Tensor-based reduced order modeling in reservoir engineering: An application to production optimization [*]

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Abstract: In this paper, a novel framework for reduced order modeling in reservoir engineering is introduced, where tensor decompositions and representations of flow profiles are used to characterize empirical features of flow simulations. The concept of classical Galerkin projection is extended to perform projections of flow equations onto empirical tensor subspaces, generating in this way, reduced order approximations of the original mass and momentum conservation equations. The methodology is applied to compute gradient-based optimal production strategies for water flooding using tensor-based reduced order adjoints.

Keywords: Tensor decompositions, Reservoir engineering, Model reduction.

1. INTRODUCTION

Simulation of multiphase flow through porous media plays a prominent role in the modern practice of reservoir engineering. Applications of reservoir simulation can be found in field development and planning, reservoir management, well location design and performance evaluation of reservoirs. Currently, the increasing computational capabilities and the advent of smart field technologies allow the design of model-based operational strategies to maximize the financial performance during the life cycle of the reservoir. For this, numerical reservoir models with large number of states (in the order of $10^4 - 10^6$) are used as equality constraints in the optimization problem, see Sarma et al. (2008), Jansen et al. (2008) and Van den Hof et al. (2012) for future perspectives of the field. Due to the computational cost of reservoir simulations, parameter estimation and model-based procedures for production optimization are limited to the computational capabilities and time constraints. Hence, there is a clear need for models with reduced dimensional complexity that can be used for efficient simulation of fluid flow through porous media.

Proper Orthogonal Decomposition (POD) has been proved to be an efficient tool for model order reduction of large scale dynamical systems. In reservoir engineering, POD techniques have been exploited by several authors Heijn et al. (2004), Markovinovic and Jansen (2006), Cardoso et al. (2009), Krogstad (2011). In production optimization, Van Doren et al. (2006) have used a projection onto POD spaces of the adjoint equations to reduce the dimension

of the linear adjoint system. Gildin et al. (2006) have designed optimal control strategies based on POD models. In parameter estimation, Kaleta et al. (2011) have used the adjoint of a reduced order linearization of a full order model to perform history matching procedures. Despite the benefits of reduced order models in simulation, production optimization and history matching, various authors have reported a recurrent limitation of the POD method: handling highly nonlinear systems with gravity terms in the flow equations generates, in most of the cases, either unstable or inaccurate reduced reservoir models. See Cardoso and Durlofsky (2007) and Heijn et al. (2004).

To overcome the limitations of POD, we exploit the spatial-temporal nature of flow patterns, taking advantage of the spatial correlations that are usually lost during the computation of the classical POD projection spaces. In this paper we propose the use of multidimensional arrays (tensors) as a natural representation of flow solutions and empirical projections. For a complete overview of the available techniques for signal and systems approximations using tensor decompositions, see Van Belzen and Weiland (2012). Tensor algebra and analysis are largely unexplored topics in reservoir engineering. Afra et al. (2014) have introduced a reduced rank approximation of permeability fields using tensor decompositions, and Insuasty et al. (2015) have presented tensor characterizations of flow profiles and evaluation of dissimilarity measures between models, with applications in the generation of control-relevant ensembles. In this work, we introduce tensor-based model order reduction techniques in reservoir engineering, with applications in production optimization of water flooding.

2. CLASSICAL POD MODEL ORDER REDUCTION

In this section we provide an introduction to POD, as a classical method for feature extraction and its application in the reduction of dynamical complexity of reservoir models, defined as the dimension of the state vector of pressures and phase saturations. For a detailed description of the philosophy and methods of classical Galerkin projections see Kirby (2000).

2.1 Classical spectral expansions

Let $\mathbb{X} \subset \mathbb{R}^{N-1}$ be the spatial domain, and let φ_i be the spatial mapping $\varphi_i : \mathbb{X} \to \mathbb{R}$ that describes a spatially distributed function. Let $\mathcal{X}$ be a separable Hilbert space of functions $\mathbb{X} \to \mathbb{R}$, with inner product $\langle \cdot, \cdot \rangle : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ and norm $\|\varphi_i\| = [\langle \varphi_i, \varphi_i \rangle]^{1/2}$. This makes $(\mathcal{X}, \|\cdot\|)$ a normed space. For every $\varphi_j, \varphi_k \in \mathcal{X}$ we consider the inner product:

$$\langle \varphi_j(\xi), \varphi_k(\xi) \rangle = \int_{\mathbb{X}} \varphi_j(\xi) \varphi_k(\xi) \mathrm{d}\xi. \tag{1}$$

so that the induced norm $\|\cdot\|$ becomes the standard $\mathcal{L}_2$ norm of functions. That is $\|\varphi_i\|_2 = \left[\int_{\mathbb{X}} |\varphi_i(\xi)|^2 \mathrm{d}\xi \right]^{\frac{1}{2}}$ and $\mathcal{X} = \mathcal{L}_2(\mathbb{X}, \mathbb{R})$. In this context, the concept of orthonormality of different elements defining a subspace is defined. Let $\mathbb{I} \subseteq \mathbb{Z}$ be a (*finite or infinite*) index set of integers. A set of elements $\{\varphi_i | i \in \mathbb{I}\}$ with $\varphi \in \mathcal{X}$ is said to be **orthonormal** if:

$$\langle \varphi_i, \varphi_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{2}$$

hold $\forall i, j \in \mathbb{I}$. In addition, the set of elements $\{\varphi_i | i \in \mathbb{I}\}$ is said to be an **orthonormal bases** of $\mathcal{X}$ if it constitutes a set of bases and if it is an orthonormal set. The linear span of all elements in such an orthonormal set is denoted **span**$\{\varphi_i | i \in \mathbb{I}\}$ and has a dimension equal to **card**$(\mathbb{I})$ the cardinality of the index set $\mathbb{I}$. Then every member $f \in \mathcal{X}$ admits an unique representation

$$f = \sum_{i=1}^{\infty} a_i \varphi_i \tag{3}$$

where $a_i = \langle f, \varphi_i \rangle$ and the equality (3) needs to be interpreted in the strong sense:

$$\lim_{I \to \infty} \left\| f - \sum_{i=1}^{I} \langle f, \varphi_i \rangle \varphi_i \right\| = 0. \tag{4}$$

2.2 POD basis functions

For spatial-temporal systems, we consider signals that evolve over space $\mathbb{X} \subset \mathbb{R}^{N-1}$, as well as time $\mathbb{T} \subset \mathbb{R}$. Let $s : \mathbb{X} \times \mathbb{T} \to \mathbb{R}$ denote such signal, and let us assume the system is described by a set of partial differential equations

$$R(s) = 0, \tag{5}$$

where R is a polynomial differential operator with differentiation in temporal and spatial variables. In reservoir engineering, the operator $R(\cdot)$ would represent oil saturation equation, and s would represent oil saturation. Let $s(\cdot, t) \in \mathcal{X}$ for all $t \in \mathbb{T}$. If $\{\varphi_i | i \in \mathbb{I}\}$ is an orthonormal basis of $\mathcal{X}$ then any solution of (5) can be expanded as $s(\xi, t) = \sum_{i=1}^{\infty} a_i(t) \varphi_i(\xi)$, where $a_i(t) = \langle s(\cdot, t), \varphi_i \rangle$ is a time varying coefficient. Let

$$s_r(\xi, t) = \sum_{i=1}^{r} a_i(t) \varphi_i(\xi) \tag{6}$$

be the $r-$th order approximation of s. Typically, $R(s_r) \neq$

0, but the Fourier coefficients $a_i(t)$ in (6) can be selected to satisfy a system of ordinary differential equations in $(a_1, \cdots, a_r)$, in such a way that the projection of $R(s_r)$ onto the space spanned by $\{\varphi_i | i \in \mathbb{I}\}$ is equal to zero. In this context, POD functions are of particular interest, mainly because they rely on experimental or simulation data from numerical models and have been successfully applied as part of model reduction techniques in fluid dynamics, structural vibrations, etc.

Let $s : \mathbb{X} \times \mathbb{T} \to \mathbb{R}$ be given with $s(\cdot, t) \in \mathcal{X}, \forall t \in \mathbb{T}$. The set of functions $\{\varphi_i(\xi)\}$ for $i = 1, 2, \cdots, r$ is called a **POD basis** for an r-dimensional subspace of $L_2(\mathbb{X}, \mathbb{R})$ associated with s, if it minimizes the following cost function:

$$J = \int_0^t \left\| s(\cdot, \tau) - \sum_{i=1}^{r} \langle s(\cdot, \tau), \varphi_i \rangle \varphi_i \right\|_2^2 d\tau$$

$$\text{s.t.} \quad \langle \varphi_i, \varphi_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}. \tag{7}$$

As it is shown in Van Belzen and Weiland (2009), (7) can be solved in terms of the correlation operator $\Phi : \mathcal{X} \to \mathcal{X}$:

$$\langle \varphi_i, \Phi \varphi_j \rangle = \int_0^t \langle \varphi_i, s(\cdot, \tau) \rangle \langle s(\cdot, \tau), \varphi_j \rangle \mathrm{d}\tau. \tag{8}$$

For $\varphi_i, \varphi_j \in L_2(\mathbb{X}, \mathbb{R})$, the correlation operator Φ is a well defined linear, bounded and non-negative operator on $L_2(\mathbb{X}, \mathbb{R})$. If $\dim(\mathcal{X}) < \infty$, the correlation operator Φ becomes a symmetric, non-negative matrix and its elements represent the correlation between the collection of signal samples in a discretized spatial-temporal domain. The problem of finding POD basis functions is equivalent to solving the singular value (*or eigenvalue*) problem for the correlation operator Φ. The POD basis correspond to the left singular vectors of the correlation operator Φ.

3. TENSOR BASED REDUCED ORDER MODELING

The previous section described classical methods for model reduction of nonlinear systems. In this section, we introduce a novel tensor formulation for the classical Galerkin projection of dynamical systems onto empirical spaces. We extend the concept of orthogonal projection of dynamical systems to the case where the empirical projection spaces span every coordinate of the spatial domain. The techniques described in this section are used for the reduction of the dynamical complexity of reservoir models.

3.1 Tensor representations and decompositions

Let us consider the spatial domain with cartesian structure $\mathbb{X} = \mathbb{X}_1 \times \cdots \times \mathbb{X}_{N-1}$, such that $\dim(\mathbb{X}_i) = R_i$ i.e., $\mathbb{X}_i = \{\xi_i^{(1)}, \cdots, \xi_i^{(R_i)}\}$ for $i = 1, \cdot, N-1$, and the temporal domain $\mathbb{T} = \{t^{(1)}, \cdots, t^{(R_N)}\}$. Moreover, $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_N$ is the result of a cartesian product of Hilbert spaces $(\mathcal{X}_i, \langle \cdot, \cdot \rangle_i)$ for $i = 1, \cdots, N$. Let us assume the sampled solution of (5), i.e. the oil saturation, is a mapping $s : \mathbb{X} \times \mathbb{T} \to \mathbb{R}$. Therefore, s has an associated multilinear mapping $S : \mathcal{X}_1 \times \cdots \times \mathcal{X}_N \to \mathbb{R}$, which can be represented with respect to canonical bases as:

$$\mathcal{S} = \sum_{l_1=1}^{R_1} \cdots \sum_{l_N=1}^{R_N} s_{l_1 \cdots l_N} e_1^{(l_1)} \otimes \cdots \otimes e_N^{(l_N)} \tag{9}$$

where the coefficient $s_{l_1 \cdots l_N} = s(\xi_1^{(l_1)}, \cdots, \xi_{N-1}^{(l_{N-1})}, t^{(l_N)})$ takes the value of s at the spatial grid point $\xi =$

$(\xi_1^{(l_1)}, \cdots, \xi_{N-1}^{(l_{N-1})})$ and time $t = t^{(l_N)}$. The $\otimes$ denotes the *tensor* or *outer product* over a vector space. For $i = 1, \cdots, N$, every $(\mathcal{X}_i, \langle \cdot, \cdot \rangle_i)$ can be spanned using a different set of orthonormal basis functions $\{\varphi_i^{(l_j)}\}_{j=1}^{R_i}$, i.e., $\langle \varphi_i^{(l_j)}, \varphi_i^{(l_k)} \rangle_i = \delta_{jk}$, where δ_{jk} is the Kronecker delta function. Hence, the tensor $\mathcal{S}$ admits a representation of the form:

$$\mathcal{S} = \sum_{l_1=1}^{R_1} \cdots \sum_{l_N=1}^{R_N} \sigma_{l_1 \cdots l_N} \varphi_1^{(l_1)} \otimes \cdots \otimes \varphi_N^{(l_N)}$$

$$= \sum_{l_1=1}^{R_1} \cdots \sum_{l_N=1}^{R_N} \sigma_{l_1 \cdots l_N} \Phi^{(l_1 l_2 \cdots l_N)} \quad (10)$$

where $\Phi^{(l_1 l_2 \cdots l_N)} = \varphi_1^{(l_1)} \otimes \cdots \otimes \varphi_N^{(l_N)}$ is a rank-1 tensor, and $\sigma_{l_1 \cdots l_N}$ is the $(l_1 \cdots l_N)$ element of a core tensor.

3.2 Empirical tensor basis and algorithms

The description of $\mathcal{S}$ in (10) does not assume ordered sets of basis functions with respect to the information content. For instance, a local transformation of coordinates is required to represent tensors in this type of bases. The problem of finding ordered sets of orthonormal bases functions in (7), can be extended for the multi-linear case as follows:

$$\min_{\{\varphi_1\} \cdots \{\varphi_N\}} \left\| \mathcal{S} - \sum_{l_1=1}^{R_1} \cdots \sum_{l_N=1}^{R_N} \sigma_{l_1 \cdots l_N} \varphi_1^{(l_1)} \otimes \cdots \otimes \varphi_N^{(l_N)} \right\|_F$$

$$\text{s.t.} \quad \langle \varphi_i^j, \varphi_i^k \rangle = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}, \quad i = 1, \cdots, N \quad (11)$$

where $\|\cdot\|_F$ is the Frobenius norm of a tensor. In literature, several algorithms have been reported to compute Tucker-type tensor decompositions and sets of tensor orthonormal basis functions. A *Tucker modal-rank decomposition* defines orthonormality for the set of vectors $\{\varphi_n^{(l_n)}\}$ for $l_n \in \mathbb{N}_{[1, R_n]}$, therefore this set constitutes an orthonormal basis for $\mathbb{R}^{R_n}$.

The *High Order SVD* (HOSVD) proposed by De Lathauwer et al. (2000a) was the first extension of a classical SVD to the multilinear case and the methodology is based on the unfolding procedure of tensors, losing the tensor structure and performing computations in the matrix plane. The *High Order Orthogonal Iteration* (HOOI) by De Lathauwer et al. (2000b), the *Tensor SVD* proposed by Weiland and Van Belzen (2010) and the *Single Directional Modal-rank decomposition* (SDM) by Shekhawat and Weiland (2014) compute singular values and vectors of tensors in a sequential way, keeping the tensor structure intact. In this work, the *SDM* method is applied for the decomposition of saturation and pressure tensors from reservoir models, in order to generate sets of tensor empirical projection spaces.

3.3 Nested Galerkin projections

Reservoir simulators describe the evolution of saturation and pressure over time and space $\mathbb{X}$, which is typically

represented with three dimensions *(N = 3)* in the Cartesian plane. For this reason, tensors are a natural way to represent and analyze data from spatial-temporal models. In this subsection, we can extend the concept of Galerkin projection onto the whole spatial domain, to a projection onto sets of orthonormal basis that span empirical spaces for the different dimensions of the spatial domain.

In the framework of Tensor MOR, we deal with every physical dimension as a separate entity, therefore, the cartesian coordinates of space can be seen as separable Hilbert spaces, which are the span of their own set of basis functions. Following the same reasoning, it is possible to represent the solution of (5) in terms of $s(\cdot)$ using tensor expansions as follows:

$$s(\xi_1^{(p_1)}, \cdots, \xi_{N-1}^{(p_{N-1})}, t^{(p_N)}) = \quad (12)$$

$$\sum_{l_1=1}^{R_1} \cdots \sum_{l_{N-1}=1}^{R_{N-1}} \sigma_{l_1 \cdots l_{N-1}} \varphi_1^{(l_1)}(\xi_1^{(p_1)}) \otimes \cdots \otimes \varphi_{N-1}^{(l_{N-1})}(\xi_{N-1}^{(p_{N-1})}),$$

where the coefficients $\sigma_{l_1 \cdots l_{N-1}}$ in (12) are functions of time. Let us truncate the sums in (12)

$$s_r(\xi_1^{(p_1)}, \cdots, \xi_{N-1}^{(p_{N-1})}, t^{(p_N)}) = \quad (13)$$

$$\sum_{l_1=1}^{r_1} \cdots \sum_{l_{N-1}=1}^{r_{N-1}} \sigma_{l_1 \cdots l_{N-1}} \varphi_1^{(l_1)}(\xi_1^{(p_1)}) \otimes \cdots \otimes \varphi_{N-1}^{(l_{N-1})}(\xi_{N-1}^{(p_{N-1})}),$$

with $r_i < R_i$ for $i = 1, \cdots, N - 1$, and we call s_r the approximation of s. Given sets of orthonormal basis functions for every spatial dimension $\{\varphi_i^{(l_i)}\}_{i=1 \cdots R_i}$, we enforce the residual equation $R(s_r)$ to be orthogonal to the basis functions that span every dimension, then, a nested Galerkin projection of (5) over the Tensor projection spaces can be defined as

$$\left\langle \varphi_1^{l_1}, \left\langle \varphi_2^{l_2}, \cdots, \left\langle \varphi_{N-1}^{(l_{N-1})}, R(s_r) \right\rangle_{N-1} \cdots \right\rangle_2 \right\rangle_1 = 0 \quad (14)$$

where $\langle \cdot \rangle_i$ is the inner product defined for the i-th dimension of the spatial coordinate, see Van Belzen and Weiland (2012) and s_r defined in (13). The expression in (14) defines an ordinary differential equation that represents the projection of $R(s_r)$ onto the basis $\varphi_1^{(l_1)}, \varphi_2^{(l_2)}, \cdots, \varphi_{N-1}^{(l_{N-1})}$. The reduced order approximation of the dynamical system in (5), can be found by generating ODEs from (14) by letting the basis indexes varying as $l_1 = 1, \cdots, r_1$, $l_2 = 1, \cdots, r_2$ until $l_{N-1} = 1, \cdots, r_{N-1}$.

In summary, the assumption of cartesian spatial $\mathbb{X}$ domain and temporal $\mathbb{T}$ domain associates a multidimensional array $\mathcal{S}$ to the solution of a dynamical system s. Then, a new representation of $\mathcal{S}$ in terms of rank-1 tensors allows the construction of empirical bases for every dimension of $\mathcal{X}$, which leads to the definition of a tensor expansion of s. Typically, $R(s_r) \neq 0$, then we use sequential projections onto the empirical bases that span every $\mathcal{X}_i$ in order to generate a reduced order model.

4. APPLICATION CASE

In this section we present an application case for the techniques illustrated in Sections 2 and 3. Classical POD and Tensor MOR techniques are applied for the reduction of the computational complexity of a reservoir model.

4.1 The model

We use a numerical model for oil-water fluid flow through heterogeneous porous media. The implicit solvers of MRST, see Lie et al. (2012), have been used to solve the pressure and saturation equations:

$$\mathbf{v} = -\lambda_t \mathbf{K} \nabla p, \qquad \nabla \cdot \mathbf{v} = q$$

$$\phi \frac{\partial}{\partial t} S_w + \nabla \cdot (f_w(S_w)(\mathbf{v} + g \nabla d)) = \frac{q_w}{\rho_w} \qquad (15)$$

where $\mathbf{v}$ is the Darcy velocity, $\mathbf{K}$ the permeability tensor, p the pressure, q volumetric rates, ϕ the porosity, S_w the water saturation, q_w water volumetric rates, ρ_w the water density, g the gravity, d depth below the surface and $f_w(S_w)$ the water fractional flow defined in terms of the relative permeabilities k_{rw} and k_{ro} as:

$$f_w(S_w) = \frac{k_{rw}(S_w)/\mu_w}{k_{rw}(S_w)/\mu_w + k_{ro}(S_w)/\mu_o} \qquad (16)$$

and μ_w and μ_o the water and oil viscosities. We assume zero flow at the boundaries. We consider a reservoir with square geometry of side length $L = 600$m, one layer of 4m thick, 5 wells (1 *injector, 4 producers*). A top view of well locations, permeability and porosity fields are provided in Fig. 1. The numerical model has 900 grid blocks of size 20m×20 × 4m and the physical parameters are presented in Table 1.
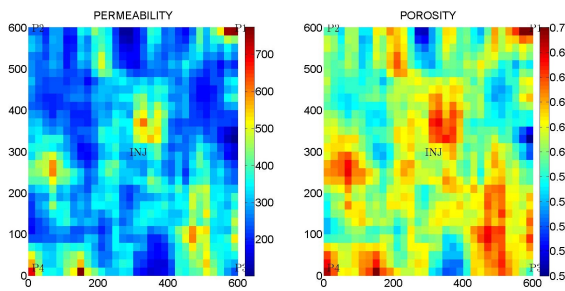

Fig. 1. Permeability, porosity for the test case.

The controls for this reservoir are the injection rate and the bottom hole pressures at producers.

4.2 Snapshots generation

A production strategy (*Initial Schedule*, Fig. 4) is used to generate 48 snapshots of pressure and saturation profiles and the variables are stored in separate multi-linear arrays. Next, the POD and Tensor projection subspaces are computed by using decomposition algorithms described in Subsections 2.2 *(SVD)* and 3.2 *(SDM)*. Some relevant POD and Tensor basis for saturation are presented in Fig. 2. These basis can be understood as the most informative directions on the snapshots data.

4.3 Model reduction

The number of dimensions for this application case is $N = 3$, 2 corresponding to the spatial domain $\mathbb{X}$, and 1 for time. POD models are obtained by projecting the (15) onto the first $n_s = 47$ and $n_p = 20$ POD bases for saturation and pressure respectively. For the Tensor-based case, we compute the inner product of the spatial basis $\varphi_1^{l_1} \otimes \varphi_2^{l_2}$ for $l_1 = 1, \cdots, r_1$, $l_2 = 1, \cdots, r_2$, $r_1 = r_2 = 7$, select the first
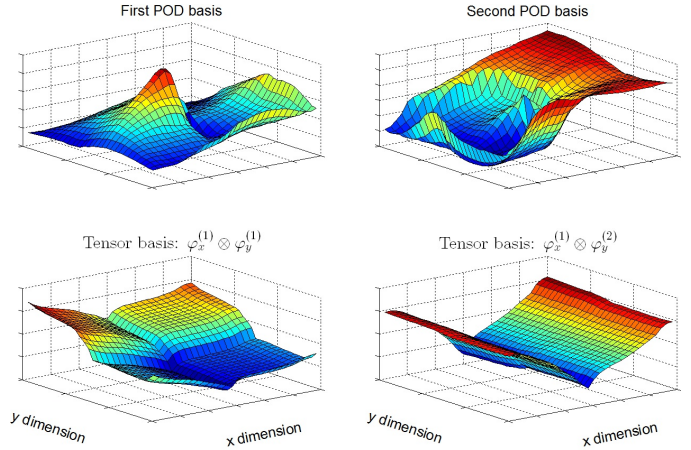

Fig. 2. POD and Tensor basis for saturation.

$n_s = 47$ and $n_p = 20$ bases for saturation and pressure respectively for the projection of (15). Production was simulated for a period of 3100 days, and some temporal snapshots of oil saturation profiles for the full order and reduced models are presented in Fig 3. On one side, it is clear that the POD approximation is not capable of generating trajectories with physical significance, and the diffusive-convective nature of the full equations is lost after the projection onto POD subspaces.
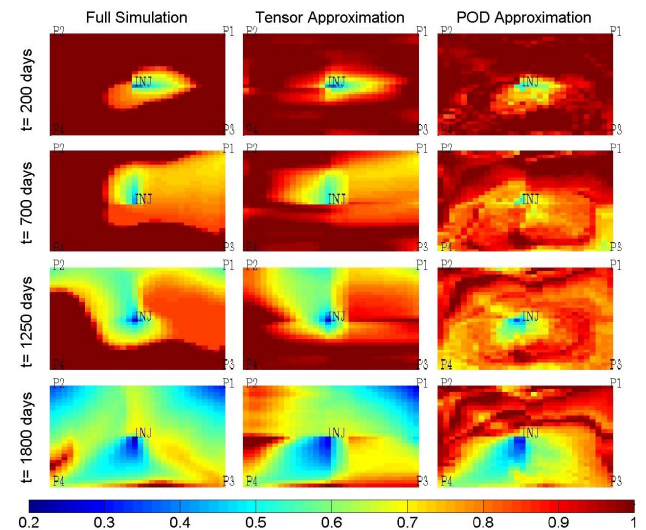

Fig. 3. Oil saturation time snapshots for full model and reduced order model.

In this application case, only $5 - 10\%$ of computational gain was obtained by employing both POD and Tensor methods for MOR. This is a known fact for projection-based methods, and it is due to the operations associated with the inner products in (14) at every iteration of the nonlinear solver. See Cardoso et al. (2010) for methods to speed-up the procedure.

5. TENSOR-BASED REDUCED ORDER ADJOINT MODELS FOR PRODUCTION OPTIMIZATION

Gradient-based production optimization of water flooding is a computationally expensive process that requires several forward reservoir and backward adjoint simulations. For instance, there is a potential use of reduced-order

Table 1. Physical parameters

Property	Value	Unit
k	$13.74 - 876.61$	mD
ϕ	$0.17 - 0.37$	-
ρ_w	1014	Kg/m^3
ρ_o	859	Kg/m^3
μ_w	1	cP
μ_o	5	cP
S_{wc}, S_{or}	0.2	-
k_{rw}^0	0.6	-
k_{ro}^0	0.9	-
n_w, n_o	3	-

modeling techniques to perform optimization routines in low dimensional spaces. In this section, we compare financial performance of the gradient-based optimal production strategies in water flooding using POD, introduced by Van Doren et al. (2006), with tensor-based optimal production strategies proposed in this paper.
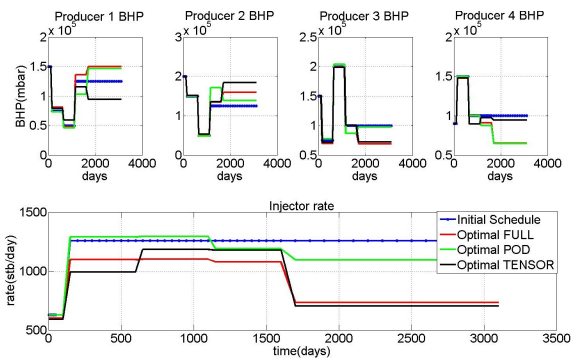

Fig. 4. Optimal production strategies for full order and reduced order approximations.

5.1 Production optimization

The purpose of model-based production optimization is to maximize a financial measure based on numerical models for the multi-phase flow through porous media. Traditionally, the financial tool used to describe performance along the life cycle of the reservoir is the Net Present Value (NPV), which is a measure of the net earnings of oil production minus the costs associated with water injection and production. The water flooding optimization problem is formulated as follows:

$$\max_u J = \sum_{k=1}^{K} J_k(x(k), u(k)) \quad (17)$$

$$\text{s.t} \quad R(u_k, x_k, x_{k+1}) = 0 \quad \text{for} \quad k = 0, \cdots, K-1$$

where:

$$J_k(k) = \frac{\Delta t_k}{(1+b)^{\frac{t_k}{\tau}}} \left[\sum_{i=1}^{N_{inj}} r_{wi} \cdot (u_{wi,i})_k + \right.$$

$$\left. \sum_{j=1}^{N_{prod}} \left[r_{wp}(y_{wp,i})_k + r_o(y_{o,j})_k \right] \right], \quad (18)$$

where x is the state vector of saturations and pressure, u the set of control inputs, K the optimization horizon, Δt_k the time step, b the discount factor, N_{inj} and N_{prod} the number of injectors and producers, r_{wi} and r_{wp} the costs of water injection and production, r_o the price of oil produced, $y_{wp,i}$ and $y_{o,j}$ the water and oil production rates.

In this work, we compute the optimal production strategy that solves (17) using gradient-based methods that used the adjoint equations, see Sarma et al. (2008) and Jansen (2011) for a detailed description.

5.2 Reduced order adjoint equations

The adjoint-based approach for production optimization is one of the most efficient methods, as it only requires one simulation for the forward and adjoint model. As it is presented in Van Doren et al. (2006), a reduced order version of the adjoint model is obtained by projecting the adjoint model equations onto empirical low order subspaces. However, the computation of the corresponding Lagrange multipliers still depends on the computation of the sensitivities of the full order model with respects to its states. In this work, with the aid of reliable reduced order models we can reformulate the optimization problem and the adjoint equations in terms of the reduced order model

$$\hat{\lambda}(k)^{\top} \frac{\partial \hat{R}(k-1)}{\partial a(k)} \Psi = -\hat{\lambda}(k+1)^{\top} \frac{\partial \hat{R}(k)}{\partial a(k)} \Psi - \frac{\partial J_k(k)}{\partial a(k)} \Psi. \quad (19)$$

For instance, the sensitivity of the augmented objective function $\mathcal{L}$ with respect to the inputs becomes:

$$\frac{\partial \mathcal{L}(k)}{\partial u(k)} = \frac{\partial J_k(k)}{\partial u(k)} + \hat{\lambda}(k+1)^{\top} \frac{\partial \hat{R}(k)}{\partial u(k)}, \quad (20)$$

and the update for the new controls is performed using steepest ascend method with

$$u_{new}(k) = u_{old}(k) + \alpha \frac{\partial \mathcal{L}(k)}{\partial u(k)}, \quad (21)$$

where $\hat{R} = R(s_r)$, a is the vector of Fourier coefficients, Ψ the stacked matrix of empirical projection bases and α is the step in the gradient direction.
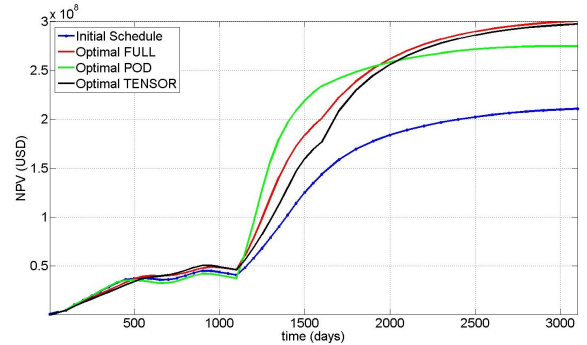

Fig. 5. NPV performance of the virtual asset with the different strategies.

5.3 Results

We compute optimal production strategies for the POD and Tensor reduced order models with the methods described in Subsections 5.1 and 5.2, where reduced-order models and adjoints are used for forward and backward simulation. The results are illustrated in Fig. 4. In order to perform the projections in (19), Ψ and $\hat{R}$ were computed for both POD and the tensor approach and coupled with the adjoint formulation provided by the fully implicit solvers of MRST.

We used 5 control steps, and the corresponding NPV build-ups for the full model operated with those strategies

are presented in Fig. 5. On one hand, it is illustrated that despite POD reduced-order models can be used for the purpose of optimization, their limitations in terms of approximation accuracy constrain the financial performance of the resulting optimal production strategy. On the other hand, the accuracy level achieved with tensor-based reduced order models affects positively their use in production optimization. For this application case, the tensor strategy shows a better financial performance compared to the POD strategy and it is close to the optimal strategy for the full order model.

6. CONCLUSION

In this work, tensor representations of flow profiles are used to generate empirical spaces where the equations of two-phase flow through porous media are projected independently in every physical dimension using the concept of nested Galerkin projections. For the application case presented in this paper, tensor models are able to approximate accurately most of the dynamical characteristics of the full order model, while the POD case experiences the limitations reported in literature. However, the accuracy of tensor models is subject to the scope of the experiment design for the generation of empirical projection spaces. The computational gains are low compared to the full order model, which is a well reported limitation of the projection methods for model order reduction, and there exist methods in literature to accelerate nonlinear reduced order models. The advantages of using tensor representations for reduced order models lies in the higher levels of approximation accuracy that can be achieved compared with the classical POD models, and in their potential use in optimization routines.

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