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A Computational Analysis of Convex Combination Models for Multidimensional Piecewise-Linear Approximation in Oil Production Optimization ⋆

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Abstract: The lift-gas allocation problem with pressure-drop constraints and well-separator routing is a mixed-integer nonlinear program considerably hard to solve. To this end, a mixed-integer linear programming formulation was developed by multidimensional piecewise-linearization of pressure drop functions using standard (CC) and logarithmic (Log) aggregated models. These models were compared by means of a computational analysis, which indicates that the logarithmic model is faster than the standard one possibly because of the reduced number of variables and constraints.

Keywords: gas-lift, pressure constraints, multidimensional piecewise-linearization, MILP.

1. INTRODUCTION

Technological innovations in hardware and software, such as tools for digital measurement and data processing, make possible the use of advanced automation techniques for production optimization of oil fields. However, scientific and technological challenges still remain to turn the concept of smart fields into a viable technology (Campos et al., 2010; Camponogara et al., 2010; Yeten et al., 2004).

Oil production is usually limited by reservoir conditions, the capacity of surface facilities, such as lift-gas availability, flow and pressure constraints in pipelines, among others. So, for an optimal daily operation, the interactions between reservoir, wells, and surface facilities should be considered simultaneously. Nowadays the operation of offshore oil fields is commonly based on sensitivity analysis using simulation tools and heuristics. Although these methods optimize oil production for particular cases, they do not necessarily determine the operational mode that maximizes daily oil production (Kosmidis et al., 2005).

Many works that use mathematical programming tools to maximize production of offshore oil fields are found in the literature (Buitrago et al., 1996; Alarcón et al., 2002; Kosmidis et al., 2005; Camponogara and de Conto, 2009; Misener et al., 2009). An aspect that still has scientific challenges is the representation of pressure constraints in flow lines connecting wells, manifolds, and separators. To this end, this work addresses the representation of pressure constraints with multidimensional piecewise-linear functions, for the problem of allocating a limited lift-gas rate to oil wells subject to routing and pressure constraints.

When surface conditions such as manifold pressure vary frequently, models based on gas-lift performance curves (GLPC) may not represent well production satisfactorily (Kosmidis et al., 2004). In such cases, the well flow depends on the lift-gas injected and the pressure of the production manifold downstream, which gathers the production of several wells sharing a common pipeline that connects the manifold to a separator. Further, the pressure drop in these pipelines depends on the total rate of gas, oil, and water.

Because such functions are multidimensional and nonconvex, their direct use in optimization models may result in a *NP-Hard* problem that could become computationally intractable (Keha et al., 2006). Litvak et al. (1997); Bieker (2007); Gunnerud and Foss (2010) proposed piecewise-linear approximations for the pressure drops based on a regular grid of breakpoints using Special Ordered Sets of Type 2 (SOS2).

This paper proposes the approximation of pressure-drop curves with multidimensional piecewise-linear functions based on mixed-integer linear models described in (Vielma et al., 2010). The pressure-drop functions are piecewise linearized with simplexes according to J1 ("Union Jack") triangulations (Todd, 1977). The first model is called convex combination (CC) because the functions are approximated with the convex combination of breakpoints and binary variables to restrict the combination to only one simplex of the J1 triangulation. The second model is called logarithmic convex combination (Log) because the number of variables and constraints is logarithmic in the set of polytopes, a result of a special branching scheme based on SOS2. These models are compared by means of computational experiments.

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2. PROBLEM FORMULATION

The problem of allocating lift-gas to oil wells and deciding upon the routing of wells to separation units, while being subject to lift-gas and facility constraints, was addressed in (Codas and Camponogara, 2012). For analyzing multi-dimensional piecewise-linear models, here we extend that work by modeling constraints on flow lines connecting wells, manifolds, and separators. The problem is cast as a mixed-integer nonlinear program:

$$P : \max f = \sum_{m\in\mathcal{M}} g(\mathbf{q}^m) - \sum_{n\in\mathcal{N}} c(q_{\mathrm{i}}^n) \quad (1a)$$

$$\text{s.t.}: \sum_{n\in\mathcal{N}} q_{\mathrm{i}}^n \le q_{\mathrm{i}}^{\max} \quad (1b)$$

For all $n \in \mathcal{N}$:

$$l_n y_n \le q_{\mathrm{i}}^n \le u_n y_n \quad (1c)$$

$$\sum_{m\in\mathcal{M}_n} z_{n,m} = y_n \quad (1d)$$

$$\mathbf{q}^{n,m} = \mathbf{q}^{n,m}(p^m, q_{\mathrm{i}}^n) z_{n,m},\ \forall m \in \mathcal{M}_n \quad (1e)$$

$$y_n \mathbf{q}^{n,\mathrm{L}} \le \sum_{m\in\mathcal{M}_n} \mathbf{q}^{n,m} \le y_n \mathbf{q}^{n,\mathrm{U}} \quad (1f)$$

$$\mathbf{q}^m = \sum_{n\in\mathcal{N}_m} \mathbf{q}^{n,m} \le \mathbf{q}^{m,\mathrm{S}},\ \forall m \in \mathcal{M} \quad (1g)$$

$$p^m = p^{m,\mathrm{S}} + \Delta p^m(\mathbf{q}^m),\ \forall m \in \mathcal{M} \quad (1h)$$

$$y_n \in \{0,1\},\ \forall n \in \mathcal{N} \quad (1i)$$

$$z_{n,m} \in \{0,1\},\ \forall n \in \mathcal{N},\ \forall m \in \mathcal{M}_n \quad (1j)$$

having the following parameters:

- N is the number of oil wells, $\mathcal{N} = \{1,\ldots,N\}$, and $\mathcal{N}_m \subseteq \mathcal{N}$ is the subset of wells whose production can be sent to manifold m;
- M is the number of manifolds, $\mathcal{M} = \{1,\ldots,M\}$, and $\mathcal{M}_n \subseteq \mathcal{M}$ is the subset of manifolds that can handle production from well n. The production of each manifold is directed to a single separator;
- $\mathcal{H} = \{\mathrm{o, g, w}\}$ has the multiphase flows: oil (o), gas (g), and water (w);
- $q_{\mathrm{i}}^{\max}$ is the lift-gas rate output by the compressors;
- l_n and u_n are bounds for lift-gas injection into well n;
- $p^{m,\mathrm{S}}$ is the operational pressure of the separator that receives production from manifold m;
- $\mathbf{q}^{n,\mathrm{L}}$ and $\mathbf{q}^{n,\mathrm{U}}$ are the lower and upper bound on the flow rate of well n;
- $\mathbf{q}^{m,\mathrm{S}}$ is the capacity of the separator of manifold m;

variables:

- q_{i}^n is the lift-gas rate allocated to well n;
- y_n is 1 when well n is producing, and 0 otherwise;
- $z_{n,m}$ takes on value 1 if the production of well n is directed to manifold m, and 0 otherwise;
- $q_h^{n,m}$ is the flow of phase $h \in \mathcal{H}$ sent from well n to manifold m and $\mathbf{q}^{n,m} = (q_h^{n,m} : h \in \mathcal{H})$ is a vector with all phase flows. The gas flow rate received by the production manifold is the sum of the lift-gas injected into well n (Inj) and the gas from the reservoir (R): $q_g^{n,m} = q_{g,\mathrm{R}}^{n,m} + q_{g,\mathrm{Inj}}^{n,m}$;
- $\mathbf{q}^m = \sum_{n\in\mathcal{N}_m} \mathbf{q}^{n,m}$ is the total flow received from the wells connected to manifold m for all phases;
- p^m is the pressure of manifold m;

and functions:

- f is a profit function defined in terms of the oil revenue from the production of each manifold, given by function g, and the cost of lift-gas injection given by function c;
- $q_h^{n,m}(p^m, q_{\mathrm{i}}^n)$ is the flow of phase h sent from well n to manifold m, given as a function of the manifold pressure and the lift-gas injected into the well, with $\mathbf{q}^{n,m}(p^m, q_{\mathrm{i}}^n) = (q_h^{n,m}(p^m, q_{\mathrm{i}}^n) : h \in \mathcal{H})$ being the vector function with all phase flows.
- $\Delta p^m(\mathbf{q}^m)$ is the pressure drop function through the line that connects manifold m and the separator that handles its production.

3. PIECEWISE-LINEAR FORMULATIONS

Piecewise-linear functions are often used to approximate non-linearities and non-convex functions. Optimization problems involving these functions can be modeled as mixed-integer linear programs (MILP), which can be solved with specialized algorithms or general-purpose solvers. Usually, the latter approach takes advantage over the first since it uses the advanced technology available for solving MILPs (Vielma et al., 2010).

This section presents mixed-integer linear formulations for multidimensional piecewise-linear functions that are used later to approximate nonlinear functions of problem P.

Notation Let $f : \mathcal{D} \to \mathbb{R}$ be a continuous function defined over a compact domain $\mathcal{D} \subseteq \mathbb{R}^d$. According to Vielma et al. (2010), f is piecewise-linear if and only if there exists a family of polytopes $\mathcal{P}$, such that $\cup_{P\in\mathcal{P}} P = \mathcal{D}$, $\{\mathbf{m}_P\}_{P\in\mathcal{P}} \subseteq \mathbb{R}^d$, and $\{c_P\}_{P\in\mathcal{P}}$, such that:

$$f(\mathbf{x}) = \mathbf{m}_P' \mathbf{x} + c_P, \quad \mathbf{x} \in P, \forall P \in \mathcal{P} \quad (2)$$

Let $V(P)$ be the set of vertices of polytope P and $\mathcal{V}(\mathcal{P}) = \cup_{P\in\mathcal{P}} V(P)$ be the set of all vertices.

The MILP formulations of piecewise-linear functions will be illustrated with the function depicted in Figure 1. The domain of this function is $\mathcal{D} = [0,4]$, which is represented by a family of polytopes $\mathcal{P} = \{P_1, P_2, P_3, P_4\}$, where $P_1 = [0,1]$, $P_2 = [1,2]$, $P_3 = [2,3]$, and $P_4 = [3,4]$, and $V(P_1) = \{0,1\}$, $V(P_2) = \{1,2\}$, $V(P_3) = \{2,3\}$, and $V(P_4) = \{3,4\}$.

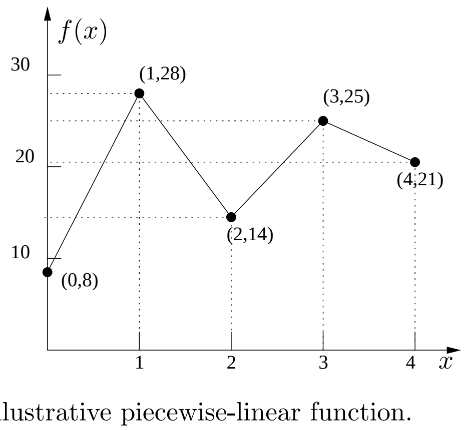

Fig. 1. Illustrative piecewise-linear function.

The formulations of concern in this paper consist of the convex combination of break points $\{(\mathbf{v}, f(\mathbf{v})) : \mathbf{v} \in$

$V(P)\}$, which in the case of the illustrative example are $\{(0,8),(1,28),(2,14),(3,25),(4,21)\}$.

3.1 Convex Combination Model (CC)

The aggregated convex combination model (CC) assigns weighting variables to each vertex $\mathbf{v} \in \mathcal{V}(\mathcal{P})$. Thus a graph point is represented by $(\mathbf{x}, f(\mathbf{x})) = \sum_{\mathbf{v}\in\mathcal{V}(\mathcal{P})} \lambda_{\mathbf{v}}(\mathbf{v}, f(\mathbf{v}))$, $\{\lambda_{\mathbf{v}}\}_{\mathbf{v}\in\mathcal{V}(\mathcal{P})} \subset \mathbb{R}_+$ such that $\sum_{\mathbf{v}\in\mathcal{V}(\mathcal{P})} \lambda_{\mathbf{v}} = 1$. The CC model is given by:

$$\sum_{\mathbf{v}\in\mathcal{V}(\mathcal{P})} \lambda_{\mathbf{v}}\mathbf{v} = \mathbf{x} \tag{3a}$$

$$\sum_{\mathbf{v}\in\mathcal{V}(\mathcal{P})} \lambda_{\mathbf{v}} f(\mathbf{v}) = f(\mathbf{x}) \tag{3b}$$

$$\lambda_{\mathbf{v}} \geq 0,\ \forall \mathbf{v} \in \mathcal{V}(\mathcal{P}) \tag{3c}$$

$$\sum_{\mathbf{v}\in\mathcal{V}(\mathcal{P})} \lambda_{\mathbf{v}} = 1 \tag{3d}$$

$$\lambda_{\mathbf{v}} \leq \sum_{P\in\mathcal{P}(\mathbf{v})} y_P,\ \forall \mathbf{v} \in \mathcal{V}(\mathcal{P}) \tag{3e}$$

$$\sum_{P\in\mathcal{P}} y_P = 1,\ y_P \in \{0,1\},\ \forall P \in \mathcal{P} \tag{3f}$$

where $\mathcal{P}(\mathbf{v}) := \{P \in \mathcal{P} : \mathbf{v} \in V(P)\}$ is the set of polytopes that contain vertex $\mathbf{v}$. This formulation was studied in the literature, in particular by Keha et al. (2004), Lee and Wilson (2001), and Padberg (2000).

3.2 Logarithmic Convex Combination Model (Log)

The logarithmic convex combination (Log) model is a variation of CC that uses only a logarithmic number of binary variables and constraints. Log associates each polytope $P \in \mathcal{P}$ with a binary vector $\mathbf{y} \in \{0,1\}^{\lceil \log_2 |\mathcal{P}| \rceil}$ through an injective function $B : \mathcal{P} \to \{0,1\}^{\lceil \log_2 |\mathcal{P}| \rceil}$ such that $B(P) = \mathbf{y}$. However, as discussed by Vielma et al. (2010), B should comply with conditions for a branching scheme on the λ variables such that the non-zero λ variables are associated with the vertices of at least one polytope of $\mathcal{P}$:

$$\exists P \in \mathcal{P} \text{ such that } \{\mathbf{v} \in \mathcal{V}(\mathcal{P}) : \lambda_{\mathbf{v}} > 0\} \subseteq V(P) \tag{4}$$

A branching scheme for (4) consists of a sequence $\{L_t, R_t\}_{t\in\mathcal{T}}$ of dichotomies defined by a finite set $\mathcal{T}$ of indices and corresponding subsets $L_t, R_t \subset \mathcal{V}(\mathcal{P})$, such that for every $P \in \mathcal{P}$ it is true that $V(P) = \cap_{t\in\mathcal{T}}(\mathcal{V}(\mathcal{P})\backslash T_t)$ where $T_t = L_t$ or $T_t = R_t$ for each $t \in \mathcal{T}$. Such branching scheme imposes (4) by fixing λ variables to zero in each side of the series, namely fixing $\lambda_{\mathbf{v}} = 0$ for all $\mathbf{v} \in L_t$ on the left branch and $\lambda_{\mathbf{v}} = 0$ for all $\mathbf{v} \in R_t$ on the right branch. For the illustrative example, a suitable branching scheme illustrated in Figure 2 is $\mathcal{T} = \{1,2\}$, $L_1 = \{2\}$, $R_1 = \{0,4\}$, $L_2 = \{3,4\}$, and $R_2 = \{0,1\}$. Given a branching scheme for (4), a valid formulation for the Log model is given by:

$$\text{Eqs. (3a)–(3d)} \tag{5a}$$

$$\sum_{\mathbf{v}\in L_t} \lambda_{\mathbf{v}} \leq y_t,\ \forall t \in \mathcal{T} \tag{5b}$$

$$\sum_{\mathbf{v}\in R_t} \lambda_{\mathbf{v}} \leq (1 - y_t),\ \forall t \in \mathcal{T} \tag{5c}$$

$$y_t \in \{0,1\},\ \forall t \in \mathcal{T} \tag{5d}$$

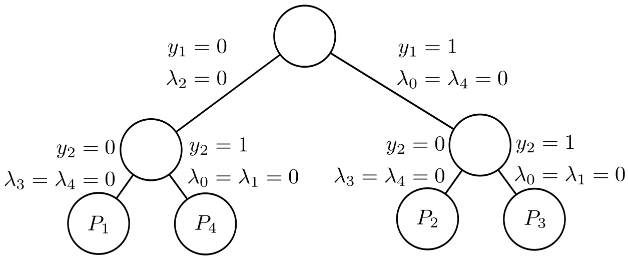

Fig. 2. Branching scheme for the Log model.

Vielma and Nemhauser (2011) developed a branching scheme with a logarithmic number of binary variables and dichotomies. Such branching scheme is valid for a polytope family $\mathcal{P}$ that is topologically equivalent or compatible with a triangulation known as J1 or "Union Jack." They also proposed a two-phase procedure branching schemes compatible with SOS2 constraints for such triangulations. The first phase consists in using disjunctive sets to limit convex combination to a hypercube, while the second phase inhibits the vertices of the hypercube so that the convex combinations are restricted to a single simplex.

Nonlinearities of problem P appear in the well production curves, which depend on the manifold pressure and lift-gas injection, and the pressure drop functions in the flow lines that connect production manifolds to separators. The computational hardness of P rests on the nonlinear, multidimensional, and non-convex nature of these functions, besides the discrete decisions involving well activation and routing. In what follows, we present two MILP reformulations of problem P that approximate the non-linear functions using the CC and Log models.

4. CC MODEL REFORMULATION

The liquid rate $\mathbf{q}^{n,m}(p^m, q_{\mathrm{i}}^n)$ of well n depends on the pressure of the manifold m to which it is connected, p^m, and the lift-gas injection rate, q_{i}^n. The liquid rate is represented by a piecewise-linear function using the CC model as follows:

For all $n \in \mathcal{N}$:

$$q_{\mathrm{i}}^n = \sum_{m\in\mathcal{M}_n} \sum_{\mathrm{q_i}\in\mathcal{K}^{n,m}} \sum_{\mathrm{p_r}\in\mathcal{R}^{n,m}} \lambda_{\mathrm{q_i},\mathrm{p_r}}^{n,m} \mathrm{q_i} \tag{6a}$$

$$\sum_{\mathrm{q_i}\in\mathcal{K}^{n,m}} \sum_{\mathrm{p_r}\in\mathcal{R}^{n,m}} \lambda_{\mathrm{q_i},\mathrm{p_r}}^{n,m} \mathrm{p_r} \leq p^m,\ \forall m \in \mathcal{M}_n \tag{6b}$$

$$p^m \leq \sum_{\mathrm{q_i}\in\mathcal{K}^{n,m}} \sum_{\mathrm{p_r}\in\mathcal{R}^{n,m}} \lambda_{\mathrm{q_i},\mathrm{p_r}}^{n,m} \mathrm{p_r} + p^{m,\max}(1 - z_{n,m}),\quad \forall m \in \mathcal{M}_n \tag{6c}$$

$$\tilde{\mathbf{q}}^{n,m} = \sum_{\mathrm{q_i}\in\mathcal{K}^{n,m}} \sum_{\mathrm{p_r}\in\mathcal{R}^{n,m}} \lambda_{\mathrm{q_i},\mathrm{p_r}}^{n,m} \mathbf{q}^{n,m}(\mathrm{q_i}, \mathrm{p_r}),\ \forall m \in \mathcal{M}_n \tag{6d}$$

$$y_n \mathbf{q}^{n,\mathrm{L}} \leq \sum_{m\in\mathcal{M}_n} \tilde{\mathbf{q}}^{n,m} \leq y_n \mathbf{q}^{n,\mathrm{U}} \tag{6e}$$

$$\lambda_{\mathrm{q_i},\mathrm{p_r}}^{n,m} \geq 0,\ \forall \mathrm{q_i} \in \mathcal{K}^{n,m},\ \forall m \in \mathcal{M}_n,\ \forall \mathrm{p_r} \in \mathcal{R}^{n,m} \tag{6f}$$

$$\sum_{\mathrm{q_i}\in\mathcal{K}^{n,m}} \sum_{\mathrm{p_r}\in\mathcal{R}^{n,m}} \lambda_{\mathrm{q_i},\mathrm{p_r}}^{n,m} = z_{n,m},\ \forall m \in \mathcal{M}_n, \tag{6g}$$

$$\lambda^{n,m}_{\mathrm{q_i},\mathrm{p_r}} \leq \sum_{P\in\mathcal{P}^{n,m}(\mathrm{q_i},\mathrm{p_r})} \delta^{n,m}_P,\ \forall m\in\mathcal{M}_n,\ \forall \mathrm{q_i}\in\mathcal{K}^{n,m},\ \forall \mathrm{p_r}\in\mathcal{R}^{n,m} \tag{6h}$$

$$\sum_{P\in\mathcal{P}^{n,m}} \delta^{n,m}_P = z_{n,m},\ \forall m\in\mathcal{M}_n \tag{6i}$$

$$\delta^{n,m}_P \in \{0,1\},\ \forall P\in\mathcal{P}^{n,m} \tag{6j}$$

having the following extra parameters:

- $\mathcal{K}^{n,m}$ and $\mathcal{R}^{n,m}$ are the set of breakpoints for the lift-gas rate and manifold pressure when well n is connected to manifold m, respectively;
- $\mathcal{P}^{n,m}$ is the set of polytopes with vertices in $\mathcal{K}^{n,m}\times\mathcal{R}^{n,m}$;
- $\mathcal{P}^{n,m}(\mathrm{q_i},\mathrm{p_r}) = \{P\in\mathcal{P}^{n,m} : (\mathrm{q_i},\mathrm{p_r})\in V(P)\}$;
- $p^{m,\max}$ is the maximum manifold pressure;

and extra variables:

- $\lambda^{n,m}_{\mathrm{q_i},\mathrm{p_r}}$ is the weighting variable of a breakpoint pair in $\mathcal{K}^{n,m}\times\mathcal{R}^{n,m}$. When manifold m receives the production of well n, $z_{n,m}$ takes on value 1 and the respective convex combination becomes active;
- $\delta^{n,m}_P$ is a binary variable associated to each polytope $P\in\mathcal{P}^{n,m}$ which assumes value 1 when the convex combination is limited to polytope P. According to constraints (6h)−(6j), only the vertices of P can be part of the convex combination that defines lift-gas injection into well n and manifold pressure p^m;
- $\tilde{\mathbf{q}}^{n,m}$ is the piecewise-linear approximation of $\mathbf{q}^{n,m}$.

The pressure drop $\Delta p^m(\mathbf{q}^m)$ for the flow line connecting manifold m to its separator is a nonlinear function, which appears in equation (1h). A common way of representing pressure drop curves is with piecewise-linear functions, which divide the decision domain in hypercubes with vertices corresponding to breakpoints and which employ SOS2 constraints to ensure that convex combinations use only vertices of a single hypercube (Gunnerud and Foss, 2010; Bieker, 2007; Kosmidis et al., 2005).

Unlike these works, we approximate the pressure drop curves with piecewise-linear forms using simplexes. For the CC model, P is reformulated by piecewise-linearizing the nonlinear constraints (1g) and (1h) which depends on the nonlinear pressure drop function $\Delta p^m(\mathbf{q}^m)$.

For all $m\in\mathcal{M}_n$:

$$\tilde{\mathbf{q}}^m = \sum_{(k^o,k^g,k^w)\in\mathcal{Q}^m} \Omega^m_{k^o,k^g,k^w}\mathbf{q}^m_{k^o,k^g,k^w} \tag{7a}$$

$$\tilde{\mathbf{q}}^m = \sum_{n\in\mathcal{N}_m} \tilde{\mathbf{q}}^{n,m} \leq \mathbf{q}^{m,\mathrm{S}} \tag{7b}$$

$$\widetilde{\Delta p}^m = \sum_{(k^o,k^g,k^w)\in\mathcal{Q}^m} \Omega^m_{k^o,k^g,k^w}\Delta p^m(\mathbf{q}^m_{k^o,k^g,k^w}) \tag{7c}$$

$$p^m = p^{m,\mathrm{S}} + \widetilde{\Delta p}^m \tag{7d}$$

$$\Omega^m_{k^o,k^g,k^w} \geq 0,\ \forall (k^o,k^g,k^w)\in\mathcal{Q}^m \tag{7e}$$

$$\Omega^m_{k^o,k^g,k^w} \leq \sum_{P\in\mathcal{P}^m(k^o,k^g,k^w)} \delta^m_P,\ \forall (k^o,k^g,k^w)\in\mathcal{Q}^m \tag{7f}$$

$$\sum_{P\in\mathcal{P}^m} \delta^m_P = y^m \tag{7g}$$

$$\sum_{(k^o,k^g,k^w)\in\mathcal{Q}^m} \Omega^m_{k^o,k^g,k^w} = y^m \tag{7h}$$

$$y^m \leq \sum_{n\in\mathcal{N}_m} z_{n,m} \tag{7i}$$

$$y^m \in \{0,1\} \tag{7j}$$

$$\delta^m_P \in \{0,1\},\ \forall P\in\mathcal{P}^m \tag{7k}$$

having the extra parameters:

- $\mathcal{Q}^m = O^m\times G^m\times W^m$ is the set of breakpoints of flow vector $\mathbf{q}^m$, where G^m, O^m, W^m are the sets of gas, oil, and water breakpoints of manifold m, respectively;
- $\mathcal{P}^m(k^o,k^g,k^w) = \{P\in\mathcal{P}^m : (k^o,k^g,k^w)\in V(P)\}$;

and following extra variables:

- $\tilde{\mathbf{q}}^m$ is the piecewise-linear approximation of $\mathbf{q}^m$;
- $\widetilde{\Delta p}^m$ is the piecewise-linear form of the pressure drop in the output flow line of manifold m;
- $\Omega^m_{k^o,k^g,k^w}$ is the weighting variable associated to a breakpoint in $\mathcal{Q}^m$;
- y^m is a binary variable that assumes value 1 when manifold m receives production from any well.

The polytopes that piecewise linearize the functions $\mathbf{q}^{n,m}(p^m,q^n_{\mathrm{i}})$ and $\Delta p^m(\mathbf{q}^m)$ are simplexes according to J1 ("Union Jack") triangulation (Todd, 1977). Finally, the objective function of problem P described in equation (1a) is then rewritten in a piecewise-linear form as follows:

$$\max\ \tilde{f} = \sum_{m\in\mathcal{M}} g(\tilde{\mathbf{q}}^m) - \sum_{n\in\mathcal{N}} c(q^n_{\mathrm{i}}) \tag{8}$$

5. LOG MODEL REFORMULATION

New concepts are introduced to implement the branching scheme proposed by Vielma and Nemhauser (2011). Let $\mathcal{S}_e = \{s_0,\ldots,s_n\}$ be the set of breakpoints on axis e and $\mathcal{I}_e := \{[s_0,s_1],\ldots,[s_{n-1},s_n]\}$ be the intervals of breakpoints. Let $\mathcal{I}_e(s) := \{\mathcal{I}\in\mathcal{I}_e : s\in\mathcal{I}\}$ be the intervals containing s. Let $\Xi_e([s_i,s_{i+1}]) = i+1$ be the index of an interval $[s_i,s_{i+1}]\in\mathcal{I}_e$. Let $B:\{1,\ldots,|\mathcal{I}_e|\}\to\{0,1\}^{\lceil\log_2(|\mathcal{I}_e|)\rceil}$ be a SOS2 compatible function, meaning that $B(i)$ and $B(i+1)$ differ only in one bit according to the Gray code property. The vertices of the domain is $\mathcal{V}(\mathcal{P}) = S_1\times\cdots\times S_d$ and d is the dimension. The first phase of the branching scheme uses the sets $J^+_{e,B,l} := \{s\in\mathcal{S}_e : B(\Xi_e(\mathcal{I}))_l = 1, \forall\mathcal{I}\in\mathcal{I}_e(s)\}$ and $J^0_{e,B,l} := \{s\in\mathcal{S}_e : B(\Xi_e(\mathcal{I}))_l = 0, \forall\mathcal{I}\in\mathcal{I}_e(s)\}$. The second phase selects a simplex of the hypercube obtained in phase one using the sets $\mathcal{L}_{r,s} = \{\mathbf{v}\in\mathcal{V}(\mathcal{P}) : \mathbf{v}_r \text{ is even and } \mathbf{v}_s \text{ is odd}\}$ and $\mathcal{R}_{r,s} = \{\mathbf{v}\in\mathcal{V}(\mathcal{P}) : \mathbf{v}_r \text{ is odd and } \mathbf{v}_s \text{ is even}\}, \forall r,s\in D = \{1,\ldots,d\}$, such that $r<s$.

The Log model piecewise-linearizes $\mathbf{q}^{n,m}(p^m,q^n_i)$ building the J1 triangulation and restricting convex combinations to a single simplex implicitly, instead of using binary variables with the following equations. For all $n\in\mathcal{N},\ m\in\mathcal{M}_n,\ l\in\Phi(\mathcal{K}^{n,m})$:

$$\sum_{\mathrm{q_i}\in J^+_{axis(\mathcal{K}^{n,m}),B,l}}\ \sum_{\mathrm{p_r}\in\mathcal{R}^{n,m}} \lambda^{n,m}_{\mathrm{q_i},\mathrm{p_r}} \leq x^{n,m}_l \tag{9a}$$

$$\sum_{\mathrm{q_i}\in J^0_{axis(\mathcal{K}^{n,m}),B,l}}\ \sum_{\mathrm{p_r}\in\mathcal{R}^{n,m}} \lambda^{n,m}_{\mathrm{q_i},\mathrm{p_r}} \leq (1-x^{n,m}_l) \tag{9b}$$

where $\Phi(\mathcal{S}) = \{1, \ldots, \lceil \log_2(|\mathcal{S}|-1) \rceil\}$. For all $n \in \mathcal{N},\ m \in \mathcal{M}_n,\ l \in \Phi(\mathcal{R}^{n,m})$:

$$\sum_{\mathrm{q_i} \in \mathcal{K}^{n,m}} \sum_{\mathrm{p_r} \in J^{+}_{axis(\mathcal{R}^{n,m}),B,l}} \lambda^{n,m}_{\mathrm{q_i},\mathrm{p_r}} \leq \check{x}^{n,m}_{l} \tag{9c}$$

$$\sum_{\mathrm{q_i} \in \mathcal{K}^{n,m}} \sum_{\mathrm{p_r} \in J^{0}_{axis(\mathcal{R}^{n,m}),B,l}} \lambda^{n,m}_{\mathrm{q_i},\mathrm{p_r}} \leq (1 - \check{x}^{n,m}_{l}) \tag{9d}$$

For all $n \in \mathcal{N},\ m \in \mathcal{M}_n,\ (r,s) \in \Gamma^{n,m}$:

$$\sum_{(\mathrm{q_i},\mathrm{p_r}) \in \mathcal{L}^{n,m}_{r,s}} \lambda^{n,m}_{\mathrm{q_i},\mathrm{p_r}} \leq y^{n,m}_{r,s} \tag{9e}$$

$$\sum_{(\mathrm{q_i},\mathrm{p_r}) \in \mathcal{R}^{n,m}_{r,s}} \lambda^{n,m}_{\mathrm{q_i},\mathrm{p_r}} \leq (1 - y^{n,m}_{r,s}) \tag{9f}$$

having the follow extra parameters:

- $axis(\mathcal{Q})$ is the axis that contains breakpoint set $\mathcal{Q}$;
- $\Gamma^{n,m} := \{(r,s) \in D^2 : r < s\}$;
- $\mathcal{L}^{n,m}_{r,s} = \{\mathbf{v} \in \mathcal{V}(\mathcal{P}^{n,m}) : \mathbf{v}_r$ is even and $\mathbf{v}_s$ is odd$\}$ and $\mathcal{R}^{n,m}_{r,s} = \{\mathbf{v} \in \mathcal{V}(\mathcal{P}^{n,m}) : \mathbf{v}_r$ is odd and $\mathbf{v}_s$ is even$\}$, $\forall r, s \in D$ such that $r < s$.

and extra variables:

- $x^{n,m}_l$ ($\check{x}^{n,m}_l$) is a binary variable that induces the first phase of Log branching for each entry $l \in \Phi(\mathcal{K}^{n,m})$ (Φ $(\mathcal{R}^{n,m})$) of B;
- $y^{n,m}_{r,s}$ is a binary variable that induces the second phase of the Log branching for each $(r,s) \in \Gamma^{n,m}$.

The nonlinear function $\Delta p^m(\mathbf{q}^m)$ is piecewise linearized as follows. For all $m \in \mathcal{M},\ \Upsilon \in \{O, G, W\},\ l \in \Phi(\Upsilon^m)$:

$$\sum_{(k^o,k^g,k^w) \in \mathcal{Q}^m : k^{\Upsilon} \in J^{+}_{axis(\Upsilon^m),B,l}} \Omega^m_{k^o,k^g,k^w} \leq x^{m,\Upsilon}_l \tag{10a}$$

$$\sum_{(k^o,k^g,k^w) \in \mathcal{Q}^m : k^{\Upsilon} \in J^{0}_{axis(\Upsilon^m),B,l}} \Omega^m_{k^o,k^g,k^w} \leq 1 - x^{m,\Upsilon}_l \tag{10b}$$

For all $m \in \mathcal{M},\ (r,s) \in \Gamma^m$:

$$\sum_{(k^o,k^g,k^w) \in \mathcal{L}^m_{r,s}} \Omega^m_{k^o,k^g,k^w} \leq y^m_{r,s} \tag{10c}$$

$$\sum_{(k^o,k^g,k^w) \in \mathcal{R}^m_{r,s}} \Omega^m_{k^o,k^g,k^w} \leq (1 - y^m_{r,s}) \tag{10d}$$

and extra variables:

- $x^{m,\Upsilon}_l$ is the binary variable that defines the first phase of Log branching for phase $\Upsilon \in \{O, G, W\}$ and for each entry $l \in \Phi(\Upsilon^m)$ of B;
- $y^m_{r,s}$ is a binary variable that builds the second phase of Log branching for each $(r,s) \in \Gamma^m$, where $\Gamma^m := \{(r,s) \in D^2 : r < s\}$.

6. COMPUTATIONAL ANALYSIS

This section presents a synthetic production system of a real-world oil field, which serves as a test bed for performance analysis of the CC and Log formulation applied to the oil production optimization problem.

6.1 Production System

The production system has $N = 16$ wells and $M = 2$ manifolds, where $\mathcal{N}_m = \mathcal{N}$ for all $m \in \mathcal{M}$. The field has two separators, one for each manifold. The production network is illustrated in Figure 3.

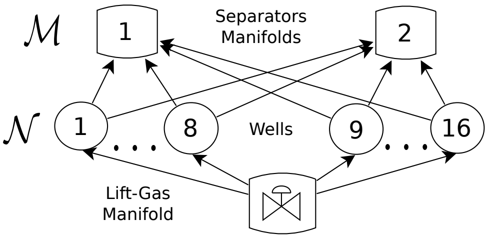

Fig. 3. Gas-lift production network instance.

Wells and manifolds are topologically divided in two groups: wells 1 to 8 are 1 km away from manifold 1, and 10 km away of manifold 2, while wells 9 to 16 are 1 km away of manifold 2 and 10 km away of manifold 1. The pipelines connecting wells to manifolds have 4 inches of inner diameter (ID) and 0.001 inches of roughness (R). With this topological structure, if well n is closer to manifold m_1 than m_2 then $\mathbf{q}^{n,m_1}(p, q_\mathrm{i}) \geq \mathbf{q}^{n,m_2}(p, q_\mathrm{i})$ for any manifold pressure p and gas injection q_i.

The wells have constant gas-oil-ratio (G) and water cut (W) for all allowed gas injections and manifold pressures. The liquid flow rate for all wells behaves according to the equation $q_l = p_i(p_r - p_{wf})$ where $q_l = q_o + q_w$, p_{wf} is the bottom hole pressure, p_i is the well production index, and p_r is the reservoir static pressure. These parameters are shown in Table 1, where the units for G, W, p_r, and p_i are sm^3/sm^3, %, psi, and STB/d/psi, respectively.

Table 1. Well parameters.

n	G	W	p_r	p_i	n	G	W	p_r	p_i
1	200	0	2100	15	9	200	10	1900	5
2	200	20	2300	2	10	200	40	2200	9
3	300	10	1950	12	11	300	0	1850	11
4	300	40	2050	15	12	300	20	2300	6
5	400	0	1750	4	13	400	10	1825	14
6	400	20	1700	9	14	400	40	2200	7
7	500	10	1700	11	15	500	0	1600	8
8	500	40	2100	10	16	500	20	1800	5

The pipeline connecting manifold 1 to its separator is 100 m long, while the pipeline from manifold 2 to its separator is 50 m long. Both have negligible elevation, $ID = 4.5$ inches, and $R = 0.001$ inches. The absolute pressure of the manifolds ranges from 300 to 800 psi depending on the operational conditions, while the nominal pressure of the separators is 300 psi.

All wells have identical tubings with the following characteristics: ID of 3 inches, perforation length of 3.7 km, depth of 2.7 km, and injection point at 2.8 km. The maximum injection allowed for each well is 8000 mscf/d.

The curves used to represent this process are available in (Silva et al., 2011) in AMPL format. This instance was obtained using Schlumberger PIPESIM software, inspired in a synthetic instance from (Kosmidis et al., 2004).

6.2 Performance Analysis

For the production optimization problem P and the synthetic instance described above, the CC and Log reformulations were compared with respect to solving time.

These formulations were expressed in AMPL and solved with CPLEX 11 in an Intel Core 2 Quad 3.0Ghz Linux workstation with 4GB of RAM. All experiments were executed with a time limit of 600 seconds (10 minutes).

Table 2 shows the running time (in seconds) to solve the production optimization problem for varying availability of lift-gas: high means that there is no limit on the lift-gas rate; medium means that the lift-gas rate is half of the rate necessary for maximum system production; and low means that the rate is sufficient to maximize the production of a single well. The results elicited the following remarks on:

Formulations: The Log formulation was solved much faster than CC in all experiments. This result was expected since the Log formulation is more compact than CC, needing fewer binary variables and constraints to express the piecewise-linear approximations. Further, the LP relaxation of the Log takes less time to be solved and its branch-and-bound tree tends to be shallower.

Lift-gas Availability: with high and medium lift-gas availability, the solver could not reach the optimal solution using CC but found an integral solution near the optimality within time limit, while it found the optimal solution and closed GAP using Log. With low availability, the solver failed to find an integral solution close to the optimal solution using CC within the time limit (600 seconds), however the optimal solution was found by the solver in less than a minute using Log.

Table 2. Comparison between formulations.

		High	Medium	Low
CC	Time (s)	600	600	600
	GAP (%)	0.07	15.96	130.22
Log	Time (s)	30	28	55
	GAP (%)	0	0	0

7. SUMMARY

This work presented the oil production optimization problem subject to well-manifold routing and pressure constraints. Owing to the nonlinear and mixed-integer nature of the problem, mixed-integer linear reformulations were developed based on two models for piecewise linearization of multidimensional non-convex functions. Both CC and Log models represent piecewise-linear functions by convex combination of breakpoints of simplexes, however CC uses one binary variable for each simplex while Log needs only a logarithmic number of variables and constraints on the number of simplexes. The effectiveness of the formulations was assessed by measuring the computational time taken to solve instances of the production optimization problem with a top-notch solver. The computational analysis corroborates the hypothesis that the Log formulation is solved faster than the standard one.

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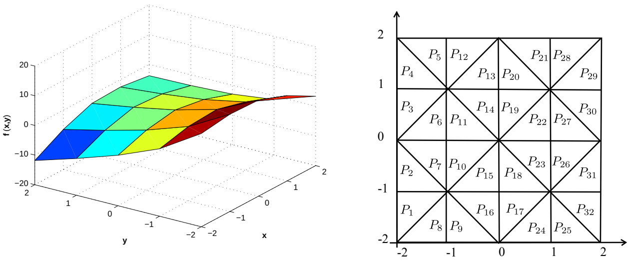
Appendix A. EXPLICIT FORMULATIONS AND ILLUSTRATIONS

For the function $f(x, y) = \mathrm{e}^{-y} + \mathrm{e}^{-x^2} - x^2 + xy - 2y$ depicted in Figure A.1(a), we present explicit CC and Log formulations.



(a) Two-dimensional PWL function. (b) J1 triangulation.

Fig. A.1. Piecewise-linear function with J1 triangulation.

The domain of function f is the Union Jack(J1) Triangulation illustrated in Figure A.1(b), which is partitioned into a set of polytopes $\mathcal{P} = \{P_1, P_2, \ldots, P_{32}\}$, a set of vertices $\mathcal{V}(\mathcal{P}) = \{v_1, v_2, \ldots, v_{25}\}$, and a set of vertices for each polytope $\mathcal{V}(P_1) = \{(-2,-2), (-2,-1), (-1,-1)\}$, $\mathcal{V}(P_2) = \{(-2,-1), (-1,-1), (-2,0)\}, \ldots,$ $\mathcal{V}(P_{32}) = \{(2,-1), (1,-1), (2,-2)\}$.

A.1 CC Formulation

The CC formulation is given by:

$$\sum_{\mathbf{v}\in\mathcal{V}(\mathcal{P})} \lambda_{\mathbf{v}}\mathbf{v} = \lambda_{\mathbf{v}_1}(-2,-2) + \lambda_{\mathbf{v}_2}(-2,-1) + \cdots + \lambda_{\mathbf{v}_{25}}(2,-2)$$

$$\sum_{\mathbf{v}\in\mathcal{V}(\mathcal{P})} \lambda_{\mathbf{v}} f(\mathbf{v}) = \lambda_{\mathbf{v}_1}11.4074 + \lambda_{\mathbf{v}_2}2.7366 + \cdots + \lambda_{\mathbf{v}_{25}}3.4074$$

$$\lambda_{\mathbf{v}_1} \geq 0,\ \lambda_{\mathbf{v}_2} \geq 0,\ \ldots,\ \lambda_{\mathbf{v}_{25}} \geq 0$$

$$\sum_{\mathbf{v}\in\mathcal{V}(\mathcal{P})} \lambda_{\mathbf{v}} = \lambda_{\mathbf{v}_1} + \lambda_{\mathbf{v}_2} + \lambda_{\mathbf{v}_3} + \cdots + \lambda_{\mathbf{v}_{25}} = 1$$

$$\lambda_{\mathbf{v}} \leq \sum_{P\in\mathcal{P}} y_P,\ \forall \mathbf{v} \in \mathcal{V}(P) \Leftrightarrow \begin{cases} \lambda_{\mathbf{v}_1} & \leq y_{P_1} + y_{P_8} \\ \lambda_{\mathbf{v}_2} & \leq y_{P_1} + y_{P_2} \\ \lambda_{\mathbf{v}_3} & \leq y_{P_1} + y_{P_2} + y_{P_7} \\ & \quad + y_{P_{10}} + y_{P_{15}} + y_{P_{16}} \\ & \quad + y_{P_9} + y_{P_8} \\ & \vdots \\ \lambda_{\mathbf{v}_{25}} & \leq y_{P_{25}} + y_{P_{32}} \end{cases}$$

$$\sum_{P\in\mathcal{P}} y_P = y_{P_1} + y_{P_2} + y_{P_3} + \cdots + y_{P_{32}} = 1$$

$$y_P \in \{0,1\}, \forall P \in \mathcal{P} \Leftrightarrow y_{P_1} \in \{0,1\},\ y_{P_2} \in \{0,1\}, \ldots, y_{P_{32}} \in \{0,1\}$$

A.2 Log Formulation

For the space dimension $d = 2$, let $D = \{1, \ldots, d\}$ be a set of indexes, and $J = [-2, \ldots, 2]^d$, be a set of vertices composing a J1 ("Union Jack") Triangulation. Let $B : \mathcal{I} \to \{0,1\}^{\lceil \log_2 |\mathcal{I}| \rceil}$ be an injective function following the Gray code property: $B([-2,-1]) = (0,0)$, $B([-1,0]) = (0,1)$, $B([0,1]) = (1,1)$, $B([1,2]) = (1,0)$, where $\mathcal{I} = \{[-2,-1], [-1,0], [0,1], [1,2]\}$ is the set of intervals. Notice that the intervals in $\mathcal{I}$ are of the form $[s_i, s_{i+1}]$ for $s_i \in \mathcal{S}\backslash\{s_n\}$, where $\mathcal{S} = \{s_0, s_1, \ldots, s_n\}$ is the set of breakpoints with $n = 4$. The Gray code property ensures that B is compatible with SOS2.

Let $J^+(e, B, l) \subseteq \mathcal{S}$ be the set of breakpoints of axis e (x or y) such that for each s the intervals $I \in I(s)$ to which it belongs have value 1 at position l of their binary code $B(I)$. For a formal definition, please refer to Section 5. The definition of $J^0(e, B, l)$ is similar to $J^+(e, B, l)$, except that the binary codes of the intervals containing a breakpoint s must have value 0 at position l instead of 1.

Because the breakpoints are identical for axis x and y, the vertex sets J^+ and J^0 are the same for both axis. For the example,

- $J^+(\mathrm{x}, B, 1) = J^+(\mathrm{y}, B, 1) = \{0\}$,
- $J^0(\mathrm{x}, B, 1) = J^0(\mathrm{y}, B, 1) = \{-2, 2\}$,
- $J^+(\mathrm{x}, B, 2) = J^+(\mathrm{y}, B, 2) = \{1, 2\}$,
- $J^0(\mathrm{x}, B, 2) = J^0(\mathrm{y}, B, 2) = \{-2, -1\}$

The first phase of Log branching consists in selecting a square within the function's domain. This branching scheme uses two binary variables in each axis to encode the intervals: $(z_2^{\mathrm{x}}, z_1^{\mathrm{x}})$ encodes the four intervals on axis x, while $(z_2^{\mathrm{y}}, z_1^{\mathrm{y}})$ encodes axis y, where z_l^a is a binary variable for position l of the code for axis a. For the example, selection of squares is implemented by the following relations:

$$\sum_{s_2=-2}^{2} \sum_{s_1\in J^+(\mathrm{x},B,1)} \lambda_{s_1,s_2} \leq z_1^{\mathrm{x}} \Leftrightarrow \begin{cases} \lambda_{0,-2} + \lambda_{0,-1} + \lambda_{0,0} \\ +\lambda_{0,1} + \lambda_{0,2} \leq z_1^{\mathrm{x}} \end{cases}$$

$$\sum_{s_2=-2}^{2} \sum_{s_1\in J^+(\mathrm{x},B,2)} \lambda_{s_1,s_2} \leq z_2^{\mathrm{x}} \Leftrightarrow \begin{cases} \lambda_{1,-2} + \lambda_{1,-1} + \lambda_{1,0} + \lambda_{1,1} \\ +\lambda_{1,2} + \lambda_{2,-2} + \lambda_{2,-1} \\ +\lambda_{2,0} + \lambda_{2,1} + \lambda_{2,2} \leq z_2^{\mathrm{x}} \end{cases}$$

$$\sum_{s_1=-2}^{2} \sum_{s_2\in J^+(\mathrm{y},B,1)} \lambda_{s_1,s_2} \leq z_1^{\mathrm{y}} \Leftrightarrow \begin{cases} \lambda_{-2,0} + \lambda_{-1,0} + \lambda_{0,0} \\ +\lambda_{1,0} + \lambda_{2,0} \leq z_1^{\mathrm{y}} \end{cases}$$

$$\sum_{s_1=-2}^{2} \sum_{s_2\in J^+(\mathrm{y},B,2)} \lambda_{s_1,s_2} \leq z_2^{\mathrm{y}} \Leftrightarrow \begin{cases} \lambda_{-2,1} + \lambda_{-1,1} + \lambda_{0,1} + \lambda_{1,1} \\ +\lambda_{2,1} + \lambda_{-2,2} + \lambda_{-1,2} + \lambda_{0,2} \\ +\lambda_{1,2} + \lambda_{2,2} \leq z_2^{\mathrm{y}} \end{cases}$$

$$\sum_{s_2=-2}^{2} \sum_{s_1\in J^0(\mathrm{x},B,1)} \lambda_{s_1,s_2} \leq (1 - z_1^{\mathrm{x}}) \Leftrightarrow \begin{cases} \lambda_{-2,-2} + \lambda_{-2,-1} + \lambda_{-2,0} \\ +\lambda_{-2,1} + \lambda_{-2,2} + \lambda_{2,-2} \\ +\lambda_{2,-1} + \lambda_{2,0} + \lambda_{2,1} \\ +\lambda_{2,2} \leq (1 - z_1^{\mathrm{x}}) \end{cases}$$

$$\sum_{s_1=-2}^{2} \sum_{s_2\in J^0(\mathrm{y},B,1)} \lambda_{s_1,s_2} \leq (1 - z_1^{\mathrm{y}}) \Leftrightarrow \begin{cases} \lambda_{-2,-2} + \lambda_{-1,-2} + \lambda_{0,-2} \\ +\lambda_{1,-2} + \lambda_{2,-2} + \lambda_{-2,2} \\ +\lambda_{-2,2} + \lambda_{-1,2} + \lambda_{0,2} \\ +\lambda_{1,2} + \lambda_{2,2} \leq (1 - z_1^{\mathrm{y}}) \end{cases}$$

$$\sum_{s_2=-2}^{2} \sum_{s_1\in J^0(\mathrm{x},B,2)} \lambda_{s_1,s_2} \leq (1 - z_2^{\mathrm{x}}) \Leftrightarrow \begin{cases} \lambda_{-2,-2} + \lambda_{-2,-1} + \lambda_{-2,0} \\ +\lambda_{-2,1} + \lambda_{-2,2} + \lambda_{-1,-2} \\ +\lambda_{-1,-1} + \lambda_{-1,0} + \lambda_{-1,1} \\ +\lambda_{-1,2} \leq (1 - z_2^{\mathrm{x}}) \end{cases}$$

$$\sum_{s_1=-2} \sum_{s_2\in J^0(\mathrm{y},B,2)} \lambda_{s_1,s_2} \leq (1 - z_2^{\mathrm{y}}) \Leftrightarrow \begin{cases} \lambda_{-2,-2} + \lambda_{-1,-2} + \lambda_{0,-2} \\ +\lambda_{1,-2} + \lambda_{2,-2} + \lambda_{-2,-1} \\ +\lambda_{-1,-1} + \lambda_{0,-1} + \lambda_{1,-1} \\ +\lambda_{2,-1} \leq (1 - z_2^{\mathrm{y}}) \end{cases}$$

$$z_1^{\mathrm{x}} \in \{0,1\},\ z_2^{\mathrm{x}} \in \{0,1\},\ z_1^{\mathrm{y}} \in \{0,1\},\ z_2^{\mathrm{y}} \in \{0,1\}$$

To choose square $[-1,0] \times [-1,0]$ the first phase defines $(z_2^{\mathrm{x}}, z_1^{\mathrm{x}})$ and $(z_2^{\mathrm{y}}, z_1^{\mathrm{y}})$. The second branching phase selects a simplex within the square chosen by the first branching phase being implemented by:

$$\tilde{\mathcal{J}} = \{(s_1, s_2) \in \mathcal{N} \times \mathcal{N} : s_1 < s_2\} = \{(1,2)\}$$

$$\mathcal{L}_{s_1,s_2} = \{\mathbf{v} \in J : \mathbf{v}_{s_1} \text{ is even}, \mathbf{v}_{s_2} \text{ is odd}\},\ \forall (s_1, s_2) \in \tilde{\mathcal{J}} \Leftrightarrow$$
$$\mathcal{L}_{1,2} = \{(-2,-1), (-2,1), (0,-1), (0,1), (2,-1), (2,1)\}$$

$$\mathcal{R}_{s_1,s_2} = \{\mathbf{v} \in J : \mathbf{v}_{s_1} \text{ is odd}, \mathbf{v}_{s_2} \text{ is even}\},\ \forall (s_1, s_2) \in \tilde{\mathcal{J}} \Leftrightarrow$$
$$\mathcal{R}_{1,2} = \{(-1,-2), (-1,0), (-1,2), (1,-2), (1,0), (1,2)\}$$

$$\sum_{(s_1,s_2)\in\mathcal{L}_{s_1,s_2}} \lambda_{s_1,s_2} \leq y_{s_1,s_2} \Leftrightarrow \begin{cases} \lambda_{-2,-1} + \lambda_{-2,1} + \lambda_{0,-1} \\ \quad +\lambda_{0,1} + \lambda_{2,-1} + \lambda_{2,1} \leq y_{1,2} \end{cases}$$

$$\sum_{(s_1,s_2)\in\mathcal{R}_{s_1,s_2}} \lambda_{s_1,s_2} \leq 1 - y_{s_1,s_2} \Leftrightarrow \begin{cases} \lambda_{-1,-2} + \lambda_{-1,0} + \lambda_{-1,2} \\ \quad +\lambda_{1,-2} + \lambda_{1,0} \\ \quad +\lambda_{1,2} \leq (1 - y_{1,2}) \end{cases}$$

$$y_{1,2} \in \{0,1\}$$