Proceedings of the 2007 American Control Conference
Marriott Marquis Hotel at Times Square
New York City, USA, July 11-13, 2007

Data-Based Modeling of Block-Diagonal Uncertainty by Convex Optimization

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***Abstract*—A procedure for deriving norm-bounded uncertainty models for MIMO systems is presented. Additive as well as multiplicative input and output uncertainty models with structured or unstructured uncertainty are treated in a unified manner. The main focus in this paper is on structured (block diagonal) uncertainty. The models are determined by matching the input-output behavior of an uncertainty model to sets of input-output data obtained, e.g., through system identification. Tight bounds are achieved by minimization of the size of an uncertainty region subject to necessary and sufficient data-matching conditions. The calculations, which are done frequency by frequency, are formulated as a convex optimization problem using LMIs as constraints. In an application to uncertainty modeling of a distillation column various structural types of uncertainty models are compared.**

I. Introduction

MANY robust control design methods require a model consisting of a linear nominal model augmented by an uncertainty description in the form of a weighted norm-bounded uncertainty. The construction of such a model with a minimum amount of conservatism is a significant problem.

A straightforward approach is first to determine a parametric model, where the parameters are known or assumed to vary within certain intervals. However, even if the model is linear in the parameters, it is usually not in the form of a norm-bounded uncertainty model. Some techniques for deriving such models from a parametric uncertainty model are described for single-input, single-output (SISO) systems in [1] and [2]. A method for multiple-input, multiple-output (MIMO) systems on state-space form was suggested in [3].

In cases where a model obtained from first principles is not available, system identification might be used. Models with probabilistic parameter bounds can then usually be obtained. Since such bounds are incompatible with the hard bounds required in a norm-bounded uncertainty description, methods for obtaining hard bounds via system identification have also been developed; see, e.g., [4].

An alternative approach in the use of system identification is to initially determine a set of models. This can facilitate the separation of noise and difficult system dynamics that cannot easily be included in a single deterministic model. Models for different operating points can also be included in the model set. The identification of ill-conditioned MIMO systems is especially troublesome. Unless special care is taken to properly excite the low-gain direction, the resulting model may easily be useless for controller design [5]. These and other considerations concerning the identification of MIMO systems make it both appealing and convenient to capture the system dynamics in a set of models.

One way of constructing a norm-bounded uncertainty model based on a set of models is to employ model-matching techniques. The goal is then to obtain an uncertainty model that can reproduce every model in the model set. Such a technique has been used in [6]. However, if the identification experiments have resulted in a set of models, it is because different input sequences and different operating points give different models. It then appears more realistic to assume that a model applies only to the input sequence used for generating the data, from which the model was determined, not to arbitrary inputs as in the case of model matching. This suggests derivation of an uncertainty model using data matching instead of model matching. Such an approach was recently used in [7] and [8]. It can be shown that data matching gives a less conservative uncertainty description than model matching [9].

In this paper we propose a technique for derivation of an uncertainty description based on data matching in the frequency domain. In particular, we consider norm-bounded uncertainty descriptions for additive, multiplicative input and multiplicative output uncertainty with a block-diagonal uncertainty. We have previously considered the unstructured case [10]. Tight bounds are achieved by minimization of the size of the worst-case uncertainty region subject to necessary and sufficient data-matching conditions. An attractive feature is that the problem can be formulated as a convex optimization problem with the data-matching constraints expressed by linear matrix inequalities (LMIs). The solution technique is thus much more tractable than the one employed in [8]. We consider various structural issues in an application to uncertainty modeling of a distillation column.

II. Problem Formulation

A. Uncertainty Description

We consider linear MIMO uncertainty models of the form

$$G(s) = G_0(s) + W_{\mathrm{a}}(s)\Delta(s)W_{\mathrm{b}}(s)\,, \quad \|\Delta\|_\infty \le 1 \qquad (1)$$

where $G_0(s)$ is a stable nominal transfer matrix model, $W_{\mathrm{a}}(s)$ and $W_{\mathrm{b}}(s)$ are stable transfer matrix filters acting as

Manuscript received September 24, 2006. Financial support from the Academy of Finland under grant number 203277 (decision number 206750) is gratefully acknowledged.

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uncertainty weights, and $\Delta(s)$ is a norm-bounded uncertainty matrix. Here the uncertainty is expressed as additive uncertainty. To include also the possibility of dealing with multiplicative uncertainty, we define

$$W_{\mathrm{a}}(s) := M_1(s)W_1(s)\,,\; W_{\mathrm{b}}(s) := W_2(s)M_2(s) \qquad (2)$$

where $M_i(s) = I$, except that $M_1(s) = G_0(s)$ in the case of multiplicative input uncertainty, and $M_2(s) = G_0(s)$ in the case of multiplicative output uncertainty.

We assume that the uncertainty matrix $\Delta(s)$ has a block-diagonal structure

$$\Delta(s) = \mathrm{diag}(\Delta_i(s);\, i = 1,\ldots,q)\,,\; \|\Delta_i\|_\infty \le 1 \qquad (3)$$

and that the uncertainty weights $W_1(s)$ and $W_2(s)$ have corresponding block-diagonal structures

$$W_1(s) = \mathrm{diag}(W_{1,i}(s);\, i = 1,\ldots,q) \qquad (4a)$$

$$W_2(s) = \mathrm{diag}(W_{2,i}(s);\, i = 1,\ldots,q) \qquad (4b)$$

of compatible dimensions such that $W_1(s)\Delta(s)W_2(s)$ is block diagonal. It is also assumed that every $W_{1,i}$ has full row rank and that every $W_{2,i}$ has full column rank.

B. Data Matching

We want to determine W_1 and/or W_2 such that the uncertainty model (1), for some admissible Δ, can reproduce known input-output data with a minimum amount of conservatism. We assume that smoothed (noise-free) input-output data $\{u_k(\mathrm{j}\omega), y_k(\mathrm{j}\omega) : \omega \in \Omega\}$, $k = 1,\ldots,N$, are available at a number of relevant frequencies $\omega \in \Omega$. In addition, a nominal model $G_0(\mathrm{j}\omega)$ may be known.

In practice, the data may be obtained from a number of identification experiments k, $k = 1,\ldots,N$. For each experiment, a model G_k is determined such that

$$z_k(s) = G_k(s)u_k(s) + n_k(s) \qquad (5)$$

where z_k is the measured output and n_k is noise. We assume that we can successfully separate noise and relevant dynamics in this stage. A special technique for this was employed in [8]. Noise-free output data can then be obtained according to

$$y_k(\mathrm{j}\omega) = G_k(\mathrm{j}\omega)u_k(\mathrm{j}\omega)\,,\; \forall\omega \in \Omega\,,\; \forall k \qquad (6)$$

Note, however, that the separation of noise and dynamics is not an issue of this paper.

The uncertainty modeling does not require a nominal model G_0 to be known initially, but if desired, a model can be obtained, e.g., by fitting to all available input-output data.

Data matching requires that there is an admissible Δ such that the uncertainty model (1) satisfies

$$y_k(\mathrm{j}\omega) = G(\mathrm{j}\omega)u_k(\mathrm{j}\omega)\,,\; \forall\omega \in \Omega\,,\; \forall k \qquad (7)$$

Note that we do not require $G(\mathrm{j}\omega) = G_k(\mathrm{j}\omega)$, which is a more stringent model-matching requirement.

C. Uncertainty Minimization

We want to minimize the conservatism of the uncertainty description by minimizing some suitable criterion with respect to W_1, W_2 and possibly G_0. We consider the size of the uncertainty region covered by the uncertainty model to be such a criterion. The minimization of the size of this region subject to (7) will result in $\bar{\sigma}(\Delta) = 1$ for some data point(s) at every frequency $\omega \in \Omega$. The uncertainty model (1) determined in this way is thus a "tight" uncertainty model.

In the next section we shall derive an expression for the size of the uncertainty region.

III. Minimization of Uncertainty

In the following, the modeling and the calculations are done frequency by frequency at a number of relevant frequencies $\omega \in \Omega$. For convenience, the argument "$\mathrm{j}\omega$" is omitted in the sequel. We do not, in this paper, discuss how to determine transfer functions $W_1(s)$, $W_2(s)$ and $G_0(s)$ from the corresponding sets of frequency-wise calculated weights and nominal model data.

A. The Region of Uncertainty

A relevant quantity in the uncertainty modeling is the difference between an output that the uncertainty model can generate and the output of the nominal model. In the case of multiplicative input uncertainty, when $M_1 = G_0$ and $M_2 = I$, the corresponding difference between inputs producing the same output is also of relevance. If we assume that M_1 is invertible, both situations can be taken into account by defining the deviation

$$e := M_1^{-1}(G - G_0)u = W_1\Delta W_{\mathrm{b}}u \qquad (8)$$

where u is an arbitrary input.

Let us now briefly consider the case of *unstructured uncertainty*. It is clear that the uncertainty block Δ can generate any vector x such that

$$e = W_1 x\,,\; \|x\| \le \|W_{\mathrm{b}}u\| \qquad (9)$$

where $\|\bullet\|$ denotes the Euclidean vector norm. When x varies over its allowable range, the deviation e covers an ellipsoidal region. The size of this region is proportional to

$$J_0(u) = \det(W_1W_1^*)^{1/2}\,\|W_{\mathrm{b}}u\|^n \qquad (10)$$

where $n = \dim(e)$. Here, * denotes complex conjugate transpose. For the worst-case input u, $\|u\| = 1$, we then have [10]

$$J_0 = \det(W_1W_1^*)^{1/2}\,\|W_{\mathrm{b}}\|^n \qquad (11)$$

In the case of *block-diagonal uncertainty*, the uncertainty block Δ_i can generate any vector x_i such that

$$e_i = W_{1,i}x_i\,,\; \|x_i\| \le \|W_{\mathrm{b},i}u\| \qquad (12)$$

where e_i denotes the deviation corresponding to $W_{1,i}$ and $W_{\mathrm{b},i} = W_{2,i}M_{2,i}$, where $M_{2,i}$ denotes the rows of M_2 corresponding to $W_{2,i}$. The size of the ellipsoidal region covered by e_i as x_i varies over its range of values is proportional to

$$J_i(u) = \det(W_{1,i}W_{1,i}^*)^{1/2} \left\|W_{b,i}u\right\|^{n_i} \quad (13)$$

where $n_i = \dim(e_i)$.

We are, of course, interested in the region covered by the entire vector e. Since every e_i can vary independently, the size of this region is proportional to

$$J(u) = \prod_{i=1}^{q} \det(W_{1,i}W_{1,i}^*)^{1/2} \left\|W_{b,i}u\right\|^{n_i} \quad (14)$$

The maximum size of the uncertainty region is given by some $u = u_*$, $\|u_*\| = 1$, i.e.

$$J = \det(W_1W_1^*)^{1/2} \prod_{i=1}^{q} \left(\left\|W_{b,i}u_*\right\|\right)^{n_i} \quad (15)$$

where it was possible to introduce $\det(W_1W_1^*)^{1/2}$ due to the block-diagonal structure of $W_1W_1^*$.

We shall next consider the data-matching conditions and how they affect the minimization of J.

B. Data-Matching Conditions

Assume that we have sets of input-output data $\{u_k, y_k\}$, $k = 1,\ldots,N$. The deviation e_k corresponding to (8) is then

$$e_k := M_1^{-1}(y_k - G_0u_k) = W_1\Delta_kW_bu_k \quad (16)$$

It is well known that in the case of unstructured uncertainty there is a Δ_k, with the maximum singular value $\bar{\sigma}(\Delta_k) \le 1$, that satisfies (16) if and only if [11], [12]

$$\begin{bmatrix} W_1W_1^* & e_k \\ e_k^* & (W_bu_k)^*(W_bu_k) \end{bmatrix} \succcurlyeq 0, \quad \forall k \quad (17)$$

where "$\succcurlyeq$" denotes semi-positive definite. For structured uncertainty we similarly obtain

$$\begin{bmatrix} W_{1,i}W_{1,i}^* & e_{i,k} \\ e_{i,k}^* & (W_{b,i}u_k)^*(W_{b,i}u_k) \end{bmatrix} \succcurlyeq 0, \quad i = 1,\ldots,q, \ \forall k \quad (18)$$

C. Convex Optimization

Unfortunately, the criterion (15) that we want to minimize is not a convex function. One reason for this is that it contains a product of factors like $\left\|W_{b,i}u_*\right\|$. However, we can define an upper bound of the criterion by using the relation $\left\|W_{b,i}u_*\right\| \le \left\|W_{b,i}\right\|$. This relation becomes an equality in certain cases. Since the quality of the uncertainty model and the cost function are not changed if the elements of $W_{1,i}$ are multiplied by a nonzero scalar and the elements of $W_{b,i}$ are divided by the same scalar, we can assign $\left\|W_{b,i}\right\|$ any constant positive value γ. This value can be attained by the use of a suitable constraint in the optimization.

Another obstacle is that the determinant is not a convex function. However, if we can write the argument of the determinant as the inverse of a positive definite matrix, it can be "convexified" by using the logarithm of the determinant [13].

To facilitate the solution of the problem, we shall now introduce some new variables. We define

$$Y_i := (W_{1,i}W_{1,i}^*)^{-1/2}, \quad Y := \mathrm{diag}(Y_i; i = 1,\ldots,q) \quad (19)$$

$$X_i := W_{2,i}^*W_{2,i}, \quad X := \mathrm{diag}(X_i; i = 1,\ldots,q) \quad (20)$$

$$Z_{1,i} := Y_i(G_0^{-1})_i, \quad Z_1 := [Z_{1,1}^{\mathrm{T}} \ \cdots \ Z_{1,q}^{\mathrm{T}}]^{\mathrm{T}} \quad (21)$$

$$Z_{2,i} := Y_iG_{0,i}, \quad Z_2 := [Z_{2,1}^{\mathrm{T}} \ \cdots \ Z_{2,q}^{\mathrm{T}}]^{\mathrm{T}} \quad (22)$$

where subscript "i" refers to the i th uncertainty block. The data-matching criterion can now be reformulated as follows.

For the case of multiplicative input uncertainty we have $M_1 = G_0$ and $M_2 = 1$. By introduction of y_k and u_k by means of (16), the data-matching criterion (18) can be written as (for details, see [10])

$$\begin{bmatrix} I & Z_{1,i}y_k - Y_iu_{i,k} \\ (Z_{1,i}y_k - Y_iu_{i,k})^* & u_{i,k}^*X_iu_{i,k} \end{bmatrix} \succcurlyeq 0, \ i = 1,\ldots,q, \ \forall k \quad (23)$$

For additive and multiplicative output uncertainty the data-matching criterion becomes

$$\begin{bmatrix} I & Y_iy_{i,k} - Z_{2,i}u_k \\ (Y_iy_{i,k} - Z_{2,i}u_k)^* & u_k^*M_{2,i}^*X_iM_{2,i}u_k \end{bmatrix} \succcurlyeq 0, \ i = 1,\ldots,q, \ \forall k \quad (24)$$

where $M_2 = I$ for additive uncertainty and $M_2 = G_0$ for multiplicative output uncertainty.

As discussed above, $W_{b,i}$ is restricted by

$$\left\|W_{b,i}\right\| = \left\|W_{2,i}M_{2,i}\right\| \le \gamma \quad (25)$$

which, with the definition in (20), can be written as

$$0 \prec M_{2,i}^*X_iM_{2,i} \preccurlyeq \gamma^2 I \quad (26)$$

Another way to write this linear matrix inequality is

$$\begin{bmatrix} X_i & X_iM_{2,i} \\ M_{2,i}^*X_i & \gamma^2 I \end{bmatrix} \succcurlyeq 0, \quad i = 1,\ldots,q \quad (27)$$

Finally, the cost function (15) can be replaced by

$$J = \log\det Y^{-1} \quad (28)$$

The general optimization problem based on input-output matching can now be formulated as follows:

$$\begin{aligned} &\underset{Y\in\mathcal{Y}, X\in\mathcal{X}, Z\in\mathcal{Z}}{\text{minimize}} \ \det Y^{-1}, \ \forall\omega\in\Omega \\ &\text{subject to (27), and (23) or (24)} \end{aligned} \quad (29)$$

where Z stands for Z_1 or Z_2 and $\mathcal{Y}$, $\mathcal{X}$ and $\mathcal{Z}$ denote allowable classes by which (further) structural restrictions may be imposed.

We note that the uncertainty weights $W_{1,i}$ and $W_{2,i}$ cannot be uniquely determined du to the quadratic forms in (19) and (20). However, this is not a serious drawback. In fact, it adds degrees of freedom to the design of filter transfer functions $W_{1,i}(s)$ and $W_{2,i}(s)$ because $W_{1,i}$ may be post-multiplied and $W_{2,i}$ premultiplied by arbitrary unitary matrices. This is allowed because Δ_i could always produce the same effect without violating the constraint $\|\Delta_i\|_\infty \le 1$. Except for such unitary matrices, $W_{1,i}$ and $W_{2,i}$ can be determined via singular value decompositions of Y_i and X_i.

The nominal model is determined by

$$G_0 = Z_1^{-1} Y \quad \text{or} \quad G_0 = Y^{-1} Z_2 \tag{30}$$

IV. Application to Distillation Modeling

A distillation column is a multivariable system usually characterized by a strong directionality, which means that the transfer matrix is ill-conditioned and nearly singular. In order to be sufficiently accurate for controller design, a model must provide a good description of the directionality properties. In identification it is therefore important to excite all directions sufficiently, especially the low-gain direction [5]. Since it may be difficult to capture in a single linear model all relevant dynamics, which tend to vary with the input direction, an appealing approach is to determine a set of models, or sets of input-output data.

A. Experiments for Generation of Input-Output Data

The distillation column of this study is a pilot-scale two-product distillation column [5]. The column has been identified by applying a series of step changes to its high- and low-gain input directions. These directions can be estimated with good accuracy from certain flow gains, which are easy to determine in practice [14]. From these experiments, a nominal model as well as six additional models were determined as transfer matrix models composed of second-order transfer functions with dead-time [5]. The model outputs provide output data, which are essentially free of noise.

B. Uncertainty Modeling

Previous studies have indicated that these sets of input-output data can be well captured by an uncertainty model of multiplicative output type [8], [10]. We shall here study how the uncertainty model can be further improved by the calculation of an optimal nominal model. We shall consider both unstructured and structured uncertainty as well as full and diagonal weight matrices. For ease of illustration, we will only show results for the steady state.

We shall consider a simple multiplicative output uncertainty model of the form

$$G = (I + W_1 \Delta) G_0 \ , \quad \|\Delta\| \le 1 \tag{31}$$

This means that $M_1 = I$, $M_2 = G_0$ and $W_2 = I$ in the previous equations. For an input u, this model can produce any output y that satisfies

$$y = G_0 u + e \ , \quad e = W_1 \Delta G_0 u \ , \ \|\Delta\| \le 1 \tag{32}$$

Our objective is to minimize the uncertainty region covered by e, due to the variation of Δ, for the worst-case input $u = u_*$, $\|u_*\| \le 1$. The solution for unstructured uncertainty

TABLE I
Steady-State Data of Individual Experiments

Exp.#	u_1	u_2	y_1	y_2
1	10.0	5.0	0.06180	−0.23315
2	−20.0	−10.0	−0.09280	0.42640
3	10.0	5.0	0.04135	−0.20590
4	0.5	−1.0	−0.11513	0.50204
5	−1.0	2.0	0.22997	−0.76869
6	0.5	−1.0	−0.17393	0.33254

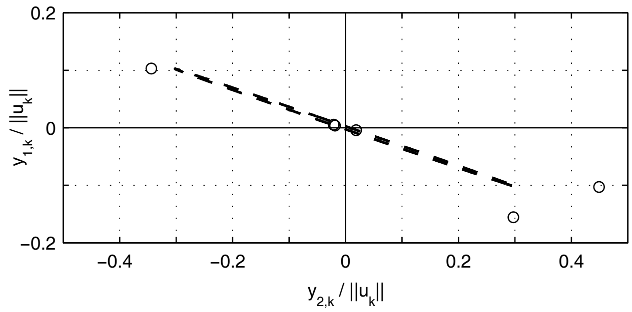

Fig. 1. Normalized experimental outputs (o) and region covered by the outputs of the original nominal model (- -).

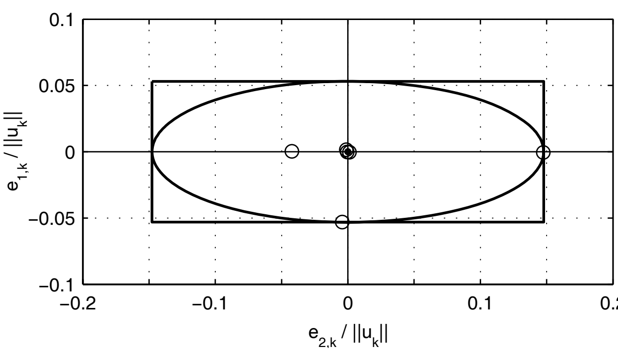

Fig. 2. Normalized experimental deviations (o) and largest uncertainty region for unstructured (ellipse) and structured (rectangle) uncertainty with the original nominal model.

is obtained by the use of $q = 1$ in the given equations. Any desired structure can be imposed on W_1; in this application W_1 is either a full matrix or a diagonal one. For the optimization, we use the YALMIP [15] software together with Matlab.

The experiments and the initial modeling have given smoothed (noise-free) sets of input-output data $\{u_k, y_k\}$, $k = 1, \ldots, 6$, and a nominal model G_0 has been determined by fitting its output to all available data [5]. The steady-state input-output data and the (initial) nominal model are given in Table I and Table II ("Fig. 3"), respectively. Because the system under study has two inputs and two outputs, the data and the optimization results can conveniently be illustrated graphically.

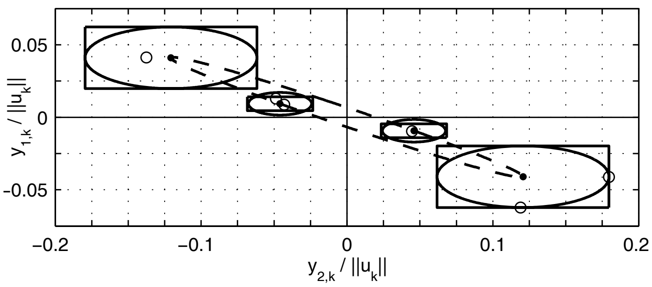

Fig. 3. Normalized experimental points (o) and uncertainty regions for unstructured (ellipse) and structured (rectangle) uncertainty at experimental points. (Transformed data.)

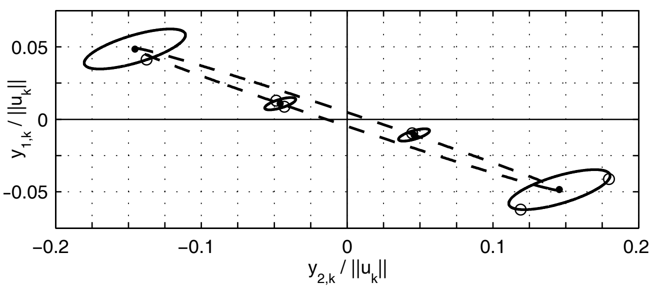

Fig. 4. Normalized experimental points (o) and uncertainty regions for unstructured uncertainty with optimal nominal model and full weight matrix. (Transformed data.)

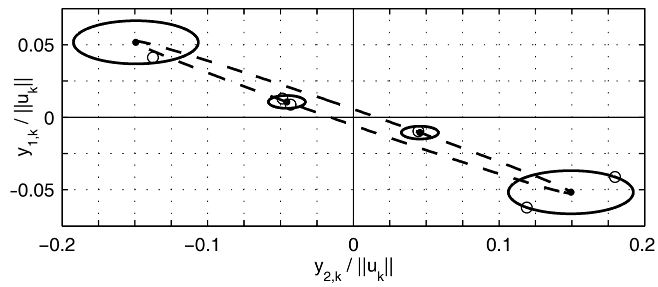

Fig. 5. Normalized experimental points (o) and uncertainty regions for unstructured uncertainty with optimal nominal model and diagonal weight matrix. (Transformed data.)

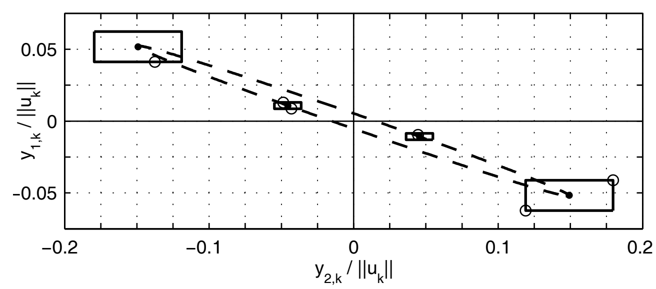

Fig. 6. Normalized experimental points (o) and uncertainty regions for structured uncertainty with optimal nominal model and diagonal weight matrix. (Transformed data.)

Figure 1 shows the experimental data outputs, scaled by the norm of the input, i.e., $y_k / \|u_k\|$, $k = 1,\ldots,6$. Included is also the elliptical region covered by the output of the nominal model for all possible inputs u. As can be seen, the model is highly ill-conditioned; its condition number is 125. It is also clear that three data points are in the high-gain region and three data points (two of which are almost identical) are in the low-gain region of the nominal model.

The data points in Fig. 2 show scaled deviations between the actual outputs and the corresponding outputs of the original nominal model. Because the data points farthest away from the origin are very close to the coordinate axes, it is sufficient to use a diagonal uncertainty weight matrix with this nominal model. The figure also shows the uncertainty regions for the worst-case input for both unstructured uncertainty (elliptical region) and diagonal uncertainty (rectangular region). As shown, unstructured uncertainty gives a smaller uncertainty region with this nominal model.

Figure 3 depicts the same solution as Fig. 2, but using actual outputs y_k (as in Fig. 1) and with more data details included. To improve the visualization, the condition number of the nominal model has been changed to 20 by changing its singular values. The inputs have also been rescaled so as to keep $G_0 u_k$ unchanged. This preserves the sizes of the uncertainty regions of model (31). However, the rescaled inputs result in a different scaling of the normalized outputs $y_k / \|u_k\|$ than in Fig. 1.

Figure 3 shows the experimental outputs as well as the uncertainty regions around the nominal outputs. Also here, the elliptical regions apply to unstructured uncertainty and the rectangular regions to diagonal uncertainty, both with a diagonal weight matrix. As required, the uncertainty regions include the experimental output(s) with the nominal output in the center of the region. There are four regions in each case because four different inputs were used in the six experiments. As can be seen, the uncertainty regions are quite large compared to the entire region covered by the output of the nominal model for all possible inputs.

In the solutions illustrated by figures 4, 5 and 6, the uncertainty regions have been minimized also with respect to the nominal model G_0. The same rescaling of inputs is used as in Fig. 3 and the new nominal models are further transformed to keep the new $G_0 u_k$ unchanged. These transformations result in the same scaling of axes in the figures, which is also implied by the fact that the experimental data points have the same positions in the figures. This means that the sizes of uncertainty regions and of the regions covered by the nominal models are directly comparable.

Figure 4 shows the solution for unstructured uncertainty with a full weight matrix. As can be seen, the uncertainty regions have the form of rotated ellipses in this case. The optimization with respect to the nominal model has reduced the size of the uncertainty regions considerably.

Figure 5 shows the solution for unstructured uncertainty

TABLE II
UNSTRUCTURED UNCERTAINTY

Fig.	Nominal Model	Weight Matrix	Area
3	$\begin{bmatrix} -0.0423 & 0.0935 \\ 0.1173 & -0.2786 \end{bmatrix}$	$\begin{bmatrix} 0.1666 & 0 \\ 0 & 0.4627 \end{bmatrix}$	0.0247
4	$\begin{bmatrix} -0.0501 & 0.1105 \\ 0.1452 & -0.3344 \end{bmatrix}$	$\begin{bmatrix} 0.0762 & 0.0477 \\ 0.0477 & 0.2215 \end{bmatrix}$	0.0068
5	$\begin{bmatrix} -0.0538 & 0.1177 \\ 0.1495 & -0.3426 \end{bmatrix}$	$\begin{bmatrix} 0.0942 & 0 \\ 0 & 0.2714 \end{bmatrix}$	0.0126

Area = Size of largest uncertainty region, $\|u\| \le 1$

TABLE III
DIAGONAL UNCERTAINTY

Fig.	Nominal Model	Weight Matrix	Area
3	$\begin{bmatrix} -0.0423 & 0.0935 \\ 0.1173 & -0.2786 \end{bmatrix}$	$\begin{bmatrix} 0.5173 & 0 \\ 0 & 0.4887 \end{bmatrix}$	0.0313
6	$\begin{bmatrix} -0.0537 & 0.1176 \\ 0.1495 & -0.3425 \end{bmatrix}$	$\begin{bmatrix} 0.2040 & 0 \\ 0 & 0.2031 \end{bmatrix}$	0.0080

Area = Size of largest uncertainty region, $\|u\| \le 1$

with a diagonal weight matrix. Even though the nominal model is adjusted to produce the best solution for this case, the sizes of the uncertainty regions have clearly increased compared to the case with a full weight matrix.

If a diagonal weight matrix is used, it seems reasonable to also use a diagonal uncertainty matrix Δ. Figure 6 depicts the solutions for this case. A comparison with Fig. 5 shows that the resulting sizes of the uncertainty regions have decreased. In fact, they are comparable in size to the uncertainty regions for unstructured uncertainty with a full weight matrix depicted in Fig. 4.

Actually, the sizes of the uncertainty regions for a diagonal uncertainty matrix could be further reduced by the use of a full weight matrix. This would result in uncertainty regions having the shape of rotated parallelograms. However, this would require the solution of a rank-constrained optimization problem, which has not been attempted.

The nominal model, the weight matrix and the size of the largest uncertainty region are shown in Table II and III for the various cases illustrated in the figures above. Here, actual units are used, not normalized ones.

V. CONCLUSION

A procedure based on convex optimization techniques for deriving norm-bounded structured (and unstructured) uncertainty models for MIMO systems has been presented. Data for the uncertainty modeling are assumed to be available as sets of input-output data. The uncertainty modeling is based on data matching in the frequency domain, for which necessary and sufficient conditions are expressed by LMIs. The minimization of the size (area, volume, etc.) of the worst-case uncertainty region can be formulated as a determinant-minimization problem, which can be further transformed into a convex optimization problem.

The modeling technique was applied to uncertainty modeling of a distillation column. Multiplicative output uncertainty models were determined both for unstructured uncertainty and diagonal uncertainty. The results show that the model uncertainty can be reduced considerably by optimizing with respect to the nominal model. In the case of unstructured uncertainty, the results also indicate a significant improvement when a full weight matrix is used instead of a diagonal one. A diagonally structured uncertainty with a diagonal weight matrix gave surprisingly good results with an optimized nominal model considering the small number of parameters used in this uncertainty description.

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