$	Angle of rotation of the rotor
$\alpha, \beta$	Angle of rotation of the plate with respect to the relevant axis

### D. Balance of energy and moment – motor

An equivalent circuit of an ideal DC series motor, is shown in Fig. 4. It consists of resistance  $R$ , inductance  $L$  and magnetic field  $M$ . Each motor is independently controlled by its own supply voltage  $U_x, U_y$  taken from a common voltage source  $U_0$  through control signal  $u_x, u_y$ . The rotor generated back electromotive force (EMF) is in reverse polarity and is proportional to the rotor angular velocity. The torque of the motor is proportional to the current  $i$ .

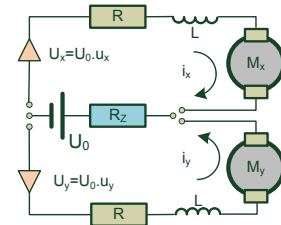


Fig. 4. Equivalent circuit of DC motor

By considering the balance of voltages (Kirchhoff's law) of the motor connected to x axis,

$$R \cdot i_x + L \frac{di_x}{dt} + k_u \Phi \frac{d\varphi_x}{dt} + R_z(i_x + i_y) = u_x \cdot U_0$$

$$L \frac{di_x}{dt} + k_u \Phi \cdot G \frac{d\alpha}{dt} + (R_z + R)i_x + R_z i_y = u_x \cdot U_0 \quad (7)$$

Similarly, by considering balance of voltages on motor connected to y axis,

$$R \cdot i_y + L \frac{di_y}{dt} + k_u \Phi \frac{d\varphi_y}{dt} + R_z(i_x + i_y) = u_y \cdot U_0$$

$$L \frac{di_y}{dt} + k_u \Phi \cdot G \frac{d\beta}{dt} + (R_z + R)i_y + R_z i_x = u_y \cdot U_0 \quad (8)$$

where,

$R$	Resistance of motor winding
$R_z$	Internal source resistance
$L$	Inductance of motor winding
$\Phi = 1$	Magnetic flux constant
$k_u$	Speed constant (voltage) of the motor
$i_x, i_y$	Current of the motor

By considering the balance of moments - moment of inertia  $M_s$ , rotational resistance proportional to rotational speed (mechanical losses)  $M_o$  and load torque  $M_x$  caused by magnetic field which is proportional to current.

$$J_m \frac{d^2\varphi_x}{dt^2} + k_o \frac{d\varphi_x}{dt} + M_x = k_m \Phi \cdot i_x$$

Substituting (6) to the above equation gives,

$$\frac{M_{Gx}}{m_b} = \frac{G \cdot k_m \Phi}{m_b} i_x - \frac{J_m G^2}{m_b} \frac{d^2\alpha}{dt^2} - \frac{k_o G^2}{m_b} \frac{d\alpha}{dt} \quad (9)$$

Similarly, considering balance of torques on motor connected to y axis,

$$J_m \frac{d^2\varphi_y}{dt^2} + k_o \frac{d\varphi_y}{dt} + M_y = k_m \Phi \cdot i_y$$

Substituting (6) gives to the above equation gives,

$$\frac{M_{Gy}}{m_b} = \frac{G \cdot k_m \Phi}{m_b} i_y - \frac{J_m G^2}{m_b} \frac{d^2\beta}{dt^2} - \frac{k_o G^2}{m_b} \frac{d\beta}{dt} \quad (10)$$

where,

$J_m$	Moment of inertia of rotating parts of the motor
$k_m$	Torque constant of motor
$k_o$	Coefficient of rotational loss of motor
$M_x, M_y$	Current load torque of motors
$\varphi_x, \varphi_y$	Current angle of rotors

### E. Combined model

The dynamical equations of ball and plate, gears and DC motors, by substituting (9-10) to (4-5), are given by,

$$\left(1 + \frac{J_b}{m_b r_b^2}\right) \frac{d^2 x}{dt^2} + \frac{dx}{dt} \left(\frac{k_b}{m_b r_b} + \frac{k_{cx}}{m_b} \left|\frac{dx}{dt}\right|\right) + x \left(\frac{d\beta}{dt}\right)^2 = -g \cdot \sin(\beta) \quad (11)$$

$$\left(1 + \frac{J_b}{m_b r_b^2}\right) \frac{d^2 y}{dt^2} + \frac{dy}{dt} \left(\frac{k_b}{m_b r_b} + \frac{k_{cy}}{m_b} \left|\frac{dy}{dt}\right|\right) + y \left(\frac{d\alpha}{dt}\right)^2 = -g \cdot \sin(\alpha) \quad (11)$$

$$\left(\frac{J_p + J_{Gx} + J_m G^2}{m_b} + y^2\right) \frac{d^2 \alpha}{dt^2} + \left(2y \frac{dy}{dt} + \frac{k_{px}}{m_b} + \frac{k_o G^2}{m_b}\right) \frac{d\alpha}{dt} = \frac{G \cdot k_m \Phi}{m_b} i_x - y \cdot g \cos(\alpha) \quad (12)$$

$$\left(\frac{J_p + J_{Gy} + J_m G^2}{m_b} + x^2\right) \frac{d^2 \beta}{dt^2} + \left(2x \frac{dx}{dt} + \frac{k_{py}}{m_b} + \frac{k_o G^2}{m_b}\right) \frac{d\beta}{dt} = \frac{G \cdot k_m \Phi}{m_b} i_y - x \cdot g \cos(\beta) \quad (12)$$

$$L \frac{di_x}{dt} + k_u \Phi \cdot G \frac{d\alpha}{dt} + (R_z + R) i_x + R_z i_y = u_x \cdot U_0 \quad (13)$$

$$L \frac{di_y}{dt} + k_u \Phi \cdot G \frac{d\beta}{dt} + (R_z + R) i_y + R_z i_x = u_y \cdot U_0 \quad (13)$$

By substituting the following parameters,

$$a_1 = 1 + \frac{J_b}{m_b r_b^2} \quad a_2 = \frac{k_b}{m_b r_b} \quad a_3 = \frac{k_{cx}}{m_b}$$

$$b_{1x} = \frac{J_p + J_{Gx} + J_m G^2}{m_b} \quad b_{2x} = \frac{k_{px}}{m_b} + \frac{k_o G^2}{m_b} \quad b_3 = \frac{G \cdot k_m \Phi}{m_b}$$

$$b_{1y} = \frac{J_p + J_{Gy} + J_m G^2}{m_b} \quad b_{2y} = \frac{k_{py}}{m_b} + \frac{k_o G^2}{m_b} \quad d_0 = \frac{U_0}{L}$$

$$d_1 = \frac{R_z}{L} \quad d_2 = \frac{R + R_z}{L} \quad d_3 = \frac{k_u}{L} \Phi \cdot G$$

The dynamics governing the ball and plate system becomes,

$$a_1 \frac{d^2 x}{dt^2} + \frac{dx}{dt} \left(a_2 + a_3 \left|\frac{dx}{dt}\right|\right) + x \left(\frac{d\beta}{dt}\right)^2 = -g \cdot \sin(\beta) \quad (14)$$

$$a_1 \frac{d^2 y}{dt^2} + \frac{dy}{dt} \left(a_2 + a_3 \left|\frac{dy}{dt}\right|\right) + y \left(\frac{d\alpha}{dt}\right)^2 = -g \cdot \sin(\alpha) \quad (15)$$

$$(b_{1x} + y^2) \frac{d^2 \alpha}{dt^2} + \left(2y \frac{dy}{dt} + b_{2x}\right) \frac{d\alpha}{dt} = b_3 i_x - y \cdot g \cos(\alpha) \quad (16)$$

$$(b_{1y} + x^2) \frac{d^2 \beta}{dt^2} + \left(2x \frac{dx}{dt} + b_{2y}\right) \frac{d\beta}{dt} = b_3 i_y - x \cdot g \cos(\beta) \quad (17)$$

$$\frac{di_x}{dt} + d_3 \frac{d\alpha}{dt} + d_2 i_x + d_1 i_y = d_0 u_x \quad (18)$$

$$\frac{di_y}{dt} + d_3 \frac{d\beta}{dt} + d_2 i_y + d_1 i_x = d_0 u_y \quad (19)$$

### III. LINEAR STATE SPACE MODEL

The non-linear dynamic equations (14-19) can be linearized around operating points  $(x_0, y_0)$  by assuming the following approximation:

1. At small angles of plate inclination:  $\sin \theta \approx \theta$ ,  $\cos \theta \approx 1$
2. At small rate of change and at initial conditions:  $v_x = \alpha = \omega_x = i_x = v_y = \beta = \omega_y = i_y = 0$

The linearized model can be represented in standard state space model in the form of,

$$\frac{d\bar{x}}{dt} = \bar{A}\bar{x} + \bar{B}u \quad (20)$$

$$y = \bar{C}\bar{x}$$

where,  $\bar{x} = [x \ v_x \ \alpha \ \omega_x \ i_x \ y \ v_y \ \beta \ \omega_y \ i_y]^T$

$$\bar{B} = \begin{bmatrix} 0 & 0 & 0 & 0 & d_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_0 \end{bmatrix}^T$$

$$\bar{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

TABLE I. PARAMETERS OF BALL AND PLATE SYSTEM

	Symbol	Unit	Value	Description
Ball	$g$	m.s <sup>-2</sup>	9.81	Gravitational constant
	$m_b$	kg	0.01	Mass of the ball (ping pong)
	$r_b$	m	0.02	Radius of the ball
	$\Delta$	m	0.001	Thickness of the ball
	$J_b$	kg.m <sup>2</sup>	$J_b = \frac{2}{5} m_b (2r_b - \Delta) \Delta$	MI of hollow ball
	$k_b$	kg.m.s <sup>-1</sup>	0.01	Coefficient of friction (ball) $F = k_b \omega_x$
	$c_x$	---	0.5	Coefficient of aerodynamic resistance ball
	$\rho_{air}$	kg.m <sup>-3</sup>	1.2	Air density
	$k_c$	kg.m <sup>-1</sup>	$k_c = \frac{1}{2} c_x \rho_{air} \pi r_b^2$	Approximation of resistance of environment
Plate	$m_p$	kg	0.4	Mass of plate
	$a$	m	0.5	Length of the pivot
	$J_p$	kg.m <sup>2</sup>	$J_p = \frac{1}{12} m_p a^2$	MI of the plate
	$k_{px}$	kg.m <sup>2</sup> .s <sup>-1</sup>	0.1	Approx of loss (platform x-axis) $M = k_{px} \frac{d\alpha}{dt}$ (estimate)
	$k_{py}$	kg.m <sup>2</sup> .s <sup>-1</sup>	0.1	Approx of loss (platform y-axis) $M = k_{py} \frac{d\beta}{dt}$ (estimate)
Drive	$J_{Gx}$	kg.m <sup>2</sup>	$J_p/3$	MI of drive (x axis) (estimate)
	$J_{Gy}$	kg.m <sup>2</sup>	$J_p/3$	MI of drive (y axis) (estimate)
	$G$	---	10	Gear ratio
Src	$U_0$	V	12	Nominal voltage of motor = Source voltage
	$R_z$	$\Omega$	0.05	Internal source resistance
Motor	$J_m$	kg.m <sup>2</sup>	45e-7	MI of rotor
	$L$	H	1.2e-3	Inductance of motor
	$\omega_0$	rad.s <sup>-1</sup>	4550* $\pi/30$	Motor ideal speed
	$i_0$	A	0.15	No load current of motor
	$M_s$	kg.m <sup>2</sup> .s <sup>-2</sup>	0.13	Moment of motor at still
	$i_s$	A	2.5	Current of motor at still
	$R$	$\Omega$	$R = \frac{U_0}{i_s}$	Winding resistance
	$k_m$	kg.m <sup>2</sup> .s <sup>-2</sup> .A <sup>-1</sup>	$k_m \Phi = \frac{M_s}{i_s}$	Torque constant of motor $M = k_m \Phi i$
	$k_u$	V.s.rad <sup>-1</sup>	$k_u \Phi = \frac{U_0 (i_s - i_0)}{i_s \omega_0}$	Rate constant of motor $U = k_u \Phi \frac{d\omega}{dt} = k_u \Phi \omega$
	$k_o$	kg.m <sup>2</sup> .s <sup>-1</sup> .rad <sup>-1</sup>	$k_o = \frac{M_s i_0}{i_s \omega_0}$	Approximation of loss (motor) $M = k_o \frac{d\omega}{dt} = k_o \omega$

$$\bar{\mathbf{A}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{a_2}{a_1} & 0 & 0 & 0 & 0 & 0 & -\frac{g}{a_1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{b_{2x}}{b_{1x}+y_0^2} & \frac{b_3}{b_{1x}+y_0^2} & -\frac{g(b_{1x}-y_0^2)}{(b_{1x}+y_0^2)^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -d_3 & -d_2 & 0 & 0 & 0 & 0 & -d_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{g}{a_1} & 0 & 0 & 0 & -\frac{a_2}{a_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -\frac{g(b_{1y}-x_0^2)}{(b_{1y}+x_0^2)^2} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{b_{2y}}{b_{1y}+x_0^2} & \frac{b_3}{b_{1y}+x_0^2} & 0 \\ 0 & 0 & 0 & 0 & -d_1 & 0 & 0 & 0 & -d_3 & -d_2 \end{bmatrix}$$

The model parameters used in the simulation is listed in Table 1. The linearized model, (20) is discretized with a sampling time of  $T_s = 0.1s$  and compared with the continuous time dynamic model (15-19) by applying a series of step control voltages. Fig. 5 shows the control voltages and ball positions in x and y axes of linear and non-linear model. Plate angles and motor currents are shown in Fig. 6. Since the system is open loop unstable and has integrating character, the quality of linearization has to be finally checked by closed loop experiments.

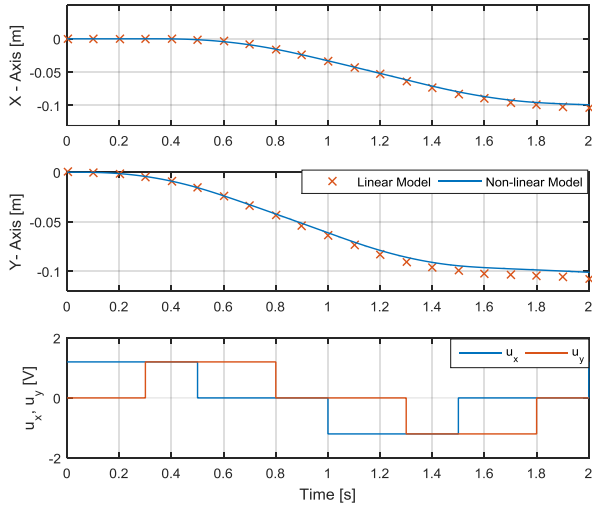


Fig. 5. Non-linear vs Linear model: outputs and inputs in open loop verification

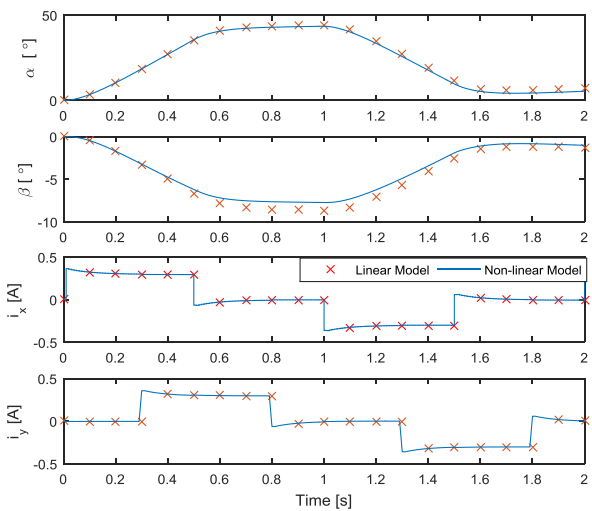


Fig. 6. Plate angles and motor currents in open loop verification

#### IV. OPTIMAL CONTROL OF BALL AND PLATE SYSTEM

Discretizing the state space model (20) with a sampling time  $T_s$  we get,

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k) \end{aligned} \quad (21)$$

With the linear state space model, an optimal LQ controller can be designed for the ball and plate system. The aim of the controller is generating optimal control voltages by minimizing the following criteria,

$$J_\infty = \sum_{i=1}^{\infty} [\mathbf{x}^T(k+i)\mathbf{Q}\mathbf{x}(k+i) + \mathbf{u}^T(k+i)\mathbf{R}\mathbf{u}(k+i)]$$

The cost function consists of penalization (weighting matrix  $\mathbf{Q}$ ) of state variables and control effort (weighting matrix  $\mathbf{R}$ ). If the state variables are able to be estimated, the optimal control actions can be calculated by,

$$\mathbf{u}(k) = -\mathbf{K}[\mathbf{x}(k) - \mathbf{x}_w(k)] \quad (22)$$

Where  $\mathbf{x}_w$  is the desired state variable for reference point at time  $t = t_k$  and  $\mathbf{K}$  is feedback gain matrix obtained by the following equation,

$$\mathbf{K} = (\mathbf{B}^T\mathbf{P}\mathbf{B} + \mathbf{R})^{-1}\mathbf{B}^T\mathbf{P}\mathbf{A}$$

The matrix  $\mathbf{P}$  is the solution of discrete Riccati equation which is given by,

$$\mathbf{P} = \mathbf{A}^T\mathbf{P}\mathbf{A} + \mathbf{Q} - \mathbf{A}^T\mathbf{P}\mathbf{B}(\mathbf{B}^T\mathbf{P}\mathbf{B} + \mathbf{R})^{-1}\mathbf{B}^T\mathbf{P}\mathbf{A}$$

In MATLAB, the feedback gain can be obtained by,

$$[\mathbf{K}, \sim, \sim] = \text{dlqr}(\mathbf{A}, \mathbf{B}, \mathbf{Q}, \mathbf{R})$$

Simulation experiments were conducted on two different trajectories: square shaped and Lissajous curve shaped trajectory. The model parameters used in the simulation are as listed in Table I, with a sampling period of  $T_s = 0.1s$ . The weighting matrices are chosen as follows,

$$\begin{aligned} \mathbf{Q} &= \text{eye}(10)/10 & \mathbf{R} &= \text{eye}(2)/10 \\ \mathbf{Q}(1,1) &= 100 & \mathbf{Q}(6,6) &= 100 \\ \mathbf{Q}(2,2) &= 100 & \mathbf{Q}(7,7) &= 10 \end{aligned}$$

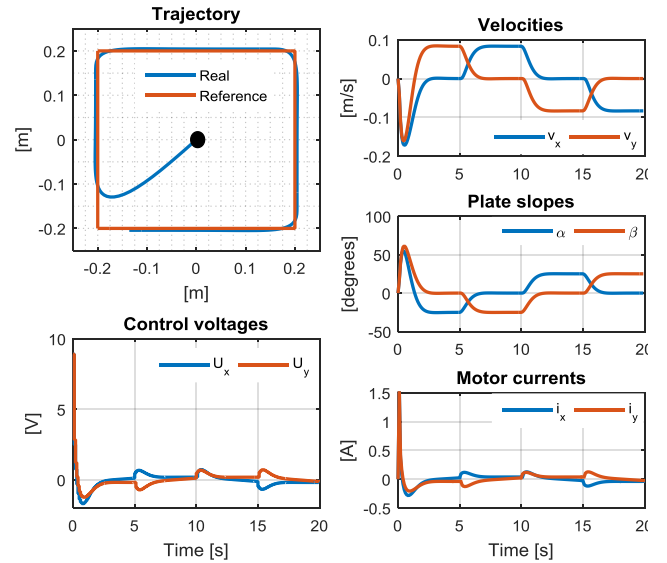


Fig. 7. Square trajectory: control voltages, velocities, plate slopes and currents

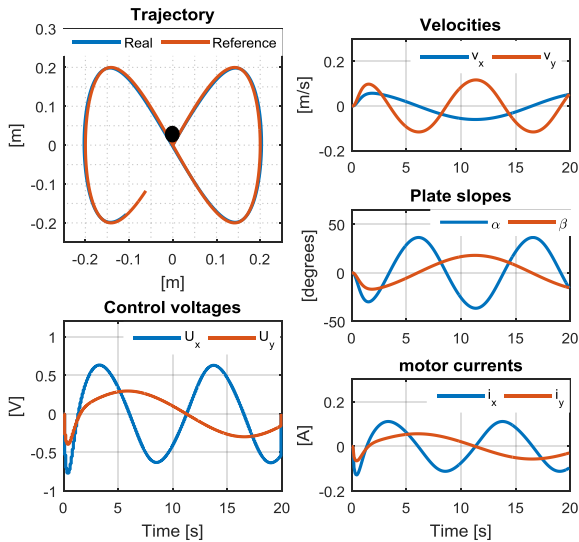


Fig. 8. Lissajous curve trajectory: control voltages, velocities, plate slopes and currents

Fig. 7 shows the simulation results of LQ control trajectory tracking, with a trajectory in the shape of square. Control voltages, ball velocities, plate slopes and motor currents are also shown. The initial location of the ball was at origin and was different from initial reference point. The controller was able to track the ball to the reference trajectory points. Fig. 8 shows the simulation results of the Lissajous curve shaped trajectory. The simulation experiments with both the trajectories show the quality of linearized model, which is derived from the non-linear model, is good for control purposes.

## V. CONCLUSION

The mathematical model of ball and plate system is derived by Newton-Euler method – considering balance of forces and torques of ball and plate, motors and gears. The non-linear

model is linearized around operating points following some approximation. Simulation of open loop model verification is performed. The linearized model is used to discrete optimal LQ control of the trajectory tracking problem of ball and plate system. The simulation result proves the quality of linearization of non-linear model.

## ACKNOWLEDGMENT

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## REFERENCES

- [1] Humusoft (1996 – 2014), CE151 Ball & Plate Apparatus: User's manual.
- [2] Nokhbeh, Mohammad, and Daniel Khashabi, "Modelling and Control of Ball-Plate system," Diss. Ph.D. thesis, Amirkabir University of Technology, 2011.
- [3] Jadlovská, A., Š. Jajčíšin, and R. Lonščák. "Modelling and PID control design of nonlinear educational model Ball & Plate," Proceedings of the 17th International Conference on Process Control. Vol. 9, 2009.
- [4] Oravec, Matej, and Anna Jadlovská, "Model Predictive Control of a Ball and Plate laboratory model," Applied Machine Intelligence and Informatics (SAMi), 2015 IEEE 13th International Symposium on. IEEE, 2015.
- [5] Liu, Dejun, Yantao Tian, and Huida Duan, "Ball and plate control system based on sliding mode control with uncertain items observe compensation," Intelligent Computing and Intelligent Systems, 2009. ICIS 2009. IEEE International Conference on, Vol. 2, IEEE, 2009.
- [6] Hauser, John, Shankar Sastry, and Petar Kokotovic, "Nonlinear control via approximate input-output linearization: The ball and beam example," IEEE transactions on automatic control vol. 37.3, pp. 392-398, 1992.
- [7] J. Šrejtr, Technická mechanika II. Kinematika 1.část.SNTL Praha, pp. 256, 1954.