A note on decoupling

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Decoupling literature

- W.L. Luyben, Distillation Decoupling, *AIChE J.*, 1970.
- K.V. Waller, Decoupling in Distillation, *AIChE J.*, 1974.
- M. Waller, J.B. Waller and K.V. Waller, Decoupling Revisited, *Ind. Eng. Chem. Res.*, 2003.
- P. Nordfeldt and T. Hägglund, Decoupler and PID controller design of TITO systems, *J. Process Control*, 2006.
- W.-J. Cai, W. Ni, M.-J. He and C.-Y. Ni, Normalized decoupling -A new approach for MIMO process control system design, *Ind. Eng. Chem. Res.*, 2008.

Introduction

- Approaches to decoupling: Ideal, simplified, normalized
- 2×2 case most often considered

Decoupling in the 2×2 case (Luyben, 1970):

$$G(s)F(s) = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & 0 \\ 0 & p_{22} \end{bmatrix} = P(s) \quad (1)$$

In ideal decoupling P is specified and, as a consequence, the structure of F can be complicated. In simplified decoupling, parts of F is specified while P is free.

Simplified decoupling in the 2×2 case

Four F-matrix candidates (Waller 1974):

$$F_{11} = \begin{bmatrix} 1 & 1 \\ f_{21} & f_{22} \end{bmatrix} \quad F_{12} = \begin{bmatrix} 1 & f_{12} \\ f_{21} & 1 \end{bmatrix} \qquad (2)$$

$$F_{21} = \begin{bmatrix} f_{11} & 1 \\ 1 & f_{22} \end{bmatrix} \quad F_{22} = \begin{bmatrix} f_{11} & f_{12} \\ 1 & 1 \end{bmatrix}$$

Structure of equation system:

$$GF_{11} = \begin{bmatrix} g_{11} + g_{12}f_{21} & g_{11} + g_{12}f_{22} \\ g_{21} + g_{22}f_{21} & g_{21} + g_{22}f_{22} \end{bmatrix} \qquad (3)$$

$$\Longrightarrow \begin{bmatrix} g_{22} & 0 \\ 0 & g_{12} \end{bmatrix} \begin{bmatrix} f_{21} \\ f_{22} \end{bmatrix} = - \begin{bmatrix} g_{21} \\ g_{11} \end{bmatrix}$$

Challenges and ongoing work

Handling systems of higher dimensions

1. A detailed analysis of the 3×3 case
2. Investigate possible generalizations

Systematic analysis and design procedure

- Mathematica software
- Discrete time representations

Discrete time representations

- Easy handling of time delays
- Realizability (causality) checked through polynomial orders
- Sampling time must be chosen considering system specification

Example (taken from Waller 1974):

$$G(s) = \begin{pmatrix} \frac{-2.2e^{-s}}{1+7s} & \frac{1.3e^{-0.3s}}{1+7s} \\ \frac{-2.28e^{-1.8s}}{1+9.5s} & \frac{4.3e^{-0.35s}}{1+9.2s} \end{pmatrix} \qquad (4)$$

With sampling time $T_s = 1\ min$ $\Longrightarrow$

$$G(z) = \begin{pmatrix} \frac{-0.293}{z(z-0.867)} & \frac{0.124z+0.0493}{z(z-0.867)} \\ \frac{-0.0588z-0.0222}{z^2(z-0.900)} & \frac{0.293z+0.150}{z(z-0.897)} \end{pmatrix} \qquad (5)$$

Simplified decoupling in the 3×3 case

- The number of F-matrix candidates for a system of dimension $n \times n$ is equal to the number of ways one element equal to 1 in each column can be chosen, i.e. n^n candidates. Thus, the 3×3 case has $3^3 = 27$ F-matrix candidates.

- Issues:
  Does a solution to the equation $GF = P$ exist?
  If so, is the resulting matrix F realizable?
  Problems with the resulting (free) dynamics?
  Impact from model uncertainties?

Simplified decoupling in the 3×3 case

$$G(s)F(s) = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} = \begin{bmatrix} p_{11} & 0 & 0 \\ 0 & p_{22} & 0 \\ 0 & 0 & p_{33} \end{bmatrix} = P(s) \quad (6)$$

One element in every column of F is set equal to 1.
Notation: F_{123} means that $f_{11} = f_{22} = f_{33} = 1$, F_{112} means that $f_{11} = f_{12} = f_{23} = 1$ etc.

Simplified decoupling in the 3×3 case

With F_{123}, the equation system to be solved becomes

$$\underbrace{\begin{bmatrix} g_{11} & 0 & 0 & 0 & 0 & g_{13} \\ 0 & g_{11} & 0 & g_{12} & 0 & 0 \\ 0 & 0 & g_{22} & 0 & g_{23} & 0 \\ 0 & g_{21} & 0 & g_{22} & 0 & 0 \\ 0 & 0 & g_{32} & 0 & g_{33} & 0 \\ g_{31} & 0 & 0 & 0 & 0 & g_{33} \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} f_{12} \\ f_{13} \\ f_{21} \\ f_{23} \\ f_{31} \\ f_{32} \end{bmatrix}}_{f} = - \underbrace{\begin{bmatrix} g_{12} \\ g_{13} \\ g_{21} \\ g_{23} \\ g_{31} \\ g_{32} \end{bmatrix}}_{g} \tag{7}$$

Solution, if M has full rank,

$$f = -M^{-1} g \tag{8}$$

Observations

- The sparse matrix M to be inverted has at most $1/3$ non-zero elements in the 3×3 and the corresponding figure for the 2×2 case is $1/2$.
  In general: For a system G of dimension $n \times n$, the coefficient matrix M will be of dimension $n(n-1) \times n(n-1)$. M will be sparse and have at most $1/n$ of its elements apart from zero.
- The invertibility of M can, for example, be investigated using a block-matrix approach. Row operations on the system $Mf = -g$ are allowed.

Observations (cont.)

- The solutions to the 3×3 case involve conditions on some 2×2-minors of the system matrix G. In total, nine such minors exists and each specific F-matrix requires that three of these are non-zero.

- A pattern between the position of the elements equal to 1 in the F-matrix and the 2×2-minors exists. Let m_{ji} be the 2×2-minor of G when excluding row j and column i. If a F-matrix with $f_{ij} = 1$ is to be used, then $m_{ji} \neq 0$ is required.

- As a result, if some minors of G are zero, the number of F-matrix candidates can be effectively reduced.

Observations (cont.)

- When a solution exists, the resulting elements of the F-matrix will be quotients of some 2×2-minors. The elements of the P-matrix will be on the form

$$p_{kk} = \pm \frac{\det G}{m_{ij}} \qquad k = 1, 2, 3 \tag{9}$$

  where the minors m_{ij} are the same found in the denominators of the F-elements.

- In addition to conditions on minors, conditions on certain elements of G are also found.

Observations (cont.)

Example: For F_{112} $(f_{11} = f_{12} = f_{23} = 1)$ a solution exists if

$$\begin{aligned} m_{11} &= g_{22}g_{33} - g_{23}g_{32} \neq 0 \\ m_{21} &= g_{12}g_{33} - g_{13}g_{32} \neq 0 \\ m_{32} &= g_{11}g_{23} - g_{13}g_{21} \neq 0 \end{aligned} \tag{10}$$

at the same time as

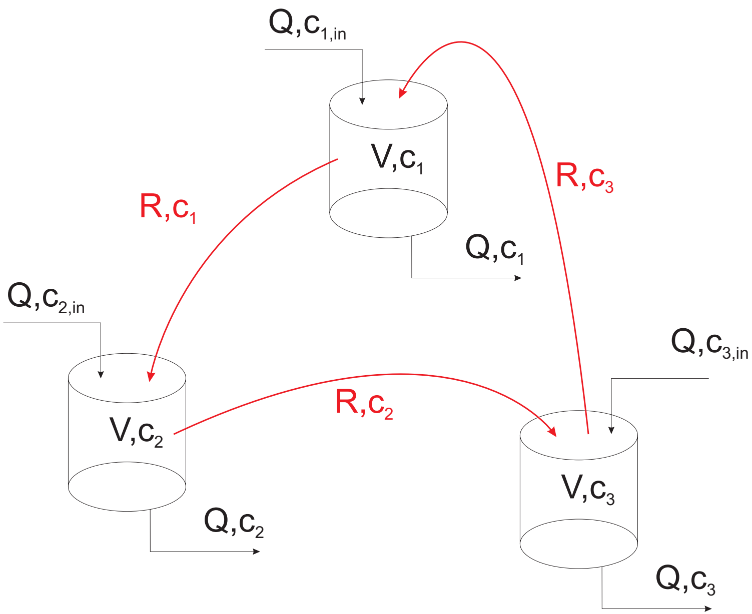
$$\begin{matrix} g_{11} \neq 0 \\ g_{22} \neq 0 \\ g_{12} \neq 0 \end{matrix} \qquad or \qquad \begin{matrix} g_{23} \neq 0 \\ g_{33} \neq 0 \end{matrix} \tag{11}$$

Observations (cont.)

- An extension to the 4×4 case indicates that minors of dimension 3×3 will have a significant role. Besides that, conditions on 2×2-minors as well as single elements of the system matrix G are found.
- The software Mathematica is suitable for symbolic solving this kind of large equation systems.

Example: Blending system of three vessels

Consider a system of three ideally stirred tanks. The volumes (V) and the flows (Q and R) are all constants.



Example: Blending system (cont.)

Introduce the time constant $\tau = V/Q$ and the relative flow $\sigma = R/Q$. From a material balance over each vessel, the following transfer function model is obtained

$$C(s) = G(s)C_{in}(s) \tag{12}$$

$$G(s) = \begin{bmatrix} 1+\sigma+\tau s & 0 & -\sigma \\ -\sigma & 1+\sigma+\tau s & 0 \\ 0 & -\sigma & 1+\sigma+\tau s \end{bmatrix}^{-1} \tag{13}$$

Example: Blending system (cont.)

Calculating the 2-times-2-minors of G gives

$$\begin{bmatrix} m_{33} & m_{32} & m_{31} \\ m_{23} & m_{22} & m_{21} \\ m_{13} & m_{12} & m_{11} \end{bmatrix} = \begin{bmatrix} \frac{1+\sigma+\tau s}{h(s)} & 0 & \frac{-\sigma}{h(s)} \\ \frac{\sigma}{h(s)} & \frac{1+\sigma+\tau s}{h(s)} & 0 \\ 0 & \frac{\sigma}{h(s)} & \frac{1+\sigma+\tau s}{h(s)} \end{bmatrix} \quad (14)$$

$$h(s) = (1+\tau s)(3\sigma^2 + 3\sigma(1+\tau s) + (1+\tau s)^2) = (\det G)^{-1}$$

As $m_{32} = m_{21} = m_{13} = 0$, F-matrices haveing f_{32}, f_{31} and/or f_{12} equal to one can not be used. As a result, the F-matrix candidates are reduced from the general 27 to 8.

Example: Blending system (cont.)

The filter F_{123} gives a realizable decoupling

$$F_{123} = \begin{bmatrix} 1 & \frac{-m_{21}}{m_{22}} & \frac{-m_{31}}{m_{33}} \\ \frac{-m_{12}}{m_{11}} & 1 & \frac{-m_{32}}{m_{33}} \\ \frac{-m_{13}}{m_{11}} & \frac{-m_{23}}{m_{22}} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{-\sigma}{1+\sigma+\tau s} \\ \frac{-\sigma}{1+\sigma+\tau s} & 1 & 0 \\ 0 & \frac{-\sigma}{1+\sigma+\tau s} & 1 \end{bmatrix} \quad (15)$$

$$P_{123} = \begin{bmatrix} \frac{1}{1+\sigma+\tau s} & 0 & 0 \\ 0 & \frac{1}{1+\sigma+\tau s} & 0 \\ 0 & 0 & \frac{1}{1+\sigma+\tau s} \end{bmatrix} \quad (16)$$

Example: Blending system (cont.)

$$F_{121} = \begin{bmatrix} 1 & 0 & 1 \\ \frac{-\sigma}{1+\sigma+\tau s} & 1 & 0 \\ 0 & \frac{-\sigma}{1+\sigma+\tau s} & \frac{-(1+\sigma+\tau s)}{\sigma} \end{bmatrix} \tag{17}$$

$$F_{121} = \begin{bmatrix} \frac{-(1+\sigma+\tau s)}{\sigma} & 0 & \frac{-\sigma}{1+\sigma+\tau s} \\ 1 & 1 & 0 \\ 0 & \frac{-\sigma}{1+\sigma+\tau s} & 1 \end{bmatrix} \tag{18}$$

As these two filters (like the remaining five for which a solution exists) include elements that are non-causal, they are not realizable and can not be used for decoupling.

Realizability

For every F-matrix that generates a solvable equation system, all resulting elements are investigated with respect to polynomial degree. Write an F-matrix element as fractions between polynomials

$$f_{ij}(z) = \frac{Q_{ij}(z)}{P_{ij}(z)} \tag{19}$$

Check the relation between the polynomials Q_{ij} and P_{ij} in terms of their highest exponent. In order for a certain F-matrix to be realizable, for all of its elements f_{ij} the highest exponent of P_{ij} must be equal to, or larger than, the highest exponent of Q_{ij}.

If no time delays are present, then the procedure can be carried out for s instead of z.

Decoupling

$$G(s) = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \dashrightarrow P(s) = \begin{bmatrix} p_{11} & 0 & 0 \\ 0 & p_{22} & 0 \\ 0 & 0 & p_{33} \end{bmatrix}$$

F-matrix candidates