A Revisit of Marino-Tomei's Result on Output Feedback Control of a Class of Non-Minimum Phase Nonlinear Systems ⋆

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Abstract: In this note, global stabilization by output feedback is investigated for a class of non-minimum-phase nonlinear systems previously considered by Marino and Tomei (2005). It is shown that it is possible to construct, via a new design method that involves no filter transformation, a globally stabilizing dynamic output feedback controller of order n, instead of $n + 2(\rho - 1)$, for the non-minimum phase nonlinear system in output feedback form Marino and Tomei (2005), under a slightly general condition (i.e., Assumption 2.2) together with the assumption that the nonlinear system is non-minimum-phase with respect to the original output, but minimum-phase with respect to a virtual linear output. Two examples are given to illustrate the simplicity of the new design approach and its effectiveness.

Keywords: Lyapunov Methods, Stabilization, Observability and Observer Design.

1. INTRODUCTION

Global stabilization of nonlinear systems by output feedback is an important topic and has received a great deal of attention in the literature. For minimum-phase nonlinear systems, extensive research has been carried out over the years and many results on global stabilization via output feedback have been reported; see, for instance, the papers Marino and Tomei (1991, 1995); Praly and Jiang (1993); Battilotti (1997); Qian and Lin (2002); Praly (2003); Yang and Lin (2005) and related references therein.

By comparison, only few research results are available devoting to the problem of global stabilization of non-minimum phase nonlinear systems by output feedback. While the paper Isidori (2000) investigated the problem of semi-global stabilization for non-minimum phase nonlinear systems in the so-called normal form, the work Karagiannis et al. (2005) proposed a globally stabilizing output feedback design method for a class of uncertain non-minimum phase systems, under the assumption that the zero-dynamics may not necessarily be stable but the inverse dynamical system satisfies a strong form of the ISS condition. In Marino and Tomei (2005), a different class of non-minimum-phase nonlinear systems is considered. In particular, it was shown that a non-minimum-phase nonlinear system in output feedback form is globally stabilizable by output feedback, if there exists a virtual output which is a linear combination of the system states, rendering the non-minimum-phase system minimum-phase with respect to the virtual output Marino and Tomei (2005). The conclusion was established by constructing a dynamic output compensator of order $n + 2(\rho - 1)$, with n being the order of the nonlinear system and ρ its relative degree. The more recent work Andrieu and Praly (2008) has studied the problem of global output feedback stabilization for non-minimum phase nonlinear systems with a strict normal form. It was shown that some previous global stabilization results on non-minimum phase nonlinear systems can be reformulated and recovered by their methods. In particular, it was pointed out that the result of Marino and Tomei (2005) can be encompassed by the work of Andrieu and Praly (2008), and there is an output feedback compensator of dimension $n - 1$ for the nonlinear system in the output feedback form.

In this note, we revisit the problem studied by Marino and Tomei and point out that for the class of non-minimum phase nonlinear systems considered in Marino and Tomei (2005), there indeed exists an n-dimensional, rather than $n+2(\rho-1)$-dimensional, globally stabilizing dynamic output feedback controller. As a matter of fact, we show that under a slight general condition than those in Marino and Tomei (2005), i.e., the nonlinear system is non-minimum-phase with respect to the measured output but minimum-phase with respect to a linear combination of the system states, one can construct explicitly an n-dimensional output feedback compensator, globally asymptotically stabilizing the non-minimum phase nonlinear system. This is accomplished by developing a new yet simpler design method that does not use a filter transformation — a technique that is commonly used in Marino and Tomei (2005, 1995).

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The paper is organized as follows. Section 2 contains the introduction of a normal form for a single-input/single-output linear system that is controllable and observable, problem statement and main results of the paper. To design an output feedback stabilizer, a high-gain observer is given in Section 3 and a step-by-step controller design is proposed in Section 4. Based on the work in Sections 3-4, a new design method is developed in Section 5, yielding a simpler solution to the problem considered previously in Marino and Tomei (2005). Two examples are given in Section 6 to illustrate the simplicity of the new design approach and its effectiveness. Concluding remarks are drawn in Section 7.

2. PRELIMINARIES AND MAIN RESULT

In this section, we first introduce a normal form of single-input/single-output (SISO) linear systems, which will play a crucial role in the design of an n-dimensional, globally stabilizing output feedback controller for a class of non-minimum phase nonlinear systems in output feedback form Marino and Tomei (2005).

2.1 A Normal Form of SISO Linear Systems

Consider a SISO linear system of the form

$$\begin{cases} \dot{x} = A_c x + bu \\ y = C_c x \end{cases} \tag{2.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $y \in \mathbb{R}$ are the system states, input and output, respectively. The vector $b = [0, \dots, 0, b_\rho, \dots, b_n]^T$ with $b_\rho \neq 0$, and the pair (A_c, C_c) is in observable canonical form

$$A_c = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix}, \quad C_c = [1 \ 0 \ \dots \ 0].$$

Clearly, the SISO system above has a relative degree ρ.

The following lemma is a direct consequence of the coordinate transform introduced in Isidori (1995).

Lemma 2.1. The state equations of the linear system (2.1) can be transformed into

$$\begin{cases} \dot{z}_b = Fz_b + gz_1, & z_b = [z_{r+1}, \dots, z_n]^T, \\ \dot{z}_1 = z_2 & \\ \vdots & F \in \mathbb{R}^{(n-r)\times(n-r)} \\ \dot{z}_{r-1} = z_r & \\ \dot{z}_r = \sum_{i=1}^{n} a_i z_i + \bar{b}_r u, & g \in \mathbb{R}^{(n-r)}, \end{cases} \tag{2.2}$$

by the following nonsingular transformation

$$z = \begin{bmatrix} z_1 \\ \vdots \\ z_r \\ z_{r+1} \\ \vdots \\ z_n \end{bmatrix} = Tx = \begin{bmatrix} t_1 \\ t_1 A_c \\ \vdots \\ t_1 A_c^{r-1} \\ t_{r+1} \\ \vdots \\ t_n \end{bmatrix} x \triangleq \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix} x, \tag{2.3}$$

with $t_1 = [t_{11}, t_{12}, \dots, t_{1n}]$, $t_1 b = t_1 A_c b = \dots = t_1 A_c^{r-2} b = 0$, $t_1 A_c^{r-1} b \neq 0$, $t_{r+1} b = \dots = t_n b = 0$, $\bar{b}_r = t_1 A_c^{r-1} b$, where r is the relative degree of the linear system (2.1) with respect to the virtual output

$$z_1 = t_1 x = [t_{11}, t_{12}, \dots, t_{1n}]\, x. \tag{2.4}$$

2.2 Main Result

We now consider the class of nonlinear systems in output feedback form Marino and Tomei (2005)

$$\begin{cases} \dot{x} = A_c x + bu + \phi(y) \\ y = C_c x, \end{cases} \tag{2.5}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $y \in \mathbb{R}$ are the system states, control input and measured output, respectively. The matrices A_c, b and C_c are defined by (2.1) and the vector field $\phi(y)$ is smooth with $\phi(0) = 0$.

Throughout this paper, we assume that the nonlinear system (2.5) is *non-minimum phase* with respect to the measured output $y = C_c x = x_1$, but there exists a virtual output of the form (2.4) so that the relative degree of the nonlinear system (2.5) with respect to the virtual output (2.4) is r. Under this hypothesis, one can employ the linear change of coordinates (2.3), as shown in Lemma 2.1, to transform the nonlinear system (2.5) into

$$\begin{cases} \dot{z}_i = z_{i+1} + \psi_i(y), & i = 1, 2, \dots, r-1, \\ \dot{z}_r = \sum_{i=1}^{n} a_i z_i + \psi_r(y) + \bar{b}_r u & \\ \dot{z}_b = Fz_b + gz_1 + \psi_b(y) & \\ y = C_c x = x_1, & \end{cases} \tag{2.6}$$

where $\psi_b(y) = [\psi_{r+1}(y), \dots, \psi_n(y)]^T$, and $\psi_i(y) = t_i \phi(y)$ for $i = 1, 2, \dots, n$.

The following condition is assumed in this note.

Assumption 2.2. The relative degree r of the nonlinear system (2.5) with respect to the virtual output (2.4) (not measured output $y = x_1$) satisfies

$$1 \le r \le \rho.$$

Notably, Assumption 2.2 reduces to the one introduced by Marino and Tomei (2005) when the relative degree $r = \rho$. On the other hand, Assumption 2.2 suggests that r can also be less than ρ — the relative degree of the nonlinear system (2.5) with respect to the measured output $y = C_c x = x_1$.

Since $1 \le r \le \rho$, the control input u would not appear when calculating the 1st, 2nd, …, $(r-1)$th derivatives of $y(t)$. This implies that

$$y = x_1 = \beta_1 z_1 + \alpha_1^T z_b \tag{2.7}$$

where $\alpha_1 = [\alpha_{11}, \alpha_{12}, \dots, \alpha_{1(n-r)}]^T$ and

$$\beta_1 = \begin{cases} \text{nonzero} & \text{when} \quad r = \rho \\ 0 & \text{when} \ 1 \le r < \rho \end{cases} \tag{2.8}$$

Similarly, it can be shown that for $k = 2, 3, \dots, r$,

$$x_k = (\beta_{k1} z_1 + \dots + \beta_{kk} z_k) + \alpha_k^T z_b, \tag{2.9}$$

where $\alpha_k = [\alpha_{k1}, \alpha_{k2}, \ldots, \alpha_{k(n-r)}]^T$, in other words, x_k is independent of $z_{k+1}, \ldots, z_r$ for $k = 1, 2, \ldots, r-1$.

From (2.7), it is clear that the zero-dynamics of system (2.6), viewing $z_1 = t_1 x$ as its output, are given by

$$\dot{z}_b = F z_b + \psi_b(\alpha_1^T z_b). \qquad (2.10)$$

For the zero-dynamics (2.10), we assume the same condition as the one introduced by Marino-Tomei Marino and Tomei (2005).

Assumption 2.3. There exists a linear virtual output (2.4),such that the zero-dynamics of the nonlinear system (2.6) with respect to the virtual output z_1, i.e. the nonlinear system (2.10), is globally exponentially stable at the equilibrium $z_b = 0$. That is, there exists a Lyapunov function $U(z_b)$ satisfying

$$\begin{aligned} h_1 \|z_b\|^2 \le U(z_b) &\le h_2 \|z_b\|^2, \\ \|\frac{\partial U}{\partial z_b}\| &\le h_4 \|z_b\|, \\ \frac{\partial U}{\partial z_b}(F z_b + \psi_b(\alpha_1^T z_b)) &\le -h_3 \|z_b\|^2, \end{aligned}$$

$\forall z_b \in \mathbb{R}^{n-r}$, where $h_i > 0, i = 1, 2, 3, 4$ are real numbers.

Under Assumptions 2.2 – 2.3, the following result can be proved.

Theorem 2.4. Under Assumptions 2.2 – 2.3, the non-minimum phase nonlinear system (2.5) is globally asymptotically stabilizable by a dynamic output feedback controller of the form

$$\begin{aligned} \dot{\omega} &= \sigma(\omega, y), \quad \omega \in \mathbb{R}^n, \\ u &= u(\omega, y), \end{aligned} \qquad (2.11)$$

where $\sigma : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ are smooth mappings, with $\sigma(0, 0) = 0$ and $u(0, 0) = 0$.

Notably, Theorem 2.4 has refined Marino and Tomei (2005)'s theorem in two aspects: i) it proves the existence of a globally stabilizing dynamic output feedback controller of order n, rather than $n+2(\rho-1)$, for the non-minimum phase nonlinear system (2.5); ii) the requirement that the relative degree r of system (2.5) with respect to the virtual linear output (2.4) needs to be equal to ρ can be relaxed and replaced by Assumption 2.2. While the former makes the design of output feedback controllers simpler, the latter allows certain non-minimum phase nonlinear systems that may not be handled by Marino and Tomei (2005) to be deal with by the proposed output feedback design method, as illustrated by Example 2.

3. A LUENBERGER-LIKE NONLINEAR OBSERVER

To design an output feedback compensator for the nonlinear system (2.5) in output feedback form, it is natural to consider the following Luenberger-like observer

$$\dot{\hat{x}} = A_c \hat{x} + bu + \phi(y) - k_0(y - C_c \hat{x}), \qquad (3.1)$$

where $k_0 \in \mathbb{R}^n$ is an observer gain, such that the matrix $(A_c + k_0 C_c)$ is Hurwitz.

Let $e = x - \hat{x}$ be the state estimate error. Then, it it straightforward to show that the error dynamical equation is given by

$$\dot{e} = (A_c + k_0 C_c) e \qquad (3.2)$$

Since $(A_c + k_0 C_c)$ is a Hurwitz matrix, there exists a $P = P^T > 0$ such that

$$(A_c + k_0 C_c)^T P + P(A_c + k_0 C_c) = -I.$$

In other words, the error dynamics (3.2) is globally exponentially stable because

$$\dot{W}_0 = -\|e\|^2, \qquad (3.3)$$

where $W_0(e) = e^T P e$ is a positive definite and proper Lyapunov function.

Observe that the nonlinear systems (2.5) is globally diffeomorphic to (2.6). As a result, the estimation of the state z can be obtained using the nonsingular transformation (2.3), i.e., $\hat{z} = T\hat{x}$. From (3.1) and (2.3), it is easy to see that for system (2.6), its observer is

$$\begin{cases} \dot{\hat{z}}_1 = \hat{z}_2 + \psi_1(y) - L_1(y - \hat{x}_1) \\ \vdots \\ \dot{\hat{z}}_{r-1} = \hat{z}_r + \psi_{r-1}(y) - L_{r-1}(y - \hat{x}_1) \\ \dot{\hat{z}}_r = \sum_{i=1}^{n} a_i \hat{z}_i + \psi_r(y) + \bar{b}_r u - L_r(y - \hat{x}_1) \\ \dot{\hat{z}}_b = F\hat{z}_b + g\hat{z}_1 + \psi_b(y) - L_b(y - \hat{x}_1) \end{cases} \qquad (3.4)$$

where $L_i = t_i k_0, i = 1, 2, \ldots, n$, and $L_b = [L_{r+1}, \ldots, L_n]^T$.

4. DESIGN OF AN N-DIMENSIONAL DYNAMIC OUTPUT COMPENSATOR

In this section, we construct a nonlinear controller based on the system (3.4), using a step-by-step recursive design procedure. Because the estimate state $\hat{z} = T\hat{x}$ and the measured output $y = x_1$ of the system (2.5) are available for feedback design, the resulting controller is implementable. As we shall see in a moment, our design involves no filter transformation and thus yielding an n-dimensional, rather than $n + 2(\rho - 1)$, dynamic output feedback controller for the non-minimum phase nonlinear system (2.5).

Step 1: Consider the $\hat{z}_1$-dynamics in (3.4). Regard $\hat{z}_2$ as a virtual control and choose

$$\begin{aligned} \hat{z}_2^* &= -\hat{z}_1 - \psi_1(y) + L_1(y - \hat{x}_1) \\ &= -\hat{z}_1 - \psi_1(y) + L_1(y - \beta_1 \hat{z}_1 - \alpha_1^T \hat{z}_b) := \gamma_1(y, \hat{z}_1, \hat{z}_b). \end{aligned}$$

with $\gamma_1(\cdot)$ being smooth and $\gamma_1(0, 0, 0) = 0$. The last relationship is a consequence of (2.7). As a such,

$$\dot{\hat{z}}_1 = -\hat{z}_1 + \hat{z}_2 - \hat{z}_2^*.$$

Define $\xi_1 = \hat{z}_1$ and $\xi_2 = \hat{z}_2 - \hat{z}_2^*$. Then,

$$\dot{\xi}_1 = -\xi_1 + \xi_2$$

Consider now the Lyapunov function $V_1(\xi_1) = \frac{1}{2}\xi_1^2$. Clearly,

$$\dot{V}_1 = -\xi_1^2 + \xi_1\xi_2 \tag{4.1}$$

Step 2: For the z_2-dynamics, note that

$$\begin{aligned}\dot{\xi}_2 &= \dot{\hat{z}}_2 - \dot{\hat{z}}_2^* \\ &= \hat{z}_3 + \psi_2(y) - L_2(y - \hat{x}_1) \\ &\quad -\frac{\partial\gamma_1}{\partial\hat{z}_1}\dot{\hat{z}}_1 - \frac{\partial\gamma_1}{\partial\hat{z}_b}\dot{\hat{z}}_b - \frac{\partial\gamma_1}{\partial y}(x_2 + \phi_1(y))\end{aligned}$$

Design the virtual controller

$$\begin{aligned}\hat{z}_3^* &= -\psi_2(y) + L_2(y - \hat{x}_1) + \frac{\partial\gamma_1}{\partial\hat{z}_1}\dot{\hat{z}}_1 + \frac{\partial\gamma_1}{\partial\hat{z}_b}\dot{\hat{z}}_b \\ &\quad + \frac{\partial\gamma_1}{\partial y}[\hat{x}_2 + \phi_1(y)] - \xi_1 - \xi_2 - \xi_2(\frac{\partial\gamma_1}{\partial y})^2 \\ &:= \gamma_2(y, \hat{z}_1, \hat{z}_2, \hat{z}_b),\end{aligned}$$

with $\gamma_2(\cdot)$ being smooth and $\gamma_2(0,0,0,0) = 0$, and denote $\xi_3 = \hat{z}_3 - \hat{z}_3^*$. Thus,

$$\dot{\xi}_2 = -\xi_1 - \xi_2 + \xi_3 - \xi_2(\frac{\partial\gamma_1}{\partial y})^2 - \frac{\partial\gamma_1}{\partial y}e_2$$

For the $(\hat{z}_1, \hat{z}_2)$-dynamics, using the Lyapunov function $V_2(\xi_1, \xi_2) = V_1(\xi_1) + \frac{1}{2}\xi_2^2$ leads to (in view of (4.1))

$$\begin{aligned}\dot{V}_2 &= -\xi_1^2 + \xi_1\xi_2 \\ &\quad -\xi_1\xi_2 - \xi_2^2 + \xi_2\xi_3 - \xi_2^2(\frac{\partial\gamma_1}{\partial y})^2 - \frac{\partial\gamma_1}{\partial y}\xi_2 e_2 \\ &\le -\xi_1^2 - \xi_2^2 + \xi_2\xi_3 + \frac{1}{4}e_2^2.\end{aligned} \tag{4.2}$$

Step m $(3 \le m \le r-1)$: Define

$$\xi_m = \hat{z}_m - \hat{z}_m^*, \quad \hat{z}_m^* := \gamma_{m-1}(y, \hat{z}_1, \ldots, \hat{z}_{m-1}, \hat{z}_b) \tag{4.3}$$

where the virtual control $\gamma_{m-1}(\cdot)$ is a smooth function with $\gamma_{m-1}(0) = 0$. Clearly,

$$\begin{aligned}\dot{\xi}_m &= \dot{\hat{z}}_m - \dot{\hat{z}}_m^* \\ &= \hat{z}_{m+1} + \psi_m(y) - L_m(y - \hat{x}_1) \\ &\quad -\sum_{i=1}^{m-1}\frac{\partial\gamma_{m-1}}{\partial\hat{z}_i}\dot{\hat{z}}_i - \frac{\partial\gamma_{m-1}}{\partial\hat{z}_b}\dot{\hat{z}}_b - \frac{\partial\gamma_{m-1}}{\partial y}(x_2 + \phi_1(y))\end{aligned}$$

Similar to the previous steps, one can design the virtual controller

$$\begin{aligned}\hat{z}_{m+1}^* &= -\xi_{m-1} - \xi_m - \psi_m(y) + L_m(y - \hat{x}_1) + \sum_{i=1}^{m-1}\frac{\partial\gamma_{m-1}}{\partial\hat{z}_i}\dot{\hat{z}}_i \\ &\quad + \frac{\partial\gamma_{m-1}}{\partial\hat{z}_b}\dot{\hat{z}}_b + \frac{\partial\gamma_{m-1}}{\partial y}(\hat{x}_2 + \phi_1(y)) - \xi_m(\frac{\partial\gamma_{m-1}}{\partial y})^2 \\ &:= \gamma_m(y, \hat{z}_1, \ldots, \hat{z}_m, \hat{z}_b).\end{aligned}$$

Let $\xi_{m+1} = \hat{z}_{m+1} - \hat{z}_{m+1}^*$. Then, we have

$$\begin{aligned}\dot{\xi}_m &= -\xi_{m-1} - \xi_m + \hat{z}_{m+1} - \hat{z}_{m+1}^* \\ &\quad -\xi_m(\frac{\partial\gamma_{m-1}}{\partial y})^2 - \frac{\partial\gamma_{m-1}}{\partial y}e_2 \\ &= -\xi_{m-1} - \xi_m + \xi_{m+1} - \xi_m(\frac{\partial\gamma_{m-1}}{\partial y})^2 - \frac{\partial\gamma_{m-1}}{\partial y}e_2\end{aligned}$$

With this in mind, consider the Lyapunov function

$$V_m(\xi_1, \ldots, \xi_m) = V_{m-1}(\xi_1, \ldots, \xi_{m-1}) + \frac{1}{2}\xi_m^2 = \frac{1}{2}\sum_{i=1}^{m}\xi_i^2$$

for the $(\hat{z}_1, \cdots, \hat{z}_m)$-subsystem. A direct calculation gives

$$\begin{aligned}\dot{V}_m &\le -\sum_{i=1}^{m-1}\xi_i^2 + \xi_{m-1}\xi_m + \frac{m-2}{4}e_2^2 \\ &\quad -\xi_{m-1}\xi_m - \xi_m^2 + \xi_m\xi_{m+1} \\ &\quad -\xi_m^2(\frac{\partial\gamma_{m-1}}{\partial y})^2 - \frac{\partial\gamma_{m-1}}{\partial y}\xi_m e_2 \\ &\le -\sum_{i=1}^{m}\xi_i^2 + \xi_m\xi_{m+1} + \frac{m-1}{4}e_2^2.\end{aligned} \tag{4.4}$$

Step r: At this step, from $\xi_r = \hat{z}_r - \hat{z}_r^*$ it follows that

$$\begin{aligned}\dot{\xi}_r &= \dot{\hat{z}}_r - \dot{\hat{z}}_r^* \\ &= \sum_{i=1}^{n}a_i\hat{z}_i + \psi_r(y) + \bar{b}_r u - L_r(y - \hat{x}_1) \\ &\quad -\sum_{i=1}^{r-1}\frac{\partial\gamma_{r-1}}{\partial\hat{z}_i}\dot{\hat{z}}_i - \frac{\partial\gamma_{r-1}}{\partial\hat{z}_b}\dot{\hat{z}}_b - \frac{\partial\gamma_{r-1}}{\partial y}(x_2 + \phi_1(y)).\end{aligned}$$

Clearly, a smooth feedback control law of the form

$$\begin{aligned}u &= \frac{1}{\bar{b}_r}\Big[-\xi_{r-1} - \xi_r - \sum_{i=1}^{n}a_i\hat{z}_i - \psi_r(y) + L_r(y - \hat{x}_1) \\ &\quad + \sum_{i=1}^{r-1}\frac{\partial\gamma_{r-1}}{\partial\hat{z}_i}\dot{\hat{z}}_i + \frac{\partial\gamma_{r-1}}{\partial\hat{z}_b}\dot{\hat{z}}_b + \frac{\partial\gamma_{r-1}}{\partial y}(\hat{x}_2 + \phi_1(y)) \\ &\quad -\xi_r(\frac{\partial\gamma_{r-1}}{\partial y})^2\Big] \\ &:= u(\hat{z}_1, \ldots, \hat{z}_r, \hat{z}_b, y) = u(\hat{z}, y), \quad u(0,0) = 0,\end{aligned} \tag{4.5}$$

is such that

$$\dot{\xi}_r = -\xi_{r-1} - \xi_r - \xi_r(\frac{\partial\gamma_{r-1}}{\partial y})^2 - \frac{\partial\gamma_{r-1}}{\partial y}e_2.$$

Now, Choose the Lyapunov function

$$V_r(\xi) = V_{r-1}(\xi_1, \cdots, \xi_{r_1}) + \frac{1}{2}\xi_r^2 = \frac{1}{2}\sum_{i=1}^{r}\xi_i^2$$

for the $(\hat{z}_1, \cdots, \hat{z}_r)$-subsystem. It is easy to show that

$$\begin{aligned}\dot{V}_r &\le -\sum_{i=1}^{r-1}\xi_i^2 + \xi_{r-1}\xi_r + \frac{r-2}{4}e_2^2 \\ &\quad -\xi_{r-1}\xi_r - \xi_r^2 - \xi_r^2(\frac{\partial\gamma_{r-1}}{\partial y})^2 - \frac{\partial\gamma_{r-1}}{\partial y}\xi_r e_2 \\ &\le -\sum_{i=1}^{r}\xi_i^2 + \frac{r-1}{4}e_2^2.\end{aligned} \tag{4.6}$$

In the next section, the Lyapunov inequality (4.6) will be used to prove the n-th order dynamic output feedback controller (3.4)-(4.5) globally asymptotically stabilizes the non-minimum phase system (2.5) or, equivalently, the system (2.6).

5. PROOF OF THEOREM 2.4

For the convenience of proof of Theorem 2.4, we observe that the closed-loop system formed by (2.6) and the dynamic output compensator (3.4)-(4.5) can be equivalently expressed as

$$\begin{cases} \dot{e} = (A_c + k_0 C_c)e, \qquad e = T^{-1}(z - \hat{z}), \\ \dot{\hat{z}}_1 = \hat{z}_2 + \psi_1(y) - L_1(y - \hat{x_1}) \\ \vdots \\ \dot{\hat{z}}_{r-1} = \hat{z}_r + \psi_{r-1}(y) - L_{r-1}(y - \hat{x_1}) \\ \dot{\hat{z}}_r = \sum_{i=1}^{n} a_i \hat{z}_i + \psi_r(y) + \bar{b}_r u(\hat{z}, y) - L_r(y - \hat{x_1}) \\ \dot{\hat{z}}_b = F\hat{z}_b + g\hat{z}_1 + \psi_b(y) - L_b(y - \hat{x}_1) \end{cases} \tag{5.1}$$

In view of (3.3) and (4.6), it is natural to choose the composite Lyapunov function

$$V(e, \xi) = (\frac{r-1}{4} + 1)W_0(e) + V_r(\xi)$$

for the $(e, \hat{z}_1, \cdots, \hat{z}_r)$-dynamics of the system (5.1). Clearly,

$$\begin{aligned} \dot{V}(e, \xi) &= (\frac{r-1}{4} + 1)\dot{W}_0(e) + \dot{V}_r(\xi) \\ &\le -(\frac{r-1}{4} + 1)||e||^2 - \sum_{i=1}^{r} \xi_i^2 + \frac{r-1}{4} e_2^2 \\ &\le -||e||^2 - \sum_{i=1}^{r} \xi_i^2. \end{aligned} \tag{5.2}$$

Keeping this in mind, we now show that the closed-loop system (5.1) is globally asymptotically stable.

To this end, recall the relation (2.7) and introduce the change of coordinate

$$\eta_b := z_b + \frac{\beta_1}{||\alpha_1||^2} \alpha_1 z_1, \tag{5.3}$$

which can be rewritten as

$$\eta_b = \hat{z}_b + t_b e + \frac{\beta_1}{||\alpha_1||^2} \alpha_1 (\hat{z}_1 + t_1 e) \tag{5.4}$$

with $t_b := [t_{r+1}, \ldots, t_n]^T$.

From (5.3) and (2.7), it is easy to verify that

$$y = x_1 = \alpha_1^T \eta_b.$$

Using this fact, together with (5.3) and z_b-dynamics of (2.6), we obtain

$$\begin{aligned} \dot{\eta}_b &= F z_b + g z_1 + \psi_b(y) + \frac{\beta_1}{||\alpha_1||^2} \alpha_1 (z_2 + \psi_1(y)) \\ &= F(\eta_b - \frac{\beta_1}{||\alpha_1||^2} \alpha_1 z_1) + g z_1 + \psi_b(\alpha_1^T \eta_b) \\ &\quad + \frac{\beta_1}{||\alpha_1||^2} \alpha_1 (-z_1 + \xi_2 + L_1 e_1 + t_1 e + t_2 e) \\ &= F\eta_b + \psi_b(\alpha_1^T \eta_b) + \bar{g}(\hat{z}_1 + t_1 e) \\ &\quad + \frac{\beta_1}{||\alpha||^2} \alpha_1 \big(\xi_2 + L_1 e_1 + (t_1 + t_2) e\big) \end{aligned} \tag{5.5}$$

where $\bar{g} = g - F \frac{\beta_1}{||\alpha||^2} \alpha_1 - \frac{\beta_1}{||\alpha||^2} \alpha_1$ is a vector in R^{n-r}.

By Assumption 2.3 and the fact that $\eta_b = z_b$ when $z_1 = 0$, it is concluded that

$$\dot{\eta}_b = F\eta_b + \psi_b(\alpha_1^T \eta_b) \tag{5.6}$$

is globally exponentially stable $\forall \eta_b \in \mathbb{R}^{n-r}$.

Note that the dynamical system (5.5) is composed of a globally exponentially stable system (5.6), driven by a linear input signal $(\hat{z}_1, \xi_2, e)$, and hence it is input to state stable (ISS). Moreover, by (5.2) the $(e, \xi_1, \cdots, \xi_r)$-dynamics of (5.1) is globally asymptotically stable at the equilibrium $(e, \xi_1, \cdots, \xi_r) = (0, 0, \ldots, 0)$. Consequently, the states $(\hat{z}_1(t), \xi_2(t), e(t)) = (\xi_1(t), \xi_2(t), e(t))$ are bounded $\forall t \ge 0$, and tend to zero as t goes to infinity. By the ISS property of the system (5.5), η_b is globally bounded and

$$\lim_{t \to \infty} \eta_b(t) = 0.$$

This, in view of (5.4), implies that state $\hat{z}_b$ is globally bounded and

$$\lim_{t \to \infty} \hat{z}_b(t, e_0, \hat{z}_0) = 0$$

for any initial condition $(e_0, \hat{z}_0) \in R^n \times R^n$. Using this fact, together with the construction of the virtual controllers (4.3), one can deduce that $\lim_{t\to\infty} \hat{z}_2^*(t) = \lim_{t\to\infty} \gamma_1(y, \hat{z}_1, \hat{z}_b) = 0$. Recursively, it can be concluded from (4.3) that $\lim_{t\to\infty} \hat{z}_m^*(t) = 0$, for $m = 1, 2, \cdots, r$. This, in turn, yields $\lim_{t\to\infty} \hat{z}_m(t) = 0$, $m = 1, 2, \cdots, r$. In conclusion, the closed-loop system (5.1) is globally asymptotically stable at the equilibrium $(e, \hat{z}) = (0, 0)$.

6. ILLUSTRATIVE EXAMPLES

In this section, we present two examples to demonstrate some interesting features of the output feedback control strategy proposed in the previous section. The first example is adopted from Marino and Tomei (2005), which illustrates how a dynamic output feedback controller of order 3, instead of 5, can be designed.

Example 1: Consider the nonlinear system of the form (2.5)

$$\begin{cases} \dot{x}_1 = x_2 - y^2 \\ \dot{x}_2 = x_3 - u + y^2 \\ \dot{x}_3 = u \\ y = x_1 \end{cases} \tag{6.1}$$

The zero-dynamics of (6.1) is obtained by setting $y = 0$. A simple calculation shows that $\dot{\eta} = \eta$ which is unstable. That is, the nonlinear system (6.1) is non-minimum phase. The system has a relative degree $\rho = 2$ with the measured output $y = x_1$.

Now, consider the virtual output

$$z_1 = x_1 + 2(x_2 + x_3).$$

The system (6.1) with respect to z_1 has a relative degree $r = 2$. The corresponding zero dynamics is $\dot{\eta} = -\eta$, and thus minimum phase. By Theorem 2.4, there is a dynamic output feedback controller of order 3, globally asymptotically stabilizing the non-minimum phase system (6.1). The output feedback controller can be designed as follows. Introduce the nonsingular transformation (see (2.3) with $r = 2$)

$$\begin{cases} z_1 = x_1 + 2(x_2 + x_3) \\ z_2 = x_2 + 2x_3 \\ z_3 = x_1 + x_2 + x_3, \end{cases} \quad (6.2)$$

which transforms system (6.1) into (see (2.6))

$$\begin{cases} \dot{z}_1 = z_2 + y^2 \\ \dot{z}_2 = -z_1 + z_2 + z_3 + y^2 + u \\ \dot{z}_3 = z_1 - z_3. \end{cases} \quad (6.3)$$

The 3rd-order dynamic output feedback controller, according to (3.1), (3.4) and (4.5), is given by

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 - y^2 + 3(y - \hat{x}_1) \\ \dot{\hat{x}}_2 &= \hat{x}_3 - u + y^2 + 3(y - \hat{x}_1) \\ \dot{\hat{x}}_3 &= u + (y - \hat{x}_1) \\ u &= 21\hat{z}_1 - 14\hat{z}_2 - 23\hat{z}_3 - 14y^2 + 6(y - \hat{x}_1) \\ &\quad -(11 + 2y)(\hat{x}_2 - y^2) \\ &\quad -(\hat{z}_2 + \hat{z}_1 + y^2 + 11(y - \hat{x}_1))(11 + 2y)^2 \end{aligned} \quad (6.4)$$

where $\hat{z}_1 = \hat{x}_1 + 2(\hat{x}_2 + \hat{x}_3)$, $\hat{z}_2 = \hat{x}_2 + 2\hat{x}_3$ and $\hat{z}_3 = \hat{x}_1 + \hat{x}_2 + \hat{x}_3$.

Example 2: Consider a 3-dimensional non-minimum phase system in the form (2.5), i.e.,

$$\begin{cases} \dot{x}_1 = x_2 - y - \frac{5}{2}y^3 \\ \dot{x}_2 = x_3 - u + y^3 \\ \dot{x}_3 = u \\ y = x_1. \end{cases} \quad (6.5)$$

It can be verified that the system above has a relative degree $\rho = 2$ and its zero-dynamics is $\dot{\eta}_1 = \eta_1$ with respect to the output $y = x_1$. Thus, the nonlinear system (6.5) is non-minimum phase.

On the other hand, if one picks the virtual output

$$z_1 = x_1 + x_2 + 2x_3,$$

the resulted zero-dynamics is

$$\begin{cases} \dot{\eta}_2 = 3\eta_2 - 2\eta_3 - \frac{3}{2}(\eta_3 - \eta_2)^3 \\ \dot{\eta}_3 = 7\eta_2 - 4\eta_3 - 4(\eta_3 - \eta_2)^3, \end{cases} \quad (6.6)$$

which is globally asymptotically stable at the equilibrium $(\eta_2, \eta_3) = (0, 0)$. Moreover, the system has a relative degree $r = 1 < \rho = 2$, with respect to z_1.

According to Theorem 2.4, there exists a dynamic output feedback controller of order 3, globally asymptotically stabilizing the non-minimum phase system (6.5). Using the following change of coordinates

$$\begin{cases} z_1 = x_1 + x_2 + 2x_3 \\ z_2 = x_1 + x_2 + x_3 \\ z_3 = 2x_1 + x_2 + x_3, \end{cases} \quad (6.7)$$

system (6.5) can be transformed into

$$\begin{cases} \dot{z}_1 = 3z_2 - 2z_3 + u - \frac{3}{2}y^3 \\ \dot{z}_2 = 3z_2 - 2z_3 - \frac{3}{2}y^3 \\ \dot{z}_3 = -z_1 + 7z_2 - 4z_3 - 4y^3. \end{cases} \quad (6.8)$$

For the nonlinear system (6.8), one can design the following dynamic output feedback controller based on (3.1), (3.4) and (4.5):

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 - y - \frac{5}{2}y^3 + 3(y - \hat{x}_1) \\ \dot{\hat{x}}_2 &= \hat{x}_3 - u + y^3 + 3(y - \hat{x}_1) \\ \dot{\hat{x}}_3 &= u + (y - \hat{x}_1) \\ u &= -\hat{z}_1 - 3\hat{z}_2 + 2\hat{z}_3 + \frac{3}{2}y^3 - 8(y - \hat{x}_1) \end{aligned} \quad (6.9)$$

where $\hat{z}_1 = \hat{x}_1 + \hat{x}_2 + 2\hat{x}_3$, $\hat{z}_2 = \hat{x}_1 + \hat{x}_2 + \hat{x}_3$ and $\hat{z}_3 = 2\hat{x}_1 + \hat{x}_2 + \hat{x}_3$.

7. CONCLUSION

In this note, we have shown the existence of an n-dimensional, rather than $n+2(\rho-1)$-dimensional, dynamic output feedback controller for a class of non-minimum-phase nonlinear systems in output feedback form (2.5) previously studied by Marino and Tomei (2005). The proof is constructive and involves no filter transformation as done in Marino and Tomei (2005), and thus leading to a simpler n-th order dynamic compensator. In addition, the proposed output feedback design method allows the relative degree r of the system with respect to the virtual output be less than or equal to the original system relative degree ρ. The simplicity of the proposed output feedback control method have been illustrated by examples.

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