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Profit and Risk Measures in Oil Production Optimization ⋆

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Abstract: In oil production optimization, we usually aim to maximize a deterministic scalar performance index such as the profit over the expected reservoir lifespan. However, when uncertainty in the parameters is considered, the profit results in a random variable that can assume a range of values depending on the value of the uncertain parameters. In this case, a problem reformulation is needed to properly define the optimization problem. In this paper we describe the concept of risk and we explore how to handle the risk by using appropriate risk measures. We provide a review on various risk measures reporting pro and cons for each of them. Finally, among the presented risk measures, we identify two of them as appropriate risk measures when minimizing the risk.

Keywords: Production Optimization, Oil production, Risk measures, Stochastic problems

1. INTRODUCTION

In oil production optimization, we are in general interested in maximizing an economic measure, like the profit or the net present value (NPV), over the expected reservoir life time. When uncertainty is taken into account, the profit is not a single quantity but has a probability distribution, i.e. the profit is described by a random variable ψ. An optimization problem involving ψ in terms of a control input u, must express ψ as a scalar quantity. Traditionally, this single quantity is the expected profit (Van Essen et al., 2009; Capolei et al., 2013). By using only the expected profit, however, we are not able to include others important indicators, that shape the profit distribution ψ, such as the profit deviation and the risk preference. The role of a measure of deviation is to quantify the variability of a random variable ψ and the uncertainty in ψ is often measured by the standard deviation of ψ, e.g. in classical portfolio theory (Markowitz, 1959), the standard deviation $\sigma(\psi)$ is used to quantify uncertainty in returns of financial portfolios. In the oil community, Bailey et al. (2005); Alhuthali et al. (2010); Yeten et al. (2003) propose to reduce the uncertainty in profit by including the standard deviation in the cost function. In many decision problems dealing with safety and reliability, risk is often interpreted as the probability of a dreadful event or disaster (Ditlevsen and Madsen, 1996; Rockafellar and Royset, 2010), and minimizing the probability of a highly undesirable event is known as the safety-first principle (Roy, 1952). In this paper we identify the risk as a measure of the risk of loss. When speaking of such a measure applied to the random profit, ψ, we have in mind that higher outcomes of ψ are welcome while lower outcomes are disliked. To reduce the risk of loss then, we seek to lower the probability of the low profits. Certainly, deviation and risk are related concepts and often these terms are used interchangeably, e.g. in finance, we can interpret the profit volatility, measured by the standard deviation, as risk. Following this idea, Capolei et al. (2015) introduce the mean-variance criterion for production optimization and suggest to use the Sharpe ratio as a systematic procedure to optimally trade-off risk and return. They interpret the standard deviation as a measure of risk. However, the mean-variance approach is more suited to reduce the profit uncertainty than to reduce the risk of loss. Fig. 1 illustrates two drawbacks of the mean-variance framework when used to measure risk preferences. First of all, the mean variance approach is insensitive to the profit shape distribution. Fig.1a is a sketch representing different profit distributions having the same values for mean and the variance. In the mean-variance framework these distributions yield the same risk preference. In Fig.1b instead, the distributions in blue have a lower standard deviation, σ, than the distribution in black. If we use the standard deviation as a risk measure, the blue distributions have a lower risk than the black profit distribution, no matter what their expected values are. Furthermore, the standard deviation as a measure of risk is symmetric, which means that it penalizes higher profits and lower profits symmetrically. This last shortcoming have been recognised by Markowitz (1959) who proposed to use the semideviation instead. However, even by using the semideviation, we still do not have common properties that make sense both for a risk

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measure like risk aversion and monotonicity, see Section 3.

In this paper we follow an axiomatic approach to define risk, i.e. we first define the principles that an appropriate risk measure should have, then we select risk measures that satisfies such principles. The risk axioms that we use are the principles that define coherent averse risk measures as introduced and defined in Artzner et al. (1999); Rockafellar (2007); Krokhmal et al. (2011); Zabarankin and Uryasev (2014). At our knowledge, this is the first time that such an axiomatic approach is used in oil production optimization.

The paper is organized as follow. Section 2 formulates the oil production optimization problem under uncertainty as a risk minimization problem. Section 3 describes the basic properties that we require from an appropriate risk measure, while a number of risk measures are discussed in Section 4. Conclusions are presented in Section 5.

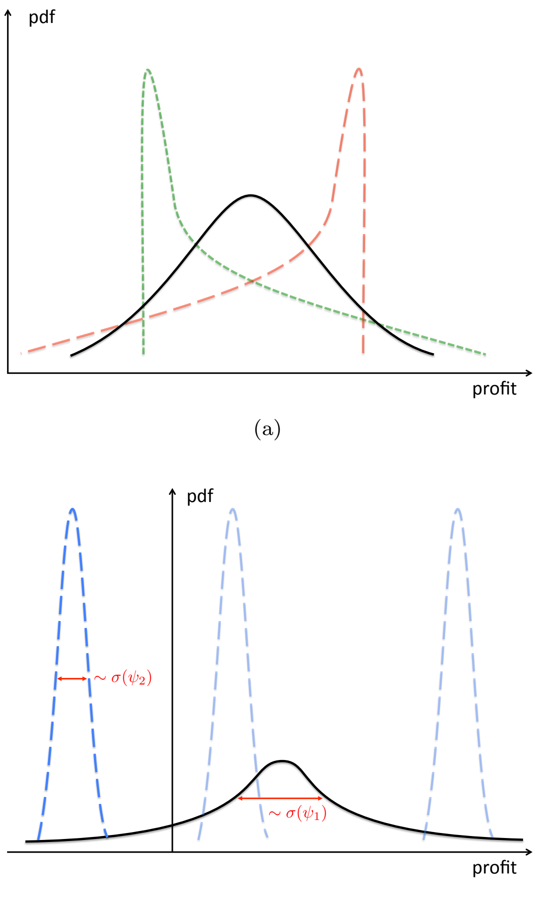


Fig. 1. The mean-variance framework is indifferent to shape distributions. Fig.1a is a sketch representing different profit distributions having the same mean and variance. In the mean-variance framework, these distributions yield the same risk preference. In Fig.1b, the distributions in blue have a lower standard deviation σ than the distribution in black. If we use the standard deviation as a risk measure, this means that the blue distributions have a lower risk than the black profit distribution, no matter what their expected values are.

2. PROBLEM FORMULATION

In oil production optimization, the profit can be visualized as a function

$$\psi = \psi(u, \theta) \tag{1}$$

of a decision vector $u \in \mathcal{U} \subset \mathcal{R}^{n_u}$ representing the control vector, with $\mathcal{U}$ expressing linear decision constraints, and a vector $\theta \subset \mathcal{R}^m$ representing the values of a number of parameters variables such as the permeability field, porosity, economic parameters, etc. The function ψ usually represents the NPV or some other performance index, and its computation typically requires the use of a reservoir simulator to solve the reservoir flow equations. If there is no uncertainty in the parameters values θ, we can maximize ψ by solving the following deterministic optimal control problem (Brouwer and Jansen, 2004; Sarma et al., 2005; Nævdal et al., 2006; Foss and Jensen, 2011; Capolei et al., 2013)

$$\max_{u \in \mathcal{U}} \psi(u, \theta) \tag{2}$$

However, in oil problems there is a high uncertainty due for example to the noisy and sparse nature of seismic data, core samples, borehole logs, and future oil prices and plant costs. Mathematically, we may represent model uncertainty by making the parameter vector θ a random variable that has some probability distribution and that belongs to some uncertainty space Θ. Consequently, the profit ψ is a random variable. Due to the complexity of real oil reservoirs and the accompanying measurement problem, we don't know the probability distribution of θ, thus, we have only incomplete informations about the uncertainty space Θ. For these reasons, the traditional way of modeling the uncertainty in oil production problems is to consider a finite set of possible scenarios for the parameters (Krokhmal et al., 2011; Van Essen et al., 2009; Capolei et al., 2013, 2015). This means that we substitute Θ with the discretized space $\Theta_d := \{\theta_1, \theta_2, \ldots, \theta_{n_d}\}$. As a consequence, a control input u, will correspond to a finite set of possible profit outcomes $\psi(u, \theta_1), \ldots, \psi(u, \theta_{n_d})$, with probabilities $p_1, \ldots, p_{n_d}$, respectively, where $p_i = Prob[\theta = \theta_i] \in [0, 1]$ and $\sum_{i=1}^{n_d} p_i = 1$. Usually the possible realizations θ_i are considered equiprobable, i.e. $p_i = 1/n_d$. It should be noted that defining the uncertainty set Θ_d is a highly interdisciplinary exercise. Furthermore, uncertainty will be updated as subsurface properties further reveal themselves from measurement surveys and production data in addition to new forecasts on oil price and costs.

When uncertainty is taken into account, the following stochastic optimization problem can be written

$$\max_{u \in \mathcal{U},\, \theta \in \Theta_d} \psi(u, \theta) \tag{3}$$

However, this formulation is not well defined. The problem is that the decision vector u must be chosen before the outcome of the distribution of θ and consequently the value of ψ, can be observed. To obtain a well defined problem, we can substitute the random variable ψ by a functional $\mathcal{R} : \psi \rightarrow \mathbb{R}$ that yields a scalar measure of ψ, and depends both on u and θ. In this way we can reformulate problem (3) as

$$\min_{u \in \mathcal{U}} \mathcal{R}(\psi(u, \theta \in \Theta_d)) \tag{4}$$

Note that we have switched to a minimization problem because we will interpret $\mathcal{R}$ as a risk measure to minimize.

$\mathcal{R}$ is a surrogate for the distribution of ψ, thus, different $\mathcal{R}$ expressions capture differents aspects of the profit distribution. So far, different measures, $\mathcal{R}$, have been proposed in the oil community such as the expected profit, i.e. $\mathcal{R} = -E_\theta[\psi]$ (Van Essen et al., 2009) or the mean-variance measure, i.e. $\mathcal{R} = -\big(\lambda E_\theta[\psi] - (1-\lambda)\sigma^2(\psi)\big)$ with $\lambda \in [0,1]$ (Capolei et al., 2015). However, for many reasons, that we will discuss in Section 4, none of these measure are satisfactory. Finally, we should stress that in this paper we focus on single objective optimization only. We do not consider important aspects that are connected to multi-objective optimization, e.g. the trade-off between long term vs short term profit (Van Essen et al., 2011). However, our analysis on risk measures can be extended to these cases. We can for example use the weighted sum method (Liu and Reynolds, 2015) to trade-off a long term profit measure $\mathcal{R}(\psi^{long})$ versus a short term profit measure $\mathcal{R}(\psi^{short})$ by solving

$$\min_{u \in \mathcal{U}} \; \lambda \, \mathcal{R}(\psi^{short}) + (1-\lambda)\, \mathcal{R}(\psi^{long}) \tag{5}$$

for different $\lambda \in [0,1]$.

3. COHERENT AVERSE MEASURES OF RISK

Measures of risk have a crucial role in oil production optimization under uncertainty, especially in coping with the losses in profit due to a too aggressive oil field development plan. The role of a risk measure is to assign to the random profit, ψ, a numerical value $R(\psi)$ that can serve as a surrogate for overall profit. The risk comparison of two choices ψ' and ψ'' then reduces to comparing $R(\psi')$ and $R(\psi'')$ and a decision maker, during the decision process, should prefer solutions that minimize the risk or that maintain the risk below a certain threshold. In this paper, we consider as appropriate risk measures, the coherent and averse measures of risk as defined by Artzner et al. (1999); Rockafellar (2007); Krokhmal et al. (2011):

Definition 1. Coherent Averse measures of risk are functionals $\mathcal{R} : \psi \to \mathbb{R}$ satisfying

(A1) Risk aversion:
- $R(c) = -c$ for constants c (constant equivalence)
- $R(\psi) > -E_\theta[\psi]$ for nonconstant ψ (averseness).

(A2) Positive homogeneity: $R(\lambda\psi) = \lambda R(\psi)$ when $\lambda > 0$.
(A3) Subadditivity: $R(\psi' + \psi'') \leq R(\psi') + R(\psi'')$ for all ψ' and ψ''.
(A4) Closure: $\forall c \in \mathbb{R}$, the set $\{\psi | R(\psi) \leq c\}$ is closed.
(A5) Monotonicity: $R(\psi') \geq R(\psi'')$ when $\psi' \leq \psi''$.

These axioms require additional explanation. Axiom (A1) formalizes the risk averse principle, see also Fig. 2. A risk-averse decision maker does not rely on the expected profit exclusively and always prefers a deterministic payoff of $E_\theta[\psi]$ over a non constant ψ. The risk of a deterministic profit is given by its negative value, i.e. $R(c) = -c$. $R(c) = -c$ implies $R(E_\theta[\psi]) = -E_\theta[\psi]$, so that the other condition $R(X) > -E_\theta[\psi]$ can be restated as $R(\psi) > R(E_\theta[\psi])$ for $\psi \neq c$ and constant c, which is the risk aversion property in terms of R. The positive homogeneity axiom (A2) ensures invariance under scaling, e.g. if the units of ψ are converted from one currency to another, then the risk is also simply scaled with the exchange rate. Finally, the positive homogeneity enables the units of measurements of $R(\psi)$ to be the same as those of ψ. The subaddivity axiom (A3) is a mathematical expression of the fundamental risk management principle of risk reduction via diversification. Also, it follows from the constant equivalence, i.e. $R(c) = -c$, and axiom (A3) that

$$R(\psi + c) = R(\psi) - c \tag{6}$$

which is called translation invariance. This is explained by its financial interpretation. If ψ is the payoff of a financial position, then adding cash to this position reduces its risk by the same amount; in particular one has

$$R(\psi + R(\psi)) = 0 = R(0) \tag{7}$$

i.e. adding a quantity of cash equal to the risk of ψ to ψ reduces to the risk of getting a deterministic zero payoff. The translation invariance principle, provides us with a natural way of defining an acceptable risk (Artzner et al., 1999; Rockafellar, 2007). We can consider a risk acceptable if its value is lower than the risk of obtaining a deterministic reference payoff c_{ref}, i.e.

$$R(\psi) \leq R(c_{ref}) = -c_{ref} \tag{8}$$

In oil problems, c_{ref} could come from the expected income by investing in an alternative project. However, typically we consider the risk as acceptable when its value is lower than the risk of a deterministic zero profit, i.e. $c_{ref} = 0$ and (8) thus becomes

$$R(\psi) \leq 0 \tag{9}$$

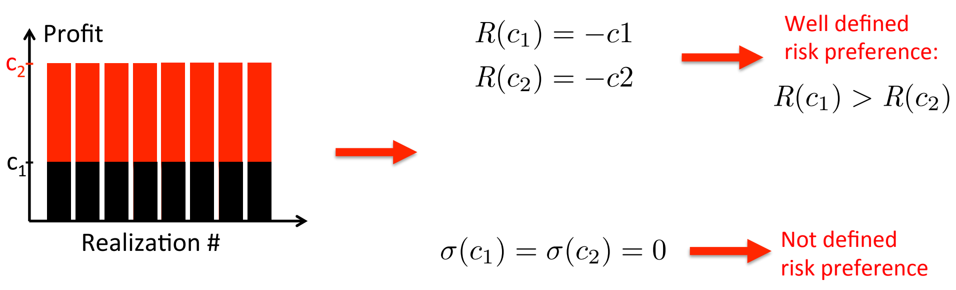
The monotonicity axiom (A5) says that we consider ψ' no less risky than ψ'' if every realization of ψ'' is no smaller than every realization of ψ', see Fig. 3. In the literature, risk measures that satisfy axioms (A1-A4) are called averse measures of risk (Rockafellar, 2007; Krokhmal et al., 2011). Risk measures that satisfy axioms (A2-A5) and the constant equivalence property are called coherent risk measures in the sense of Artzner (Artzner et al., 1999; Krokhmal et al., 2011). A final note on risk measures principles is that the positive homogeneity and the subadditivity imply convexity of the risk measure $R(.)$ (Rockafellar, 2007; Krokhmal et al., 2011). The convexity property is important when we want to minimize the risk, R, because it allows the optimizer to find solutions that are globally optimal. Therefore, we would like to formulate convex oil production optimization problems. In oil optimization problems, however, $\psi = J(u,\theta)$ is computed by a function J that is non convex with respect to the decision vector u. As a consequence, $R(J(u,\theta))$ is non convex and the optimizer can only yield local minima solutions. Unfortunately, the non convexity originates from the physics of the problem itself. However, future research in this field may allow to efficiently approximate and reformulate the problem by using convex surrogate models, e.g. piecewise linear models where there, however, is a need to include integer variables.

4. RISK MEASURES IN PRODUCTION OPTIMIZATION

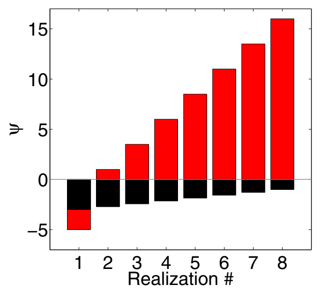
In this section, we review some traditional approaches of measuring risk. For each approach we will discuss if and how it adheres to the risk measure definition of Section 3.

4.1 Nominal profit

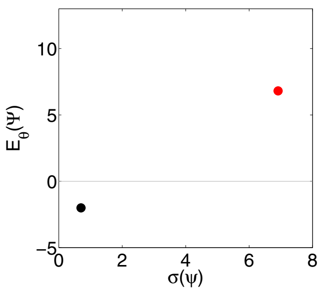
This approach is based on selecting one single realization, θ_i, of the uncertainty space Θ_d, as outcome for the future value of θ:



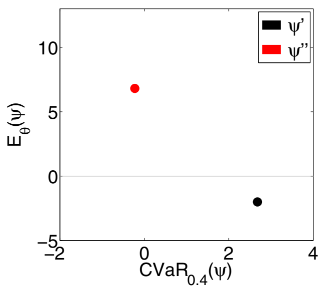
(a) Constant equivalence property



(b) Profit realizations

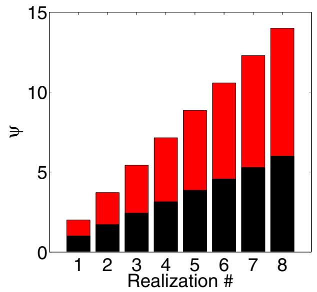


(c) Standard deviation

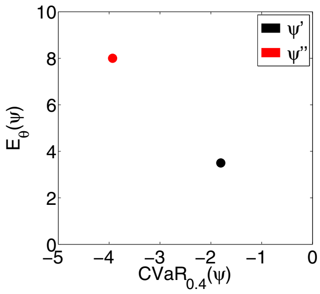


(d) Coherent averse Risk measure example

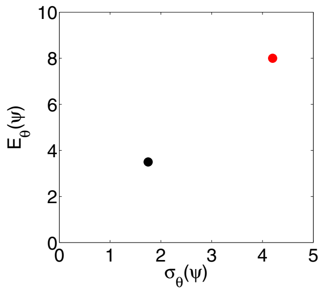
Fig. 2. Aversity axiom. The constant equivalence property in Fig. 2a states that for constant profit distributions, a risk measure as defined in Section 3 yields a well defined risk preference. Instead, when we use the standard deviation, σ, the risk preference is not well defined. Fig. 2b shows two profit distributions ψ' and ψ''. ψ' yields negative profits and also negative expected profit. ψ'' yields a negative profit only in the first realization and has positive expected profit. Despite for the first realization results $\psi'' < \psi'$, the probability of lower profits is much larger for the profit distribution ψ' than for ψ''. However, if we use the standard deviation as a measure of risk, ψ' would result with lower risk, see Fig. 2c, because it has a lower standard deviation. Instead, a coherent averse risk measure would allow to choice a risk preference for which ψ' yields the lowest risk, see Fig. 2d.



(a) Profit realizations



(b) Coherent averse Risk measure example



(c) Standard deviation

Fig. 3. The monotonicity axiom states that if we have two profit distributions ψ' and ψ'', and for every realization $\psi' \leq \psi''$, see Fig.3a, then a risk measure should yield a higher risk for ψ' than for ψ'', see Fig.3b. For the standard deviation, see Fig.3c, the monotonicity axiom does not hold.

$$R(\psi) = -\psi(u, \theta_i). \tag{10}$$

Despite the fact that this approach is coherent (not averse), it is open to criticism since it relies on one parameter sample only. Thus, one may argue that this approach is closer to guess work than analysis. With this approach, a profit distribution is acceptable in the sense of (9) when the profit for the chosen realization θ_i is greater than zero.

4.2 Certainty Equivalent profit

This approach is based on using the expected value of the parameters, $E_\theta[\theta]$, as outcome for the future value of θ:

$$R(\psi) = -\psi(u, E_\theta[\theta]) \tag{11}$$

This approach was used in Capolei et al. (2013, 2015). The idea is that the expected value of the parameters yields a good representation of the model. However, there are some problems with this approach. First of all, the expected value of θ may not be a possible realization of Θ, i.e. it could be that $E_\theta[\theta] \notin \Theta$. Secondly, as a risk measure it is neither averse nor coherent. In fact, the aversity and the monotonicity axioms are not valid because of the nonlinearity and nonconvexity of ψ. With this approach, a profit distribution is acceptable in the sense of (9) when the deterministic system with the expected value of θ for the parameters, has a profit greater than zero.

4.3 Expected profit

$$R(\psi) = -E_\theta[\psi] \tag{12}$$

This is a coherent measure of risk commonly used in oil production optimization where it is called robust optimization (Van Essen et al., 2009; Capolei et al., 2013, 2015). With this risk measure, a profit distribution is acceptable in the sense of (9) when the expected profit is greater than zero. Despite its widespread use, this risk measure has some important drawbacks. These are its risk neutrality and the fact that there is no control on the possible outcome of very low profit realizations. The use of the expected profit as risk measure in (4) is justified when the following two assumptions hold (Krokhmal et al., 2011):

- Long run assumption: The control found as a solution of the stochastic problem will be employed repeatedly under identical or similar conditions.
- Low variability of the profit realizations. If the profit is highly variable, then the possible outcome of very low profit realizations could be disastrous.

However, in oil production optimization we do not apply the control to many oil reservoirs that are in similar conditions and that share the same uncertainty distribution of the parameters, i.e. we do not have repetition. So, the long run assumption does not hold. Further, the variability in the profit distribution ψ could be high due to high uncertainty in the parameters. For these reasons, the expected value of the profit does not seems to be the right choice for risk management in oil production optimization.

4.4 Worst-case scenario

$$R(\psi) = -\inf_\theta\{\psi\} \tag{13}$$

This is a coherent and averse measure of risk used for example in Alhuthali et al. (2010). However, often it provides a too conservative risk measure. In fact, its major drawback is that it does not take into account the probability distribution of ψ. With this risk measure, a profit distribution is acceptable in the sense of (9) when the lowest possible profit is greater than zero.

4.5 Standard deviation and variance

$$R(\psi) = -\sigma_\theta(\psi), \quad R(\psi) = -\sigma_\theta^2(\psi) \tag{14}$$

By definition, $\sigma_\theta(\psi)$ measures the deviation of the profit from the expected value. The risk preference with this risk measure does not obey the constant equivalence principle. In fact, taken two constant profit distributions c_1, c_2 we have $\sigma_\theta(c_1) = \sigma_\theta(c_2) = 0$, see Fig. 2a. Consequently, with this risk measure it makes no sense to talk of acceptable risk in the sense of (9). Further, the standard deviation misses risk averseness and monotonicity (see the axioms in Section 3). In general, this measure lacks aversity and coherency, see Fig. 2 and Fig. 3. The standard deviation was used as an uncertainty measure, for example in Yeten et al. (2003); Bailey et al. (2005); Alhuthali et al. (2010); Capolei et al. (2013) and interepreted as a risk measure in Capolei et al. (2015). The variance, σ^2, compared to σ, misses also the positive homogeneity and subadditivity.

4.6 Safety margin

$$R(\psi) = -\big(E_\theta[\psi] - \lambda\,\sigma_\theta(\psi)\big), \quad \lambda > 0 \tag{15}$$

This is an averse risk measure, but it is not coherent because it does not satisfy the monotonicity axiom. With this risk measure, a risk is acceptable, in the sense of (9), when the expected value of ψ is λ units larger than the standard deviation of ψ, i.e.

$$E_\theta[\psi] \geq \lambda\,\sigma_\theta(\psi), \quad \lambda > 0 \tag{16}$$

The safety margin risk measure (15) was used by Yeten et al. (2003); Bailey et al. (2005); Alhuthali et al. (2010). However, they did not use (15) directly to measure the risk. Rather, they used the λ parameter as a weight to find solutions with different profit-uncertainty trade-offs. By varying λ, they were able to compute different $(E_\theta[\psi], \sigma_\theta[\psi])$ solution pairs.

4.7 Mean-Variance

$$R(\psi) = -\big(\lambda E_\theta[\psi] - (1-\lambda)\,\sigma_\theta^2(\psi)\big), \quad \lambda \in [0,1] \tag{17}$$

For $\lambda = 1$ we find the expected profit measure (12) and for $\lambda = 0$ we find the variance (14). For $\lambda \in (0,1)$, (17) satisfies the risk aversion axiom, however, it does not satisfies the positive homogeneity, the subadditivity, and the monocity axioms. Then, for $\lambda \in (0,1)$, (17) is neither averse nor coherent. Because (17) does not satisfies the translation invariance principle (6), we cannot give a sense to the risk acceptance constraint (9). The mean-variance risk measure (17) was used by Capolei et al. (2015). However, they did not use directly (17) to measure the risk. Rather, they used the λ parameter as a weight to find Pareto solutions with different profit-risk trade-offs. They used $E_\theta[\psi]$ as a measure of profit and $\sigma_\theta^2(\psi)$ as a measure of risk.

4.8 Value at Risk (VaR_α)

Value-at-Risk (VaR)) is one of the most widely used risk measures in the area of financial risk management and is a major competitor to the standard deviation measure (JP Morgan, 1994; Jorion, 2006). Given a profit distribution ψ, $VaR_\alpha(\psi)$ is defined as the negative α-quantile

$$VaR_\alpha(\psi) = -q_\psi(\alpha), \quad \alpha \in (0,1) \tag{18}$$

where

$$q_\psi(\alpha) = \inf\{z \big| Prob[\psi \leq z] > \alpha\}. \tag{19}$$

The quantile with α confidence level, denoted as $q_\psi(\alpha)$, is the value for which the probability that the profit ψ is lower than $q_\psi(\alpha)$ is no greater than α, see Fig. 4c. The Value-at-Risk concept has its counterparts in the form of probabilistic, or chance constraints, that were introduced by Cooper and Symonds (1958), and since then have been widely used in many disciplines as operations research and stochastic programming, systems reliability theory, reliability-based design and optimization, and others (Ditlevsen and Madsen, 1996; Rockafellar and Royset, 2010). With a chance constraint, we may declare that a profit ψ should exceed a certain predefined level c_{ref} with probability of at least $1-\alpha$, with $\alpha \in (0,1)$:

$$Prob[\psi \geq c_{ref}] \geq 1 - \alpha \tag{20}$$

whereas in the case of $\alpha = 0$, constraint (20) reduces to the worst case approach in Section 4.4. From a risk reduction point of view, the probabilistic constraint (20) has a dual aspect. One aspect is that for a fixed α, we would like to find the highest value of c_{ref} such that (20) is satisfied.

This ensures that with a probability greater than $1-\alpha$, the profit lower bound is the highest possible. On the other hand for a fixed c_{ref} value, we would like to have α as low as possible to increase the probability of having profits larger than c_{ref}. The chance constraint (20) is also called the failure probability constraint in reliability theory (Ditlevsen and Madsen, 1996; Rockafellar and Royset, 2010). The probabilistic constraint (20) is equivalent to

$$Prob[\psi < c_{ref}] \leq \alpha \tag{21}$$

and it can be expressed as a constraint on the Value-at-Risk of ψ (Krokhmal et al., 2011; Zabarankin and Uryasev, 2014):

$$VaR_\alpha(\psi) \leq -c_{ref} \tag{22}$$

One of the major deficiencies of VaR_α is that it does not take into account the tail of the profit distribution beyond the $\alpha-$quantile level. Even more importantly, VaR_α does not satisfies the subadditivity axiom (A3) (Artzner et al., 1999). In addition, VaR_α is discontinuous with respect to the confidence level α, see Fig. 4c, meaning that small changes in the values of α can lead to significant jumps in the risk estimates provided by VaR_α. Despite VaR_α does not satisfies the translation invariance principle (6), because VaR_α is not subadditive, we can still give a sense to the risk acceptance constraint (9) by using the probabilistic interpretation (21). Then, a profit distribution is acceptable in the sense of (21) when

$$VaR_\alpha(\psi) \leq 0 \tag{23}$$

The meaning of (23) is to limit with α the probability of having negative profits (loss), i.e.

$$Prob[\psi < 0] \leq \alpha. \tag{24}$$

4.9 Conditional Value at Risk ($CVaR_\alpha$)

$$R(\psi) = CVaR_\alpha(\psi(u,\theta)) \tag{25}$$

As a measure of risk, $VaR_\alpha(\psi)$ lacks continuity with respect to α, provides no information of how significant losses in the α-tail could be and it is not subadditive, see axiom (A3) in Section 3. These VaR's deficiencies are resolved by the conditional value-at-risk ($CVaR$) (Rockafellar and Uryasev, 2002) defined as the average of VaR on $[0,\alpha]$:

$$CVaR_\alpha(\psi) = \frac{1}{\alpha}\int_0^\alpha VaR_s(\psi)ds, \quad \alpha \in (0,1) \tag{26}$$

Since $CVaR_\alpha(\psi)$ dominates $VaR_\alpha(\psi)$ i.e. $CVaR_\alpha(\psi) \geq VaR_\alpha(\psi)$ (Krokhmal et al., 2011; Zabarankin and Uryasev, 2014), we can approximate the chance constraint (20) with

$$CVaR_\alpha(\psi) \leq -c_{ref}. \tag{27}$$

$CVaR_\alpha$ yields a convex upper bound approximation for the failure probability. This was used in structural engineering problems by Rockafellar and Royset (2010). In the oil community, $CVaR_\alpha$ has been used by Valladao et al. (2013) as a deviation measure. They used a λ parameter as a weight to find Pareto solutions with different expected profit - profit deviation trade-offs. They used $E_\theta[\psi]$ as a measure of profit and $\mathcal{D} = E_\theta[\psi] + CVaR_\alpha[\psi]$ as a deviation measure. Note that this deviation measure $\mathcal{D}$, similarly to the standard deviation σ, does not satisfy the risk aversion and the monotonicity axioms. In the limit of α approaching zero, $CVaR_\alpha$ reduces to the worst-case measure (13). In fact, we have

$$\lim_{\alpha\to 0^+} CVaR_\alpha(\psi) = -\inf_\theta\{\psi\} \tag{28}$$

and in the limit of α approaching one we find the expected-profit measure 4.3. In fact, we have

$$\lim_{\alpha\to 1^-} CVaR_\alpha(\psi) = -E_\theta(\psi) \tag{29}$$

Finally, $CVaR_\alpha$ for $\alpha \in (0,1)$ satisfies all the axioms of a coherent averse measure of risk, given in Definition 1. By using $CVaR_\alpha$ as a risk measure, a profit distribution is acceptable in the sense of (9) when

$$CVaR_\alpha(\psi) \leq 0 \tag{30}$$

Considering that $CVaR_\alpha$ is an approximation of the chance constraint (21), the meaning of (30) is to limit with α the probability of having negative profits (loss), i.e.

$$Prob[\psi < 0] \leq \alpha. \tag{31}$$

However, differently from VaR_α, $CVaR_\alpha$ considers information about the tail of the profit distribution beyond the $\alpha-$quantile level, and yields a smooth and convex approximation of (20). In Fig. 4d we compute $CVaR_\alpha$ for a test profit distribution resulting from the scenario with 8 profit realizations of Fig. 4a.

5. CONCLUSIONS

When we look at the oil production optimization problem as a risk minimization problem, we find that common risk measures used in the oil community are non satisfactory. Instead, we propose the conditional value at risk and the worst-case scenario as appropriate risk measures for risk minimization.

REFERENCES

Alhuthali, A.H., Datta-Gupta, A., Yuen, B., and Fontanilla, J.P. (2010). Optimizing smart well controls under geologic uncertainty. *Journal of Petroleum Science and Engineering*, 73, 107–121.

Artzner, P., Delbaen, F., Eber, J.M., and Heath, D. (1999). Coherent measures of risk. *Mathematical Finance*, 9(3), 203–228.

Bailey, W.J., Couët, B., and Wilkinson, D. (2005). Framework for field optimization to maximize asset value. *SPE Reservoir Evaluation & Engineering*, 8(1), 7–21.

Brouwer, D.R. and Jansen, J.D. (2004). Dynamic optimization of waterflooding with smart wells using optimal control theory. *SPE Journal*, 9(4), 391–402. SPE-78278-PA.

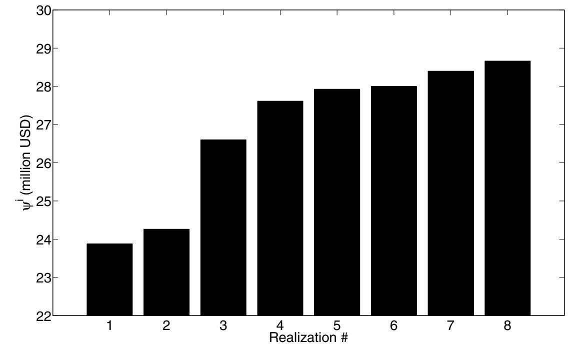
Capolei, A., Suwartadi, E., Foss, B., and Jørgensen, J.B. (2013). Waterflooding optimization in uncertain geological scenarios. *Computational Geosciences*, 17(6), 991–1013.

Capolei, A., Suwartadi, E., Foss, B., and Jørgensen, J.B. (2015). A mean-variance objective for robust production optimization in uncertain geological scenarios. *Journal of Petroleum Science and Engineering*, 125, 23–37.

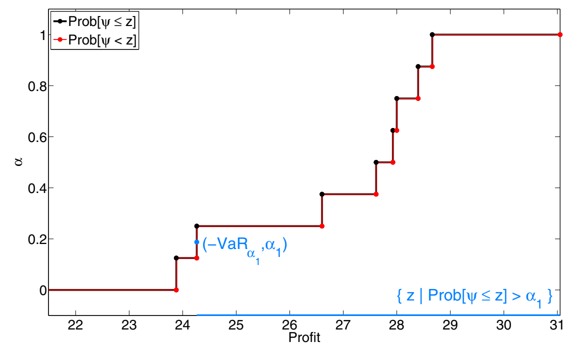
Cooper, A. and Symonds, G. (1958). Cost horizons and certainty equivalents: an approach to stochastic programming of heating oil. *Management Science*, 4, 235–263.

Ditlevsen, O. and Madsen, H.O. (1996). *Structural Reliability Methods*. Wiley.

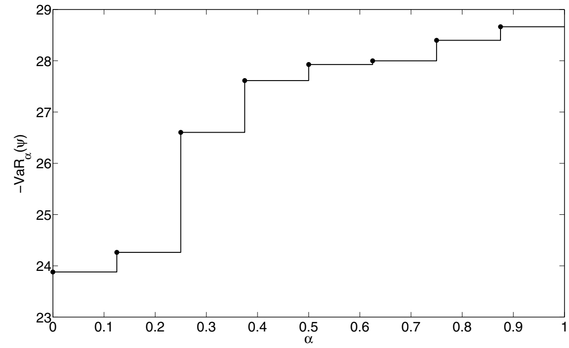
Foss, B. and Jensen, J.P. (2011). Performance analysis for closed-loop reservoir management. *SPE Journal*, 16(1), 183–190. doi:10.2118/138891-PA. SPE-138891-PA.



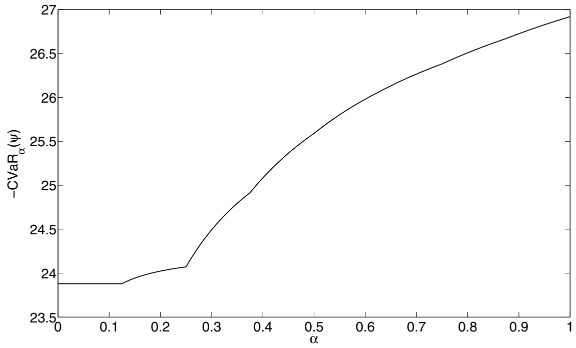
(a) Profit for each realization of the ensemble.



(b) Probability functions and αquantile.



(c) Value at Risk funtion



(d) Conditional Value at Risk

Fig. 4. Illustration of the concepts of quantile, value at risk and conditional value at risk by using an example of 8 realizations. Fig. 4a shows the profit for each realization in the ensemble. Fig. 4b illustrates the $Prob[\psi \leq z], Prob[\psi < z]$ functions. Fig. 4c depicts the discontinuous VaR_α function while Fig. 4d shows the continuous $CVaR_\alpha$ function.

Jorion, P. (2006). *Value at Risk: The New Benchmark for Managing Financial Risk*. McGraw-Hill, 3rd edition.

JP Morgan (1994). *Riskmetrics*. New York.

Krokhmal, P., Zabarankinb, M., and Uryasev, S. (2011). Modeling and optimization of risk. *Surveys in Operations Research and Management Science*, 16, 49–66.

Liu, X. and Reynolds, A.C. (2015). Multiobjective optimization for maximizing expectation and minimizing uncertainty or risk with application to optimal well control. In *SPE Reservoir Simulation Symposium*, SPE-173216-MS.

Markowitz, H.M. (1959). *Portfolio selection: efficient diversification of investments*. John Wiley.

Nævdal, G., Brouwer, D.R., and Jansen, J.D. (2006). Waterflooding using closed-loop control. *Computational Geosciences*, 10, 37–60.

Rockafellar, R.T. and Royset, J.O. (2010). On buffered failure probability in design and optimization of structures. *Reliability Engineering and System Safety*, 95, 499–510.

Rockafellar, R. (2007). *Coherent Approaches to Risk in Optimization Under Uncertainty*, 38–61. Tutorials in Operations Research INFORMS.

Rockafellar, R. and Uryasev, S. (2002). Conditional value-at-risk for general loss distributions. *J. Bank. Financ.*, 26(7).

Roy, A. (1952). Safety first and the holding of assets. *Econometrica*, 20(3), 431–449.

Sarma, P., Aziz, K., and Durlofsky, L.J. (2005). Implementation of adjoint solution for optimal control of smart wells. In *SPE Reservoir Simulation Symposium, 31 January-2 Feburary 2005, The Woodlands, Texas*, 67–83.

Valladao, D.M., Torrado, R.R., Flach, B., and Embid, S. (2013). On the stochastic response surface methodology for the determination of the development plan of an oil & gas field. In *SPE Middle East Intelligent Energy Conference and Exhibition*, SPE-167446-MS.

Van Essen, G.M., Van den Hof, P.M.J., and Jansen, J.D. (2011). Hierarchical long-term and short-term production optimization. *SPE Journal*, 16(1), 191–199. SPE-124332-PA.

Van Essen, G.M., Zandvliet, M.J., Van den Hof, P.M.J., Bosgra, O.H., and Jansen, J.D. (2009). Robust waterflooding optimazation of multiple geological scenarios. *SPE Journal*, 14(1), 202–210. SPE-102913-PA.

Yeten, B., Durlofsky, L.J., and Aziz, K. (2003). Optimization of nonconventional well type, location, and trajectory. *SPE Journal*, 8(3), 200–210.

Zabarankin, M. and Uryasev, S. (2014). *Statistical Decision Problems*. Springer.

