Simulation of Subsurface Two-Phase Flow in an Oil Reservoir

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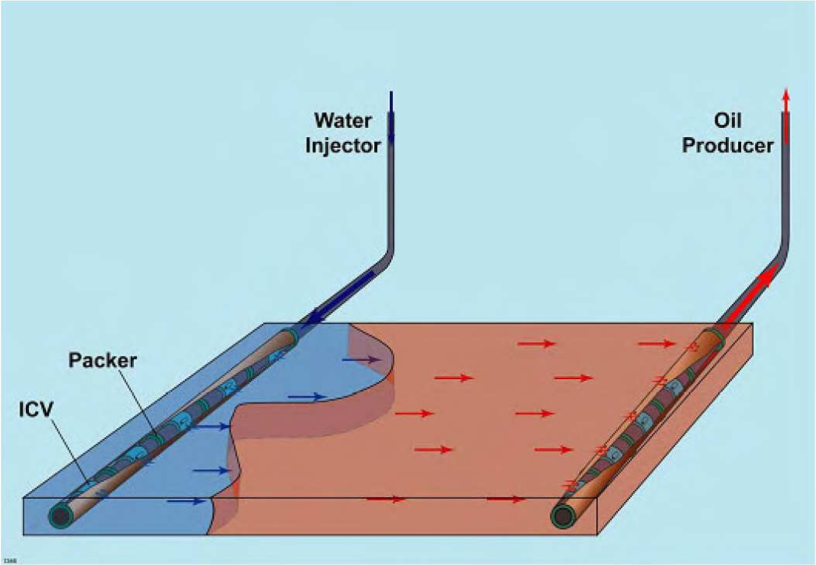
Outline

2D Reservoir



Optimizing Production

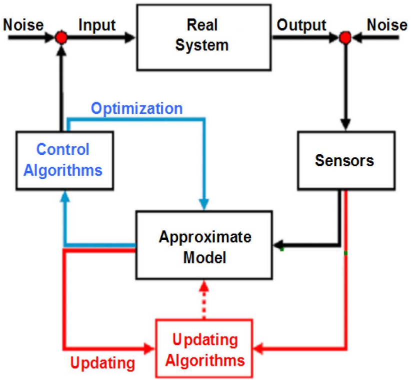
Closed loop optimizer:

Maximizing net present value (NPV):

$$\max_{u} \quad \text{NPV} = \int_{t_0}^{t} l(x(t), u)dt$$

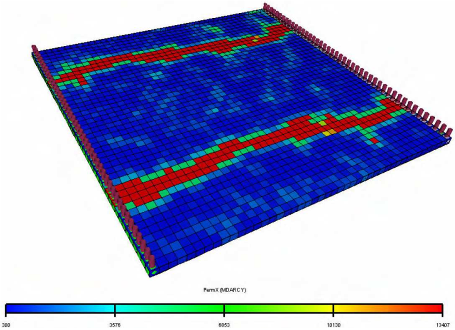
$$\text{s.t.} \quad \frac{d}{dt}g(x(t)) = f(x(t), u)$$

$$x(t_0) = x_0$$

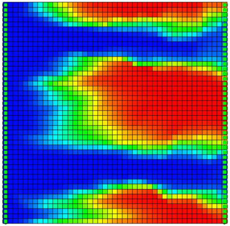


Water Flooding without/with Optimal Control

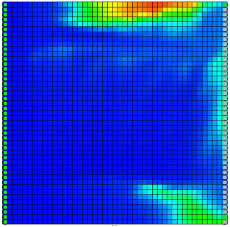
Permeability field with two streaks:



Without optimal control:



With optimal control:



Conservation Equations

Mass conservation of water and oil:

$$\frac{\partial}{\partial t} C_w(P_w, S_w) = -\nabla N_w(P_w, S_w) + Q_w$$
$$\frac{\partial}{\partial t} C_o(P_o, S_o) = -\nabla N_o(P_o, S_o) + Q_o$$

- No flow potential due to gravitation.
- Homogenous permeability field.
- Capillary pressure neglected.
- Incompressible rock.

Mass concentrations:

$$C_w = \phi \rho_w(P_w) S_w$$
$$C_o = \phi \rho_o(P_o) S_o$$

Fluxes through the porous medium:

$$N_w = \rho_w(P_w) u_w(P_w, S_w)$$
$$N_o = \rho_o(P_o) u_o(P_o, S_o)$$

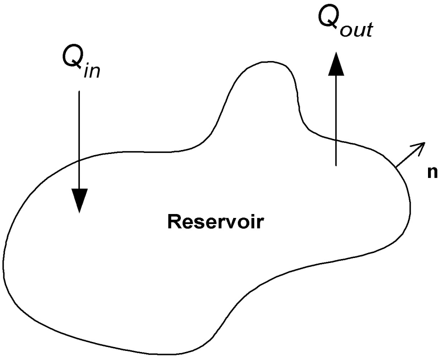
Darcy Velocity and Boundary Conditions

Pressure driven flow, Darcy's law:

$$u_\alpha = -\mathbf{k}\frac{k_{r\alpha}(S_\alpha)}{\mu_\alpha}\nabla P_\alpha \quad \alpha = o, w$$

No flow across boundaries:

$$\mathbf{u}_\alpha \cdot \mathbf{n} = 0 \quad \text{and} \quad \mathbf{u}_\alpha \cdot \mathbf{n} = 0$$



Internal sources/sinks due to wells:

- Water is injected to maintain pressure and replace the oil.
- Oil and water are produced.

Reduction Of State Variables:

Water saturation (volume fraction):

$$S_w + S_o = 1$$

Pressure difference due to capillary pressure:

$$P_{cow} = P_o - P_w$$

Reduction of variables:

$$\begin{aligned} S_w = 1 - S_o \quad &\Rightarrow \quad S = S_w = 1 - S_o \\ P_{cow} = 0 \quad &\Rightarrow \quad P = P_w = P_o \end{aligned}$$

State variables $(S, P) = (S_w, P_o)$:

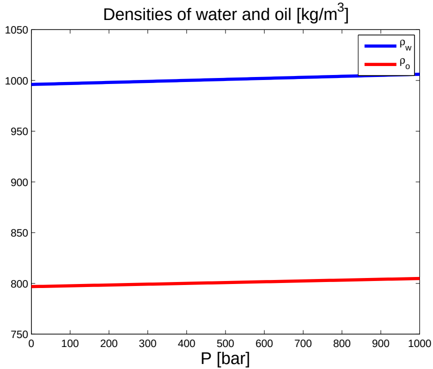
$$\begin{aligned} S &= S(t, s) \\ P &= P(t, s) \end{aligned}$$

Density and compressibility

Compressible fluids:

$$\rho_w = \rho_{w0} e^{P - P_{w0}}$$

$$\rho_o = \rho_{o0} e^{P - P_{o0}}$$



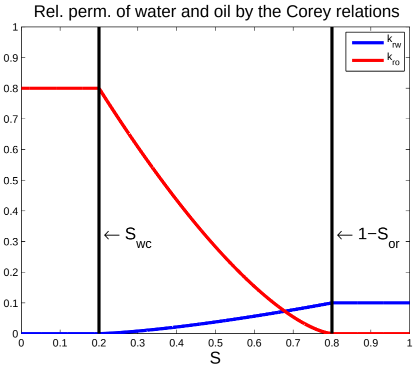
Relative Permeabilities by The Corey Relations

Relative permeabilities:

$$k_{rw} = k_{rw0} s^{n_w}$$
$$k_{ro} = k_{ro0} (1 - s)^{n_w}$$

Reduced water saturation:

$$s = \frac{S - S_{wc}}{1 - S_{wc} - S_{or}}$$



Different Formulation

Partial differential equation (PDE) model:

$$\frac{\partial}{\partial t} C_w(P_w, S_w) = -\nabla N_w(P_w, S_w) + Q_w$$
$$\frac{\partial}{\partial t} C_o(P_o, S_o) = -\nabla N_o(P_o, S_o) + Q_o$$

Different formulation of an ordinary differential equation (ODE) model after discretizing spatially:

$$\frac{d}{dt} g(x(t)) = f(t, x(t)) \quad x(t_0) = x_0$$

Runge-Kutta Methods

Tailored formulation of an s-stage Runge-Kutta method:

$$T_i = t_n + c_i h_n \quad i = 1, 2, \ldots, s$$

$$g(X_i) = g(x_n) + h_n \sum_{j=1}^{s} a_{ij} f(T_j, X_j) \quad i = 1, 2, \ldots, s$$

$$g(x_{n+1}) = g(x_n) + h_n \sum_{j=1}^{s} b_j f(T_j, X_j)$$

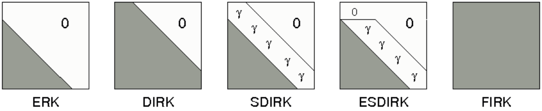
$$g(\hat{x}_{n+1}) = g(x_n) + h_n \sum_{j=1}^{s} \hat{b}_j f(T_j, X_j)$$

$$e_{n+1} = g(x_{n+1}) - g(\hat{x}_{n+1}) = h_n \sum_{j=1}^{s} d_j f(T_j, X_j) \quad d_j = b_j - \hat{b}_j$$

Fast Integration by ESDIRK

Only $s-1$ implicit stages:

$$
\begin{array}{c|ccccc}
0 & 0 & & & & \\
c_2 & a_{21} & \gamma & & & \\
c_3 & a_{31} & a_{32} & \gamma & & \\
\vdots & \vdots & & & \ddots & \\
1 & b_1 & b_2 & b_3 & \cdots & \gamma \\
\hline
 & b_1 & b_2 & b_3 & \cdots & \gamma \\
 & \hat{b}_1 & \hat{b}_2 & \hat{b}_3 & \cdots & \hat{b}_s \\
\hline
 & d_1 & d_2 & d_3 & \cdots & d_s
\end{array}
$$



Modified Newton Step

The state values X_i are obtained by sequential solution of the residual:

$$R(X_i) = g(X_i) - h_n \gamma f(T_i, X_i) - \psi_i = 0 \quad i = 2, 3, \ldots, s$$

$$\psi_i = g(x_n) + h_n \sum_{j=1}^{i-1} a_{ij} f(T_j, X_j) \quad i = 2, 3, \ldots, s$$

The Jacobian of the residual $R(X_i)$:

$$\begin{aligned} J(X_i) = \frac{\partial R}{\partial X_i}(X_i) &= \frac{\partial g}{\partial x}(X_i) - h_n \gamma \frac{\partial f}{\partial x}(T_i, X_i) \\ &\approx \frac{\partial g}{\partial x}(x_m) - h_m \gamma \frac{\partial f}{\partial x}(t_m, x_m) \\ &= J(x_m) = LU \end{aligned}$$

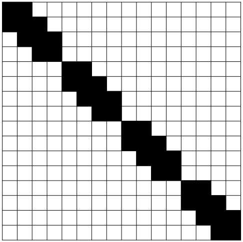
Only updating the Jacobian by slow convergence or divergence:

$$LU \Delta X_i = R(X_i)$$

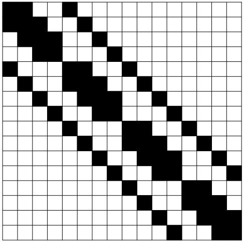
$$X_i := X_i - \Delta X_i$$

Jacobian Structure

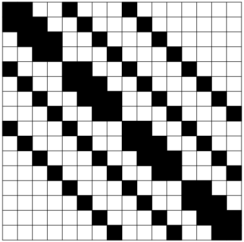
1D: 3 Non-zeros:



2D: 5 Non-zeros:



3D: 7 Non-zeros:



Solving the linear equations:

- Sparse direct solver: LU factorization and back sustitution.
- Iterative solver: GMRES.

Adaptive Time Stepping

In most commercial simulators:

- Simple heuritics implemented e.g. maximum variation of saturations.

ESDIRK, embedded error estimator:

$$\hat{e}_{n+1} = g(x_{n+1}) - g(\hat{x}_{n+1}) = h_n \sum_{i=1}^{s} d_i f(T_i, X_i)$$

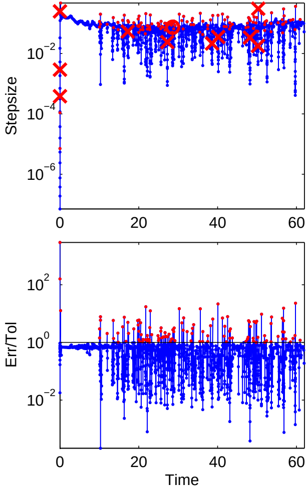
Measures of the error may be controlled adjusting the time step according to

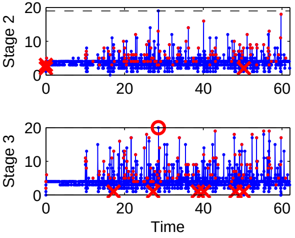
$$h_{n+1} = \frac{h_n}{h_{n-1}} \left(\frac{\varepsilon}{\hat{r}_{n+1}} \right)^{k_2/k} \left(\frac{\hat{r}_n}{\hat{r}_{n+1}} \right)^{k_1/k} h_n$$

- $\hat{e}_{n+1}$ is an error estimate of the conserved quantities $g(x_{n+1})$.

ESDIRK Performance

ESDIRK23 performance and statistics, 45×45 grid blocks:



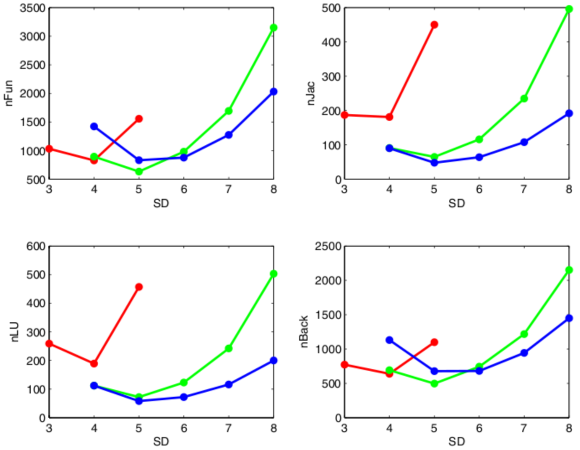



```
Method : ESDIRK23   nIter 2 : 4734
absTol : 4050x1     nIter 3 : 5506
relTol : 4050x1

nStep : 1289        nFun   : 12815
nFail : 129         nJac   : 1279
nDiv  : 10          nLU    : 1290
nSlow : 1           nBack  : 10240
```

Performance by Significant Digits

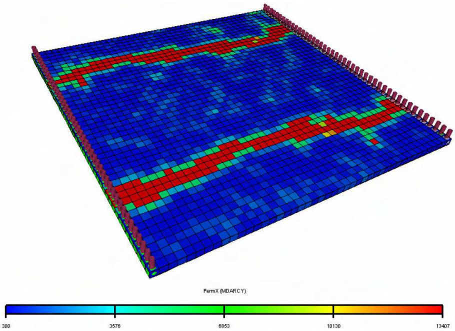
ESDIRK performance on 1D case, 1000 grid blocks:



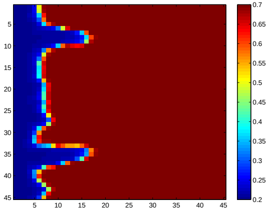
- ESDIRK12 = red, ESDIRK23 = green, ESDIRK34 = blue.

2D Test Case, 45×45 Grid Blocks

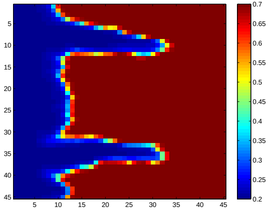
Permeability field with two streaks:



Oil saturation after 31 days:



Oil saturation after 62 days:

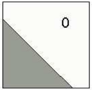


Butcher Tableau's of Runge-Kutta Methods

The explicit Runge-Kutta (ERK) method:

$$
\begin{array}{c|ccccc}
0 & 0 & & & & \\
c_2 & a_{21} & 0 & & & \\
c_3 & a_{31} & a_{32} & 0 & & \\
\vdots & \vdots & & & \ddots & \\
c_s & a_{s1} & a_{s2} & a_{s3} & \cdots & 0 \\
\hline
 & b_1 & b_2 & b_3 & \cdots & b_s \\
 & \hat{b}_1 & \hat{b}_2 & \hat{b}_3 & \cdots & \hat{b}_s \\
\hline
 & d_1 & d_2 & d_3 & \cdots & d_s
\end{array}
$$

The A-matrix in Runge-Kutta methods:



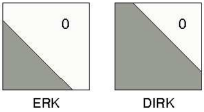
ERK

Butcher Tableau's of Runge-Kutta Methods

The diagonally implicit Runge-Kutta (DIRK) method:

$$
\begin{array}{c|ccccc}
c_1 & a_{11} & & & & \\
c_2 & a_{21} & a_{22} & & & \\
c_3 & a_{31} & a_{32} & a_{33} & & \\
\vdots & \vdots & & & \ddots & \\
c_s & a_{s1} & a_{s2} & a_{s3} & \cdots & a_{ss} \\
\hline
 & b_1 & b_2 & b_3 & \cdots & b_s \\
 & \hat{b}_1 & \hat{b}_2 & \hat{b}_3 & \cdots & \hat{b}_s \\
\hline
 & d_1 & d_2 & d_3 & \cdots & d_s
\end{array}
$$

The A-matrix in Runge-Kutta methods:

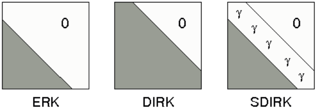


Butcher Tableau's of Runge-Kutta Methods

The singly diagonally implicit Runge-Kutta (SDIRK) method:

$$
\begin{array}{c|ccccc}
c_1 & \gamma & & & & \\
c_2 & a_{21} & \gamma & & & \\
c_3 & a_{31} & a_{32} & \gamma & & \\
\vdots & \vdots & & & \ddots & \\
c_s & a_{s1} & a_{s2} & a_{s3} & \cdots & \gamma \\
\hline
 & b_1 & b_2 & b_3 & \cdots & b_s \\
 & \hat{b}_1 & \hat{b}_2 & \hat{b}_3 & \cdots & \hat{b}_s \\
\hline
 & d_1 & d_2 & d_3 & \cdots & d_s
\end{array}
$$

The A-matrix in Runge-Kutta methods:

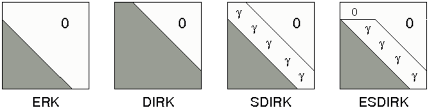


Butcher Tableau's of Runge-Kutta Methods

The explicit singly diagonally implicit Runge-Kutta (ESDIRK) method:

$$
\begin{array}{c|ccccc}
0 & 0 & & & & \\
c_2 & a_{21} & \gamma & & & \\
c_3 & a_{31} & a_{32} & \gamma & & \\
\vdots & \vdots & & & \ddots & \\
c_s & a_{s1} & a_{s2} & a_{s2} & \cdots & \gamma \\
\hline
 & b_1 & b_2 & b_3 & \cdots & b_s \\
 & \hat{b}_1 & \hat{b}_2 & \hat{b}_3 & \cdots & \hat{b}_s \\
\hline
 & d_1 & d_2 & d_3 & \cdots & d_s
\end{array}
$$

The A-matrix in Runge-Kutta methods:

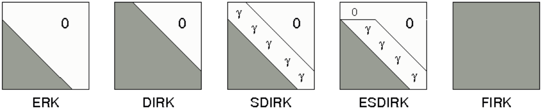


Butcher Tableau's of Runge-Kutta Methods

The fully implicit Runge-Kutta (FIRK) method:

$$
\begin{array}{c|ccccc}
c_1 & a_{11} & a_{12} & a_{13} & \cdots & a_{1s} \\
c_2 & a_{21} & a_{22} & a_{23} & & a_{2s} \\
c_3 & a_{31} & a_{32} & a_{33} & & a_{3s} \\
\vdots & \vdots & & & \ddots & \vdots \\
c_s & a_{s1} & a_{s2} & a_{s3} & \cdots & a_{ss} \\
\hline
 & b_1 & b_2 & b_3 & \cdots & b_s \\
 & \hat{b}_1 & \hat{b}_2 & \hat{b}_3 & \cdots & \hat{b}_s \\
\hline
 & d_1 & d_2 & d_3 & \cdots & d_s
\end{array}
$$

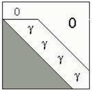
The A-matrix in Runge-Kutta methods:



Fast Integration by ESDIRK

Only $s-1$ implicit stages:

$$
\begin{array}{c|ccccc}
0 & 0 & & & & \\
c_2 & a_{21} & \gamma & & & \\
c_3 & a_{31} & a_{32} & \gamma & & \\
\vdots & \vdots & & & \ddots & \\
1 & b_1 & b_2 & b_3 & \cdots & \gamma \\
\hline
 & b_1 & b_2 & b_3 & \cdots & \gamma \\
 & \hat{b}_1 & \hat{b}_2 & \hat{b}_3 & \cdots & \hat{b}_s \\
\hline
 & d_1 & d_2 & d_3 & \cdots & d_s
\end{array}
$$



ESDIRK

Fast Integration by ESDIRK

Only $s-1$ implicit stages:

$$\begin{array}{c|ccccc}
0 & 0 & & & & \\
c_2 & a_{21} & \gamma & & & \\
c_3 & a_{31} & a_{32} & \gamma & & \\
\vdots & \vdots & & & \ddots & \\
1 & b_1 & b_2 & b_3 & \cdots & \gamma \\
\hline
 & b_1 & b_2 & b_3 & \cdots & \gamma \\
 & \hat{b}_1 & \hat{b}_2 & \hat{b}_3 & \cdots & \hat{b}_s \\
\hline
 & d_1 & d_2 & d_3 & \cdots & d_s
\end{array}$$

The first stage is explicit, which implies that:

$$X_1 = x_n$$

$$x_{n+1} = X_s$$

Fast Integration by ESDIRK

Only $s-1$ implicit stages:

$$\begin{array}{c|ccccc}
0 & 0 \\
c_2 & a_{21} & \gamma \\
c_3 & a_{31} & a_{32} & \gamma \\
\vdots & \vdots & & & \ddots \\
1 & b_1 & b_2 & b_3 & \cdots & \gamma \\
\hline
 & b_1 & b_2 & b_3 & \cdots & \gamma \\
 & \hat{b}_1 & \hat{b}_2 & \hat{b}_3 & \cdots & \hat{b}_s \\
\hline
 & d_1 & d_2 & d_3 & \cdots & d_s
\end{array}$$

The Butcher tableau is constructed such that:

$$X_1 = x_n$$
$$x_{n+1} = X_s$$

Fast Integration by ESDIRK

Only $s-1$ implicit stages:

$$\begin{array}{c|ccccc} 0 & 0 & & & & \\ c_2 & a_{21} & \gamma & & & \\ c_3 & a_{31} & a_{32} & \gamma & & \\ \vdots & \vdots & & & \ddots & \\ 1 & b_1 & b_2 & b_3 & \cdots & \gamma \\ \hline & b_1 & b_2 & b_3 & \cdots & \gamma \\ & \hat{b}_1 & \hat{b}_2 & \hat{b}_3 & \cdots & \hat{b}_s \\ \hline & d_1 & d_2 & d_3 & \cdots & d_s \end{array}$$

The state values X_i are obtained by sequential solution of the residual:

$$R(X_i) = g(X_i) - h_n \gamma f(T_i, X_i) - \psi_i = 0 \quad i = 2, 3, \ldots, s$$

$$\psi_i = g(x_n) + h_n \sum_{j=1}^{i-1} a_{ij} f(T_j, X_j) \quad i = 2, 3, \ldots, s$$