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Digital control of Ball & Plate model using LQ controller

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***Abstract*—This paper proposes the design of linear quadratic (LQ) digital controller in Ball & Plate model in experimental environment. The non-linear mathematical model of Ball & Plate structure is presented and adequately linearized. Polynomial approach to controller design for two degrees of freedom (2DOF) controller structure is introduced as the main tool for determination of unknown parameters. This method requires placing poles of characteristic polynomial, which are semi-optimally determined using minimization of linear quadratic criterion. This criterion is minimized by spectral factorization with the aid of the Polynomial Toolbox for MATLAB. Experiments have proved that this type of controller is able to stabilize the ball in desired position on the plate, reject external disturbances and follow reference path without much effort. In addition, a simple maze was created on the plate to extend possibilities of the choice of reference signal. The algorithm is able to determine correct path through the maze and navigate the ball along this path.**

I. Introduction

The Ball & Plate model has two inputs and two outputs, integrating properties and is suitable for testing designed control algorithms for unstable processes in real environment. The polynomial approach to controller design is used in this paper, as it simplifies the design problem to operations on algebraic polynomial (Diophantine) equations [1]. Controller parameters are derived by minimizing linear quadratic (LQ) criterion, which leads to semi-optimal solution (half of poles of characteristic polynomial have to be user-defined) [2]. This method can be applied to various controller structures and this paper discusses the use of 2 degrees of freedom (2DOF) controller structure, which provides separation of feed-back part (responsible for stabilization and disturbance rejection) and feed-forward part (responsible for reference tracking) [3]. The same real model was used in [4], where PID/PSD control in closed-loop feedback structure was applied. Butterworth, Graham-Lathrop and Naslin's methods were used for calculating controller parameters and results were promising. A double feedback loop structure based on fuzzy logic is tested in [5]. Fuzzy supervision and sliding control are proposed in [6] and a non-linear switching is described in [7].

The paper is organized as follows. A brief description of mathematical model of the Ball & Plate structure is in Section II. The design of LQ controller is shown in Section III. Section IV contains results of simulation and experiments on the real model. Section V concludes the paper.

II. Ball & Plate Mathematical Model

The mathematical model of the real system has to be acquired in the first place, thus initial assumptions are needed to simplify and linearize the model used for controller design.

A. Initial Assumptions

It is assumed that servomotors of the model are not influenced by the motion of the plate or the ball. Thus the mathematical model can be divided into ball-plate model and servo motor model separately. The Ball & Plate part is shown in Fig. 1.

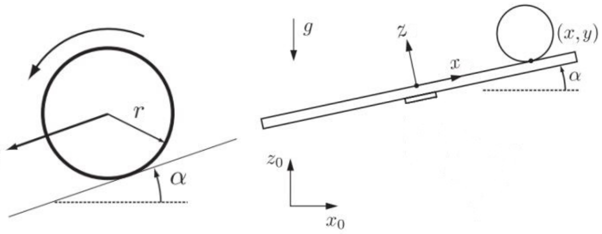

Fig. 1. Ball & Plate mathematical setup [8]

Other assumptions taken into account are:

- No slip between the ball and the plate,
- perfect contact between the ball and the plate,
- friction is omitted (e.g. from air or ball-plate contact),
- the ball is an ideal spherical shell,
- the plate has no boundaries.

B. Nonlinear System Equations

The derivation of system equations starts with general form of Euler-Lagrange equation of the second kind [9]:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i \qquad (1)$$

where T is kinetic energy of the system, V is potential energy, Q_i is i-th generalized force and q_i is i-th generalized coordinate. This system has 4 generalized coordinates in total (two ball position coordinates x, y and two plate inclinations α, β). Forces acting on the system are gravitational force and forces in the form of torque acting on the plate τ_x and τ_y. The derivation of specific equations from (1) is not the purpose of

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this paper, thus only final result will be presented. This result consists of a system of 4 ordinary second-order differential equations, but equations describing inclination angles α and β can be omitted, because it is assumed that stepper motors do not lose any step and load does not affect their performance, thus these angles are direct system inputs and the model is simplified to two equations of coordinates x and y:

$$x\colon \left(m + \frac{I_b}{r^2}\right)\ddot{x} - m\left(\dot{\alpha}\dot{\beta}y + \dot{\alpha}^2 x\right) + mg\sin\alpha = 0 \quad (2)$$

$$y\colon \left(m + \frac{I_b}{r^2}\right)\ddot{y} - m\left(\dot{\alpha}\dot{\beta}x + \dot{\beta}^2 y\right) + mg\sin\beta = 0 \quad (3)$$

where m, r and I_b are mass, radius and moment of inertia of the ball respectively, g is gravitational acceleration, α, β are plate angles (α changes x coordinate and β changes y coordinate), $\dot{\alpha}$, $\dot{\beta}$ are first time derivatives of plate angles, x, y are coordinates of the ball from center of the plate and $\ddot{x}$, $\ddot{y}$ are second time derivatives of ball coordinates.

C. Linearized Model

For small angles of the plate, one can write $\sin\alpha \approx \alpha$ and $\sin\beta \approx \beta$. It is also assumed that the rate of change in plate inclination is small around the linearization point, thus $\dot{\alpha}\dot{\beta} \approx 0$, $\dot{\alpha}^2 \approx 0$ and $\dot{\beta}^2 \approx 0$. The moment of inertia of a sphere (or a hollow sphere - spherical shell) can be ideally expressed as $I_{sphere} = \frac{2}{5}mr^2 \left(I_{shell} = \frac{2}{3}mr^2\right)$. These simplifications applied to (2) and (3) result in

$$x\colon \quad \ddot{x} = K_b\alpha \quad (4)$$

$$y\colon \quad \ddot{y} = K_b\beta \quad (5)$$

where K_b is constant dependent only on the gravitational acceleration g and the type of ball (whether it is spherical shell or sphere).

Because servo motors dynamics was neglected, it will be approximated by first-order transfer function

$$G_m(s) = \frac{K_m}{\tau_m s + 1} \quad (6)$$

where $K_m = 0.1878$ and $\tau_m = 0.187$ are gain and time constants of the motor respectively. These constants were obtained from model's manual pages [10].

It is obvious that the problem is symmetric. It is then possible to express the mathematical model (Ball & Plate model together with approximated model of servo motors) in one continuous transfer function $G(s)$ with generalized coordinate as output $Y(s)$:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K}{s^2(\tau_m s + 1)} = \frac{K}{\tau_m s^3 + s^2} \quad (7)$$

where $K = K_b K_m C_x$ is velocity gain of the integrating system ($C_x = -\frac{1}{0.2} = -5m^{-1}$ is conversion coefficient from meters to normalized coordinates specific for the real model).

Equation (7) can be discretized into:

$$G(z^{-1}) = \frac{B(z^{-1})}{A(z^{-1})} = \frac{b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}} \quad (8)$$

where $B(z^{-1})$ and $A(z^{-1})$ are polynomials with unknown coefficients. Since Ball & Plate model has double integrator, discrete transfer function (8) can be simplified to:

$$G(z^{-1}) = \frac{b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{(1 - z^{-1})^2 (1 - c_1 z^{-1})} \quad (9)$$

where c_1 is the only unknown pole of the discrete transfer function.

III. LQ Controller Design

A. Control Law

The controller is designed for two degree of freedom (2DOF) closed-loop control system shown in Fig. 2, where G is controlled plant, C_f and C_b are feed-forward and feed-back parts of the controller respectively, $1/K(z^{-1}) = 1/(1 - z^{-1})$ is the summation part of the controller (it is extracted from denominators of C_f and C_b for practical reasons), $w(k)$ is reference signal, $y(k)$ is output of the system, $u(k)$ is output of the controller, $n(k)$ is load disturbance and $v(k)$ is disturbance signal. It is assumed that no disturbances act on the system. This is obviously not true for real system, but it simplifies the design and structure of the controller. As mentioned, the

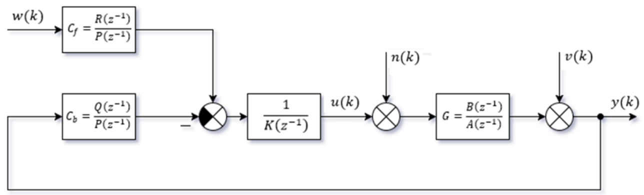

Fig. 2. Structure of 2DOF controller

controller is designed using polynomial approach. By taking signals from Fig. 2 in their discrete forms (and omitting z^{-1} in polynomials' notation), one can write

$$Y(z^{-1}) = \frac{BR}{AKP + BQ} W(z^{-1}) \quad (10)$$

The characteristic polynomial $D(z^{-1})$ can be extracted from (10) creating a Diophantine equation:

$$D = AKP + BQ \quad (11)$$

All polynomials in transfer functions will be called by their respective letter from now on, because omitting "(z^{-1})" will simplify the notation. Degree of polynomials Q, R and P can be determined by choosing the degree of the characteristic polynomial D, as described in [2], from where it should be 6 for this specific case:

$$D = \sum_{i=0}^{6} d_i z^{-i} \quad (12)$$

thus controllers C_b and C_f are

$$C_b(z^{-1}) = \frac{Q}{P} = \frac{q_0 + q_1 z^{-1} + q_2 z^{-2} + q_3 z^{-3}}{1 + p_1 z^{-1} + p_2 z^{-2}} \quad (13)$$

$$C_f(z^{-1}) = \frac{R}{P} = \frac{r_0}{1 + p_1 z^{-1} + p_2 z^{-2}} \tag{14}$$

where Q and P are polynomials with unknown coefficients, computed from (11) by method of undetermined coefficients. Polynomial R has one unknown coefficient r_0, which can be calculated for step-changing signal (see [2]) as:

$$r_0 = \frac{d_0 + d_1 + d_2 + d_3 + d_4 + d_5 + d_6}{b_1 + b_2 + b_3} = \sum_{i=0}^{3} q_i \tag{15}$$

B. Minimization of LQ Criterion

It is possible to obtain a semi-optimal solution by minimizing linear quadratic (LQ) criterion, which has the following form, as described in [2]:

$$J = \sum_{k=0}^{\infty} \left\{ [e(k)]^2 + q_u [u(k)]^2 \right\} \tag{16}$$

where $e(k) = w(k) - y(k)$ is error, $u(k)$ is controller output and q_u is penalization constant, which influences the controller output during minimization process. Standard minimization of LQ criterion is conducted in state-space description and leads to the solution of algebraic Riccati equation. According to [2], this criterion can be also minimized for input-output description of the model by applying spectral factorization on the following equation:

$$A(z^{-1}) q_u A(z) + B(z^{-1}) B(z) = D(z^{-1}) \delta D(z) \tag{17}$$

where δ is chosen so that coefficient $d_0 = 0$ for the sake of simplification. Spectral factorization for polynomials with degree higher than 2 has no analytical solution and has to be solved numerically by iterative methods. The Polynomial Toolbox for MATLAB [11] can be used for this task, specifically using function `spf.m`. The result of this spectral factorization offers 3 roots of characteristic polynomial (12) that are optimal. Remaining 3 roots (poles) have to be user-defined. These user-defined poles could be all equal to zero to obtain a fully optimal solution, but that would make the controller less robust, thus these 3 user-defined poles will be non-zero (actually they need to be placed closer to a unit circle on the Z-plane to bound the output of the controller). By choosing appropriate poles, coefficients of the polynomial (12) can be obtained and unknown coefficients of polynomial Q, P and R can be computed from (11) and (15).

IV. Identification and Experimental Results

The real model, which will be used for experiments is Ball & Plate model CE151 from Humusoft s.r.o. [10] with camera for tracking the position of the ball. Because the algorithm will be applied to real model, it was properly identified. The ball was placed to the center of the plate and after step-changing plate inclination ball's position was recorded. Multiple measurements were taken and averaged to compensate errors of measurement. Parameters K and τ_m in model (7) were identified by minimizing the sum of squared errors between measurement and unknown model. Resulting model in continuous form is:

$$G(s) = \frac{5.7402}{s^2\,(0.1877s + 1)} \tag{18}$$

Its discrete form follows the general discrete transfer function (8). The sampling period T_s was based on the limitation of camera's sampling frequency of 10 fps, thus the continuous transfer function was discretized for $T_s = 0.1s$:

$$G(z^{-1}) = \frac{0.00449z^{-1} + 0.01579z^{-2} + 0.00344z^{-3}}{1 - 2.5870z^{-1} + 2.1741z^{-2} - 0.5870z^{-3}} \tag{19}$$

Three optimal poles were computed from (17) for $q_u = 1$: $0.8391 \pm 0.1491i$ and 0.5811. Remaining poles were defined: 0.8, 0.8 and 0.88. These 6 poles are roots of polynomial (12). Zeros and poles of discrete transfer function (19) are shown in Fig. 3.

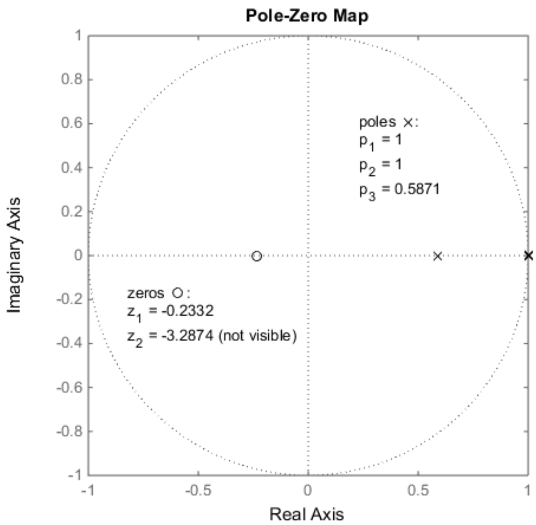

Fig. 3. Pole-zero map of the system

Coefficients of controller were computed from (11) and (15) and substituted into (13) and (14):

$$C_b = \frac{2.2372 - 5.5540z^{-1} + 4.5040z^{-2} - 1.1831z^{-3}}{1 - 1.1796z^{-1} + 0.4187z^{-2}} \tag{20}$$

$$C_f = \frac{0.0041}{1 - 1.1796z^{-1} + 0.4187z^{-2}} \tag{21}$$

Designed controller is used for both inclination angles of the plate (α and β), because it is still assumed that the system is symmetric. This assumption was supported by experimental results, which showed that measured data were similar with relatively minor differences. Fig. 4 shows results of tracking of reference signal, which was set to $(x, y) = (0, 0)$, which corresponds to the center of the plate.

As can be seen from plots of α and β, the angle generated by the controller is not equal to zero (which obviously should be, because the ball is steady only when the plate is in horizontal position). This is caused by unmeasurable errors of the real model. Its constructional design has flaws which cause stepper motors to lose steps. This ultimately leads to errors in the output that controller compensates by generating non-zero angle of the plate. Each initialization of the model removes this problem, so that the effect will not stack. However, the error of angle moved from 0 to almost half of the maximum value in only few seconds, which is not very convenient. Fig. 5 shows the same measurement, but on the x-y plane as seen by user.

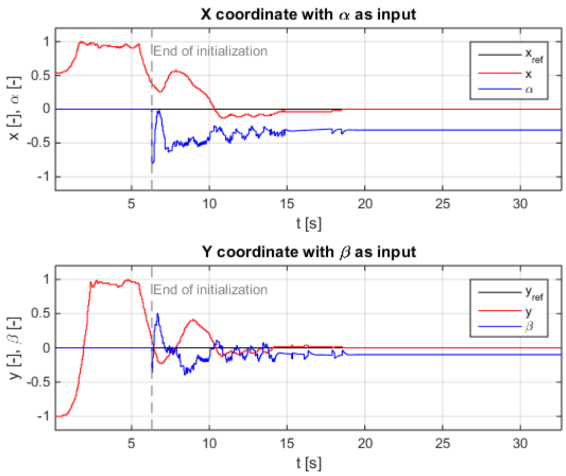

Fig. 4. Reference tracking

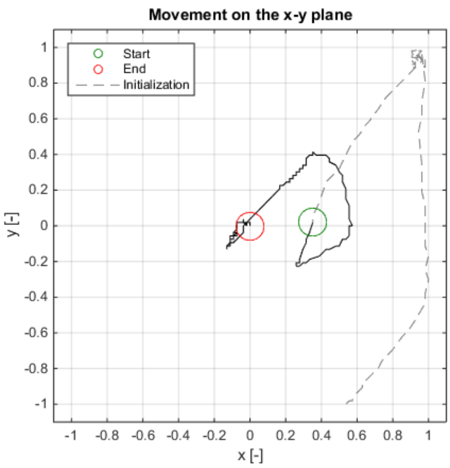

Fig. 5. Reference tracking on x-y plane

In the next step, the disturbance was introduced after the ball stabilized in the same reference value $(0, 0)$ as in previous case. This disturbance was created by continuous blowing to the ball from different directions, which caused random movements of the ball. The controller successfully compensated this disturbance, which can be easily seen in Fig. 6 and Fig. 7 (the error is within 4 cm radius or 0.2 in normalized units).

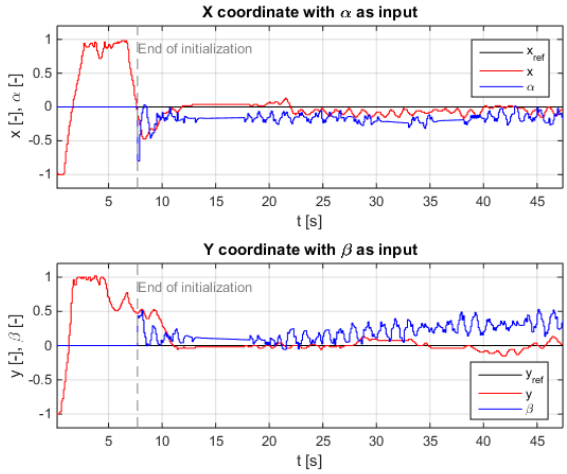

Fig. 6. Disturbance rejection

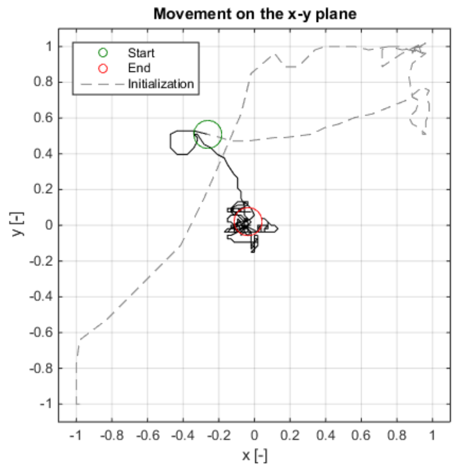

Fig. 7. Disturbance rejection on x-y plane

A simple maze was also constructed on the plate using colored tape (Fig. 8), therefore making the choice of reference value more interesting and challenging.

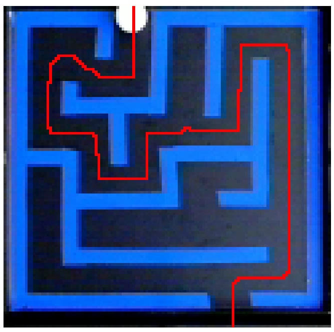

Fig. 8. Maze and computed path

Correct path through maze is determined and transformed into set of reference values. Controller navigates the ball from start at the top to exit in the bottom of Fig. 8. The resulting position of the ball and outputs of controllers are shown in Fig. 9 and position of the ball in x-y plane is in Fig. 10.

It is obvious that the ball diverts from its path mostly near edges of the plate, because the model is most reliable and rigid in the center. It is worth noting that maze walls have basically no height, hence are only 2D. Their purpose is purely visual and they provide no support for the ball whatsoever.

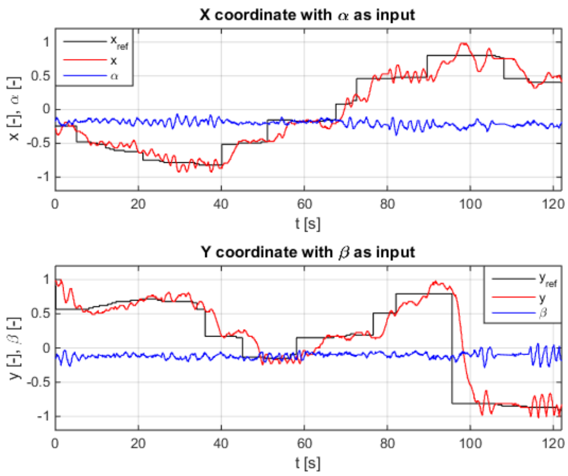

Fig. 9. Maze navigation

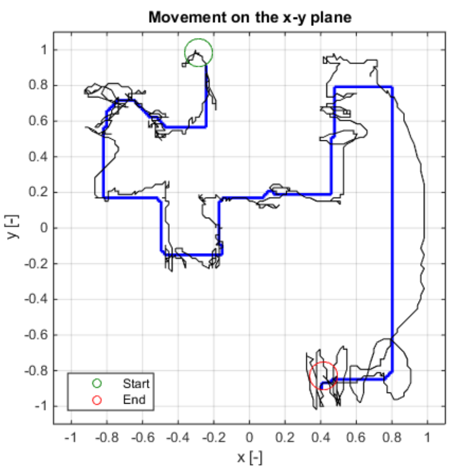

Fig. 10. Maze navigation on x-y plane

The method used for path determination is so-called watershed transform, which considers a grayscale image as a topographical relief. In image processing, it was introduced as a tool for segmenting grayscale images by S. Beucher and C. Lantujoul in the late 70s [12]. The `watershed` function in MATLAB [13] detects these segmented regions and outputs them in a matrix.

Fig. 11 shows control with harmonic reference signal, resulting in circular path of the ball (Fig. 12). Time range of the measurement is greater, because the controller was designed for step-changing reference value and harmonic signal in this scale can be essentially considered a sequence of steps (this property drops with rising frequency of harmonic signal). The design of the controller for harmonic change would increase its order and complexity. The structure of the real model also plays a significant role in the design, as the system is quite prone to elastic deformations, which causes undesirable vibrations during continuous changes in plate angle.

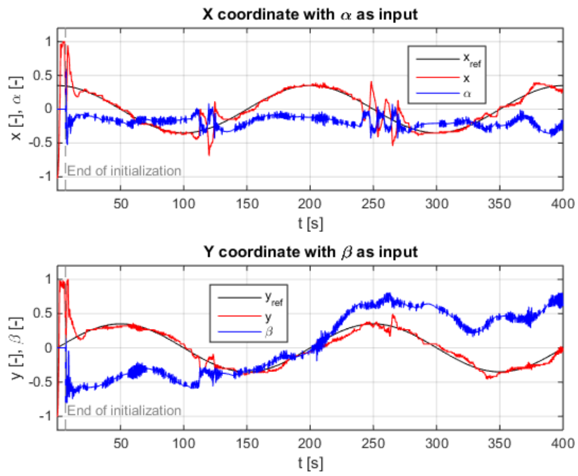

Fig. 11. Circular reference tracking

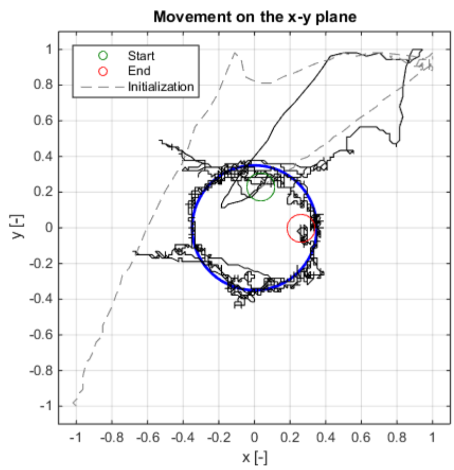

Fig. 12. Circular reference tracking on x-y plane

A simple graphical user interface (GUI) was designed to encapsulate control algorithms and schemes in MATLAB/Simulink environment. It provides several options and plots to quickly analyze desired trajectories and model behavior, as shown in Fig. 13.

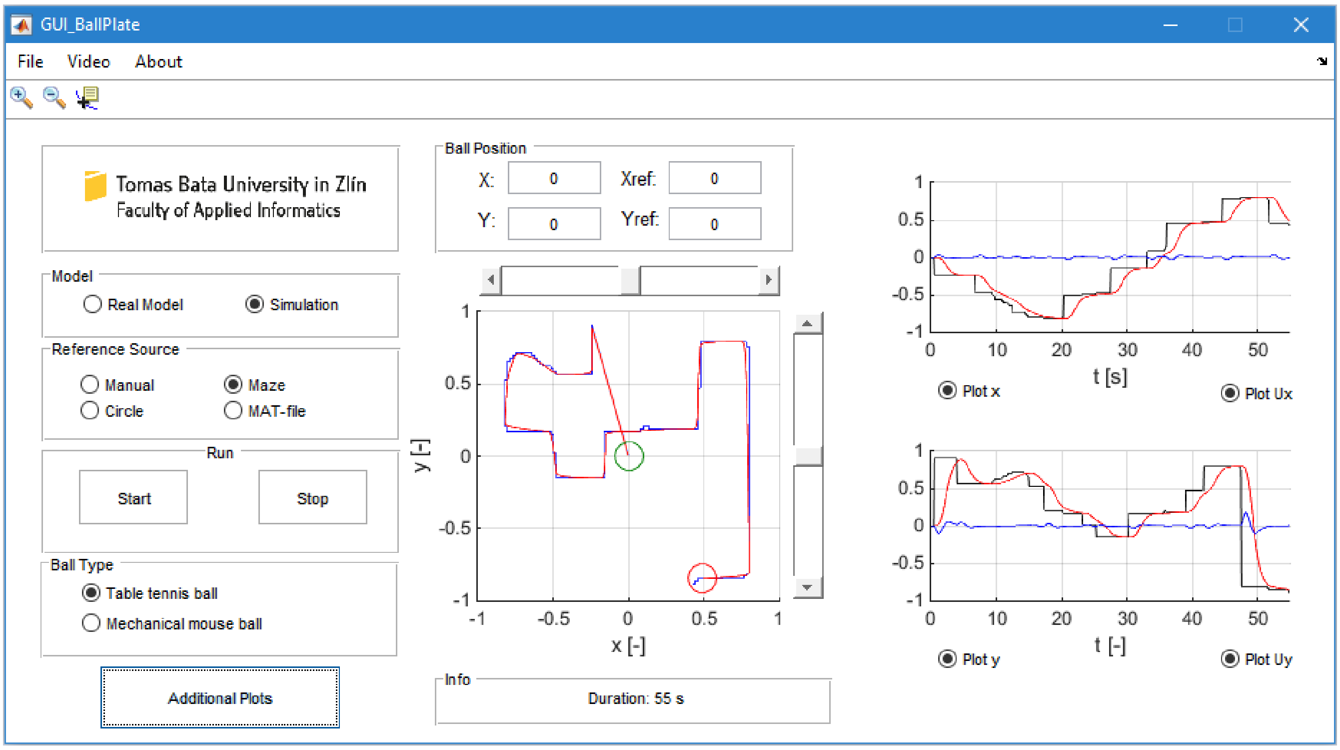

Fig. 13. Graphical user interface for Ball & Plate model

V. Conclusion

A digital LQ 2DOF controller for Ball & Plate model has been described in this paper. The controller was designed based on linearized mathematical model and polynomial approach for input/output form of the model. Experiments were carried out on CE151 educational model designed and built by Humusoft. This model is unfortunately very sensitive to vibrations caused by rapid movements of servomotors connected to plate using steel wire. This wire is prone to elastic deformations, which is the root cause of these vibrations. The controller is able to partially compensate these flaws, although it had to be designed to generate rapid changes as rarely as possible. This was achieved by choosing user-defined poles in polynomial method algorithm near the unit circle, which slowed the whole process, but resulted in subtle changes in plate inclination. The minimization of LQ criterion provided rest of poles in an optimal solution, which successfully compensated system dynamics. Experiments showed that this type of controller can easily stabilize the ball on the plate in desired position, reject introduced disturbances and navigate the ball through maze constructed on the plate.

Acknowledgment

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