Design of multivariable LQ-optimal PID controllers based on convex optimization

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Problem formulation

Consider the following LTI system model:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + n^y \end{aligned} \tag{1}$$

Here $x \in \mathbb{R}^{n_x}$ are the states, $u \in \mathbb{R}^{n_u}$ are the inputs, $y \in \mathbb{R}^{n_y}$ are the outputs we want to control, and $n^y \in \mathbb{R}^{n^y}$ is a vector of additive noise.

In this work we present a method for design of multivariable LQ-optimal PID controllers based on convex optimization for systems that can be described by (1).

Theory

A key result, which is the basis for this paper, is the *nullspace theorem* [Alstad et al., 2008]:

Theorem 1. *(Loss by introducing linear constraint for noisy quadratic optimization problem) Consider the unconstrained optimization problem*

$$\min_u J(u,d) = \begin{bmatrix} u \\ d \end{bmatrix}^T \begin{bmatrix} J_{uu} & J_{ud} \\ J_{ud}^T & J_{dd} \end{bmatrix} \begin{bmatrix} u \\ d \end{bmatrix} \tag{2}$$

and a set of noisy measurements $y_m = y + n^y$, where $y = G^y u + G_d^y d$. Assume that $n_c = n_u$ constraints $c = Hy_m = c_s$, with $\operatorname{rank}(H) = n_c$, are added to the problem, which will result in a non-optimal solution with a loss $L = J(u,d) - J_{opt}(d)$. Consider disturbances d and noise n^y with magnitudes

$$d = W_d d'; \quad n^y = W_{n^y} n^{y'}; \quad \left\| \begin{bmatrix} d' \\ n^{y'} \end{bmatrix} \right\|_2 \leq 1. \tag{3}$$

Then for a given H, the worst-case loss introduced by adding the constraint $c = Hy$ is $L_{wc} = \bar{\sigma}(M)/2$, where M is

$$\begin{aligned} M &\triangleq \begin{bmatrix} M_d & M_{n^y} \end{bmatrix} \\ M_d &= -J_{uu}^{1/2}(HG^y)^{-1} HFW_d \\ M_n &= -J_{uu}^{1/2}(HG^y)^{-1} HW_{n^y} \end{aligned} \tag{4}$$

The optimal H that minimizes the loss can be found by solving the convex *optimization problem*

$$\begin{aligned} &\min_H \|H\widetilde{F}\|_F \\ &\textit{subject to } HG^y = J_{uu}^{1/2} \end{aligned} \tag{5}$$

Here $\widetilde{F} = \begin{bmatrix} FW_d & W_{n^y} \end{bmatrix}$ and $F = -(G^y J_{uu}^{-1} J_{ud} - G_d^y)$. The reason for using the Frobenius norm is that minimization of this norm also minimizes $\bar{\sigma}(M)$ Kariwala et al. [2008].

Derivation of multivariable PID controller

Assuming that the available "measurements" in y include the present, integrated, and derivative value of the output, Theorem 1 can be used for design of multivariable PID controllers. The following procedure is proposed:

1. To include integral action in the LQ problem formulation, augment the plant with $n_d = n_y$ disturbances such that offset-free tracking is guaranteed, i.e. by using the rank-conditions from [Pannocchia and Rawlings, 2003], and $n_\sigma = n_d$ integrators that belongs to the controller. The augmented plant becomes:

$$\begin{bmatrix} \dot{x} \\ \dot{\sigma} \\ \dot{d} \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 & B_d \\ C & 0 & C_d \\ 0 & 0 & 0 \end{bmatrix}}_{\widetilde{A}} \begin{bmatrix} x \\ \sigma \\ d \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix}}_{\widetilde{B}} u$$
$$\begin{bmatrix} y^P \\ y^I \\ y^D \end{bmatrix} = \underbrace{\begin{bmatrix} C & 0 & 0 \\ 0 & I & 0 \\ CA & 0 & CB_d \end{bmatrix}}_{\widetilde{C}} \begin{bmatrix} x \\ \sigma \\ d \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ CB \end{bmatrix}}_{\widetilde{D}} u + \begin{bmatrix} n_P^y \\ n_I^y \\ n_D^y \end{bmatrix} \tag{6}$$

   This system can be discretized to

$$\begin{aligned} \widetilde{x}_{k+1} &= \Phi \widetilde{x}_k + \Gamma u_k \\ \widetilde{y}_k &= \widetilde{C}\widetilde{x}_k + \widetilde{D} u_k + \widetilde{n}^y \end{aligned} \tag{7}$$

   Here $\widetilde{x}_k = (x_k, \sigma_k, d_k)$, $\hat{y}_k = (y_k^P, y_k^I, y_k^D)$ and $\widetilde{n}^y = (n_P^y, n_I^y, n_D^y)$.

2. Define the LQ-objective for the control problem,

$$\begin{aligned} \min_U J(U, x(0)) = \sum_{i=0}^{\infty} x_k^{\mathrm{T}} Q x_k + \Delta u_k^{\mathrm{T}} R_\Delta \Delta u_k \\ \text{subject to } x_0 = x(0) \text{ and} \\ \text{equation (7) for } \ k = 0, 1, 2, \ldots, \end{aligned} \tag{8}$$

   where $U \triangleq (u_0, u_1, u_2, \ldots)$.

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3. Convert (8) to a finite optimization problem by using for $k \geq N$, $u_k = -K_{\mathrm{LQR}} x_k$. This gives an objective function on the form

$$J(u_0, u_1, \ldots, u_{N-1}, x_0) = x_N^{\mathrm{T}} P x_N + \sum_{i=0}^{N-1} x_k^{\mathrm{T}} Q x_k + \Delta u_k^{\mathrm{T}} R_\Delta \Delta u_k, \tag{9}$$

where P is a solution of a Lyapunov equation, see Scokaert and Rawlings [1998].

4. Substitute the model equations into the objective function, to get an objective on the form (2), with

$$\frac{J_{uu}}{2} = \begin{bmatrix} \Gamma^{\mathrm{T}} P \Gamma & \Gamma^{\mathrm{T}} \Phi^{\mathrm{T}} K \Gamma & \ldots & \Gamma^{\mathrm{T}} (\Phi^{N-1})^{\mathrm{T}} P \Gamma \\ \Gamma^{\mathrm{T}} P \Phi \Gamma & \Gamma^{\mathrm{T}} P \Gamma & \ldots & \Gamma^{\mathrm{T}} (\Phi^{N-2})^{\mathrm{T}} P \Gamma \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma^{\mathrm{T}} P \Phi^{N-1} \Gamma & \Gamma^{\mathrm{T}} P \Phi^{N-2} \Gamma & \ldots & \Gamma^{\mathrm{T}} P \Gamma \end{bmatrix} + M^{\mathrm{T}} \begin{bmatrix} R_\Delta & & \\ & \ddots & \\ & & R_\Delta \end{bmatrix} M, \tag{10}$$

where

$$M = \begin{bmatrix} -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix} \in \mathbb{R}^{n_u(N-1) \times n_u N}, \tag{11}$$

and

$$\frac{J_{ud}}{2} = \begin{bmatrix} \Gamma^{\mathrm{T}} & & & \\ & \Gamma^{\mathrm{T}} & & \\ & & \ddots & \\ & & & \Gamma^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} P \\ P\Phi \\ \vdots \\ P\Phi^{N-1} \end{bmatrix} \Phi \tag{12}$$

Here $u = (u_0, u_1, \ldots, u_{N-1})$ and $d = x(0)$.

5. We now let the "measurements" in Theorem 1 include the process outputs and the inputs, $y = (y_k^P, y_k^I, y_k^D, u_k, \ldots, u_{k+N-1})$. These variables can be written as

$$y = G^y u + G_d^y d \tag{13}$$

with

$$G^y = \begin{bmatrix} \widetilde{D} & 0 \\ I & 0 \\ 0 & I \end{bmatrix}; \; G_d^y = \begin{bmatrix} \widetilde{C} & 0 \end{bmatrix}, \tag{14}$$

where 0 is a matrix of zeros of appropriate dimensions and I is an identity matrix of appropriate dimensions.

6. We can now compute the sensitivity matrix $F = -(G^y J_{uu}^{-1} J_{ud} - G_d^y)$ and use (5) in Theorem 1 to find the optimal H. This convex optimization problem can be solved for example with `cvx`, a package for specifying and solving convex programs [Grant and Boyd, 2008], with the following Matlab$^{\mathrm{TM}}$ code:

```
cvx_begin
   variable H(N*nu,ny+nu*N);
   minimize norm(H*Ftilde,'fro')
   subject to
      H*Gy == sqrtm(Juu);
cvx_end
```

The optimal H combines Hy such that when controlled to the constant setpoint of 0 gives minimum operational loss from the optimal solution, which is defined by the solution of (8) when the full state vector $x(0)$ is available for measurement.

7. From Alstad et al. [2008] we have that for an optimal $\widetilde{H}$, $H = D\widetilde{H}$ will still be optimal with respect to the optimization problem in (5) provided that the $n_c \times n_c$ D-matrix is non-singular. Let $\widetilde{H} = [H^y \; H^u]$. For linearly independent inputs we have that H^u is non-singular, hence another optimal H is $H = (H^u)^{-1}\widetilde{H} = [(H^u)^{-1} H^y \; I]$.

The H matrix is a $Nn_u \times (3n_y + Nn_u)$ matrix. The first n_u rows of $Hy = 0$ has this information:

$$K_P y_k^P + K_I y_k^I + K_D y_k^D + I u_k + 0 u_{k+1} + \cdots + 0 u_{k+N-1} = 0 \tag{15}$$

We solve for u_k and finally get the LQ-optimal multivariable PID controller:

$$u_k = -(K_P y_k^P + K_I y_k^I + K_D y_k^D) \tag{16}$$

This is the MIMO PID approximation of the original LQ problem. To guarantee closed loop stability a separate analysis is required.

Conclusions

In this extended abstract we outlined how to find a multivariable LQ-optimal PID controller based on convex optimization. This is a significant contribution because previous work indicates that this problem is non-convex. Examples will be given in the presentation.

References

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