

Control of Uncertain Nonlinear Systems using Ellipsoidal Reachability Calculus

L. Asselborn* D. Groß* O. Stursberg*

* *Institute of Control and System Theory, Dept. of Electrical Eng. and
Computer Science, University of Kassel (Germany).*
Email: {l.asselborn, dgross, stursberg}@uni-kassel.de

Abstract: This paper proposes an approach to algorithmically synthesize control strategies for discrete-time nonlinear uncertain systems based on reachable set computations using the ellipsoidal calculus. For given ellipsoidal initial sets and bounded ellipsoidal disturbances, the proposed algorithm iterates over conservatively approximating and LMI-constrained optimization problems to compute stabilizing controllers. The method uses first-order Taylor approximation of the nonlinear dynamics and a conservative approximation of the Lagrange remainder. An example for illustration is included.

Keywords: Ellipsoidal calculus, nonlinear uncertain dynamics, conservative linearization, reach set approximation.

1. INTRODUCTION

Since controller synthesis for nonlinear systems with uncertainties by use of analytical techniques is in most cases restricted to specific system structures, synthesis based on algorithmic computation of reachable sets may appear as a suitable alternative. In particular, algorithmic reachability analysis can explicitly account for the effects of bounded disturbances or parametric uncertainties. Within the context of algorithmic verification of formal properties like safety, considerable effort has been spent in recent years to compute or (more often) conservatively approximate reachable sets for different types of systems.

Over-approximating reachable sets for linear systems by the use of zonotopes was studied e.g. in [12], based on support functions in [13], and by the use of ellipsoids in [15]. The reachability problem of nonlinear systems was addressed in [16], [19], [20], among others, with different techniques to propagate reachable sets, represented mainly by polyhedra, forward in time. System uncertainties were considered for linear dynamics in [6] and [9] using reach set representations by ellipsoids and polytopes, and for nonlinear dynamics in [7] with reachable sets specified by zonotopes, respectively. The methods in [17] and [18] have in common that the reach set over-approximation is based on linearizations of the nonlinear dynamics around a current estimate of the state combined with conservative approximations of the linearization error using interval arithmetics.

In this paper, we use a similar linearization method for nonlinear systems with disturbances bounded to ellipsoidal sets. In contrast to [17] and [18], we use an conservative ellipsoidal over-approximation of the linearization error. The substitute system dynamics with ellipsoidal reachable state sets allows us to apply the well-known ellipsoidal calculus [2]. These techniques are used to solve the control problem of stabilizing the uncertain nonlinear system into

a given target set. The idea is to locally and conservatively linearize the system, and to specify an algorithm which solves an LMI-constrained optimization problem in any iteration to obtain a stabilizing controller for all disturbances. In order to show the convergence towards the terminal set, the principle of flexible Lyapunov functions, see e.g. [1], is used.

The paper is organized as follows: Preliminaries on set representation and calculation are contained in Sec. 2, and the considered uncertain nonlinear system together with the linearization procedure are introduced in Sec. 3. Section 4 states the control problem formally, and the original problem is recast into an optimization-based solution procedure in Sec. 5. The results are illustrated by an example in Sec. 6, and Sec. 7 draws conclusions.

2. SET REPRESENTATION AND CALCULATION

The notation used later in context of the ellipsoidal calculus is introduced first.

Definition 1. An *ellipsoid* $\varepsilon(q, Q)$ is parametrized by its center point q and its shape matrix Q , and is defined as:

$$\varepsilon(q, Q) = \{x \in \mathbb{R}^n \mid (x - q)^T Q^{-1} (x - q) \leq 1\} \quad (1)$$

Definition 2. A *convex polytope* \mathcal{P} is the intersection of n_p halfspaces, such that $\mathcal{P} = \{x \in \mathbb{R}^n \mid Kx \leq b, K \in \mathbb{R}^{n_p \times n}, b \in \mathbb{R}^{n_p}\}$. The geometrical center of a bounded polytope is determined by the following function:

$$\eta := \text{centroid}(\mathcal{P}) \quad (2)$$

The control procedure to be proposed uses at some point an enclosing hyperbox of a set, determined as follows:

Definition 3. A function *intval* takes an arbitrary bounded and connected set $\mathcal{W} \subset \mathbb{R}^n$ as its argument and returns intervals in each dimension, denoted by $[x]$, which can be interpreted as the smallest hyperbox, in which \mathcal{W} is contained:

$$[x] := \text{intval}(\mathcal{W}) = \begin{bmatrix} [\min_{x \in \mathcal{W}} x_1, \max_{x \in \mathcal{W}} x_1] \\ \vdots \\ [\min_{x \in \mathcal{W}} x_n, \max_{x \in \mathcal{W}} x_n] \end{bmatrix} \supseteq \mathcal{W} \quad (3)$$

An *affine transformation* of an ellipsoid $\varepsilon(q, Q)$ by a matrix $A \in \mathbb{R}^{n \times n}$ and a vector $b \in \mathbb{R}^{n \times 1}$ leads again to an ellipsoid:

$$A \cdot \varepsilon(q, Q) + b = \varepsilon(A \cdot q + b, AQA^T) \quad (4)$$

The *Minkowski sum* $\mathcal{W} \oplus \mathcal{M}$ of two arbitrary but bounded sets $\mathcal{W} \subset \mathbb{R}^n$, $\mathcal{M} \subset \mathbb{R}^n$ is given by:

$$\mathcal{W} \oplus \mathcal{M} := \{w + m \mid w \in \mathcal{W}, m \in \mathcal{M}\} \quad (5)$$

The Minkowski sum of two ellipsoids $\varepsilon(q_1, Q_1)$, $\varepsilon(q_2, Q_2)$ is, in general, not an ellipsoid, but it can be outer approximated by an ellipsoid $\varepsilon(q_1 + q_2, \bar{Q})$. The shape matrix \bar{Q} is a function of the generalized eigenvalues of Q_1 and Q_2 . The Minkowski sum of an ellipsoid $\varepsilon(q_1, Q_1)$ and a polytope \mathcal{P} is obviously not an ellipsoid, but tight outer ellipsoidal approximations can be computed [2].

3. SYSTEM DEFINITION AND TRANSFORMATION

We consider the following time-invariant discrete-time nonlinear dynamic system with additive uncertainty and time-invariant input and disturbance constraints:

$$\begin{aligned} x_{k+1} &= f(x_k, u_k) + Gv_k, \\ x_0 \in \mathcal{X}_0 &= \varepsilon(q_0, Q_0) \subset \mathbb{R}^n, \\ u_k \in \mathcal{U} &= \{u \in \mathbb{R}^m \mid Ru_k \leq b\}, \\ v_k \in \mathcal{V} &= \varepsilon(0, \Sigma) \subseteq \mathbb{R}^n \end{aligned} \quad (6)$$

with state $x_k \in \mathbb{R}^n$, control input $u_k \in \mathbb{R}^m$, and disturbance input $v_k \in \mathbb{R}^n$. The ellipsoidal initial set of states is \mathcal{X}_0 , and \mathcal{U} denotes the time-invariant polytope of admissible control inputs with $R \in \mathbb{R}^{n_c \times m}$, $b \in \mathbb{R}^{n_c}$, and the number n_c of faces of the polytope. The disturbance input v_k is bounded to the ellipsoidal set \mathcal{V} , with $\Sigma = \Sigma^T > 0$, and its effect on the dynamics is parametrized by $G \in \mathbb{R}^{n \times n}$. The state transfer function $f(x_k, u_k)$ is assumed to be twice continuously differentiable in its arguments.

Furthermore, we assume that the undisturbed part of system (6) has an equilibrium point in the origin $0 = f(0, 0)$, and let \mathcal{D} denote its stability domain (not to be computed explicitly).

The one-step reachable set of (6) at time $k+1$ given a set $\mathcal{X}_k \subset \mathbb{R}^n$ is:

$$\mathcal{X}_{k+1} = \{x \mid x_k \in \mathcal{X}_k, u_k \in \mathcal{U}, \dots, v_k \in \mathcal{V} : x_{k+1} = f(x_k, u_k) + Gv_k\}, \quad (7)$$

i.e. it contains all states reachable in one step from \mathcal{X}_k by a control input in \mathcal{U} and a disturbance $v_k \in \mathcal{V}$. The set-valued operation of mapping \mathcal{X}_k into the set of successor states is briefly denoted by:

$$\mathcal{X}_{k+1} = F(\mathcal{X}_k, \mathcal{U}) \oplus G\mathcal{V} \quad (8)$$

where:

$$F(\mathcal{X}_k, \mathcal{U}) := \{x \mid x_k \in \mathcal{X}_k, u_k \in \mathcal{U} : x_{k+1} = f(x_k, u_k)\}. \quad (9)$$

The considered problem in this contribution is formulated as follows:

Problem 1. Determine a set-valued control law $\tilde{\mathcal{U}}_{k+1} = \kappa(\mathcal{X}_k, k)$, for which it holds that:

$$\tilde{\mathcal{U}}_k \subseteq \mathcal{U} \quad \forall k \in \{0, 1, \dots, N-1\}, \quad N \in \mathbb{N}$$

and that it stabilizes the nonlinear system (6) from an initial set $\mathcal{X}_0 = \varepsilon(q_0, Q_0)$ in a finite number N of time steps into an ellipsoidal terminal set $\mathbb{T} = \varepsilon(0, T)$ which is parametrized by $T \in \mathbb{R}^{n \times n}$ and is centered in 0:

$$\begin{aligned} \exists N \geq 0 : \mathcal{X}_{k+1} &= F(\mathcal{X}_k, \tilde{\mathcal{U}}_k) \oplus G\mathcal{V} \\ \mathcal{X}_N &\subseteq \mathbb{T} = \varepsilon(0, T), \quad k \in \{0, 1, \dots, N-1\} \quad \square \end{aligned}$$

The underlying principle here is that stability has to be achieved by steering the center point q_k of \mathcal{X}_k towards zero and to parametrize \mathcal{X}_{k+N} such in size that it is contained in \mathbb{T} in finitely many steps.

Assumption 1. We assume the existence of a stabilizing *terminal* controller which renders the set \mathbb{T} robustly forward invariant for all disturbances $v_k \in \mathcal{V}$ while satisfying the input constraints in (6). ■

The exact computation and representation of the sets \mathcal{X}_k in Problem 1 is impossible for arbitrary nonlinear dynamics. Thus, we resort to approximate computation of \mathcal{X}_k in an conservative manner, i.e. the solution of a reformulated problem to be introduced next solves Problem 1, too.

The function $f(x_k, u_k)$ can be approximated by a first order Taylor series with a Lagrange remainder $\mathcal{L}(\xi_k, z)$. For this purpose, we define a combined vector $\xi_k = [x_k, u_k]^T \in (\mathcal{X}_k \times \mathcal{U})$ and a linearization point $\bar{\xi}_k = [\bar{x}_k, \bar{u}_k]^T$. Given ξ_k in a neighborhood of $\bar{\xi}_k$, a point $z \in \{\alpha\xi_k + (1-\alpha)\bar{\xi}_k \mid \alpha \in [0, 1]\}$ exist according to Taylor's theorem such that:

$$f(\xi_k) = f(\bar{\xi}_k) + \left. \frac{\partial f(\xi_k)}{\partial \xi_k} \right|_{\xi_k = \bar{\xi}_k} (\xi_k - \bar{\xi}_k) + \mathcal{L}(\bar{\xi}_k, z) \quad (10)$$

holds, where $\mathcal{L}(\xi_k, z) \in \mathbb{R}^n$ denotes the Lagrange remainder and its i -th component $\mathcal{L}_i(\xi_k, z)$ is a second-order Taylor polynomial and describes the linearization error:

$$\mathcal{L}_i(\bar{\xi}_k, z_i) = \frac{1}{2} (\xi_k - \bar{\xi}_k)^T \left. \frac{\partial^2 f_i(\xi_k)}{\partial^2 \xi_k} \right|_{\xi_k = z_i} (\xi_k - \bar{\xi}_k). \quad (11)$$

According to the mean-value theorem, the approximation of the original function becomes exact for a unique $z_i \in \{\alpha\xi_k + (1-\alpha)\bar{\xi}_k \mid \alpha \in [0, 1]\}$, for all $i \in \{1, \dots, n\}$. In other words, the Lagrange remainder accounts for all terms of order 2 and higher, see [3]. The system dynamics can then be written as:

$$x_{k+1} = A_k(x_k - \bar{x}_k) + B_k(u_k - \bar{u}_k) + \mathcal{L}(\bar{\xi}_k, z) \dots + f(\bar{x}_k, \bar{u}_k) + Gv_k \quad (12)$$

with matrices A_k , B_k denoting the first-order derivatives of f evaluated at \bar{x}_k and \bar{u}_k . The Lagrange remainder can be over-approximated by means of interval arithmetics [4]. Since ξ_k may take any value in $\mathcal{X}_k \times \mathcal{U}$, the reach set \mathcal{X}_k and the input space \mathcal{U} are over-approximated by two intervals by applying the aforementioned function *intval*:

$$[x_k] = \text{intval}(\mathcal{X}_k), \quad [u] = \text{intval}(\mathcal{U}) \quad (13)$$

$$[\xi_k] = [[x_k]^T, [u]^T]^T$$

With the aim to minimize the linearization error, the center point q_k of the ellipsoid \mathcal{X}_k is used as the linearization point of the state set. The linearization point of the

input set \mathcal{U} is chosen to be the centroid of the input set: $p = \text{centroid}(\mathcal{U})$. The combined linearization point is then $\bar{\xi}_k = [\bar{x}_k, \bar{u}_k]^T = [q_k, p]^T$, and the linearization error can be described by the over-approximating interval box:

$$\mathcal{L}_{box}(\bar{\xi}_k) \supseteq \{\mathcal{L}(\xi_k, z) \mid z = \alpha\xi_k + (1-\alpha)\bar{\xi}_k, \dots, \alpha \in [0, 1], \xi_k \in \mathcal{X}_k \times \mathcal{U}\} \quad (14)$$

To be able to apply the ellipsoidal calculus subsequently, the error box is tightly enclosed by an ellipsoid $\mathcal{L}_{ell}(\xi_k) = \varepsilon(l_k, L_k) \supseteq \mathcal{L}_{box}(\bar{\xi}_k)$.

With these prerequisites, (12) is transformed into

$$x_{k+1} = A_k(x_k - q_k) + B_k(u_k - p) + \mathcal{L}(\bar{\xi}_k, z) \dots + f(q_k, p) + Gv_k, \quad (15)$$

again with $z \in \{\alpha\xi_k + (1-\alpha)\bar{\xi}_k, \alpha \in [0, 1]\}$. The reachable set \mathcal{X}_{k+1} given by (8) can be over-approximated by:

$$\tilde{\mathcal{X}}_{k+1} = A_k(\tilde{\mathcal{X}}_k - q_k) \oplus B_k(\mathcal{U} - p) \oplus G\mathcal{V} \dots \oplus \mathcal{L}_{ell}(\bar{\xi}_k) + f(q_k, p) \quad (16)$$

Proposition 1. The true reach set \mathcal{X}_{k+1} given by (8) is contained in an ellipsoid $\hat{\mathcal{X}}_{k+1}$ with:

$$\tilde{\mathcal{X}}_{k+1} \subseteq \hat{\mathcal{X}}_{k+1} \quad (17)$$

Proof 1. The nonlinear set valued function $F(\mathcal{X}_k, \mathcal{U})$ is approximated by the linearized dynamic given in (12). Since all possible linearization errors are considered in an over-approximating, and thus conservative, manner by using the ellipsoid $\mathcal{L}_{ell}(\bar{\xi}_k)$, the Minkowski addition yields an over-approximation of the true reach set:

$$\mathcal{X}_{k+1} \subseteq \tilde{\mathcal{X}}_{k+1} \quad (18)$$

By the use of *Lemma 2.2.1* in [2], it is possible to find a tight ellipsoidal approximation $\hat{\mathcal{X}}_{k+1}$, which contains the result of the Minkowski addition, which yields:

$$\mathcal{X}_{k+1} \subseteq \tilde{\mathcal{X}}_{k+1} \subseteq \hat{\mathcal{X}}_{k+1} \quad (19)$$

□

The true reach set \mathcal{X}_k is over-approximated in two steps. First the nonlinear dynamics is conservatively linearized, thus $\tilde{\mathcal{X}}_k$ can be computed through affine transformations of ellipsoids and Minkowski additions. $\tilde{\mathcal{X}}_k$ is in general not an ellipsoid, but the compact and convex set can be over-approximated by an ellipsoid $\hat{\mathcal{X}}_k \supseteq \tilde{\mathcal{X}}_k$. For the initial set, the relation $\hat{\mathcal{X}}_0 = \mathcal{X}_0$ is valid.

4. ALGORITHMIC SOLUTION APPROACH

The aim is to design a method that guarantees to stabilize the system (6) from an initial set with the given input constraints and the bounded disturbances. By using ellipsoidal sets, LMI formulations are a possible approach for solution.

Let us assume a state feedback control law of the structure:

$$u_k = H_k e_k + d_k, \quad e_k = x_k - q_k \quad (20)$$

The error vector e_k describes the difference between the current state x_k and the center point q_k of the reach set \mathcal{X}_k . The error ellipsoid \mathcal{E}_k is defined to be the following ellipsoid centered in the origin: $\mathcal{E}_k = \hat{\mathcal{X}}_k - q_k = \varepsilon(0, Q_k)$. The set-valued mapping of (20) results in:

$$\tilde{\mathcal{U}}_k = H_k \mathcal{E}_k + d_k \subseteq \mathcal{U}, \quad (21)$$

with which the closed loop dynamics of the linearized system follows:

$$\begin{aligned} \tilde{\mathcal{X}}_{k+1} &= [A_k(\hat{\mathcal{X}}_k - q_k) + B_k(\tilde{\mathcal{U}}_k - p)] \oplus G\mathcal{V} \dots \\ &\quad \oplus \mathcal{L}_{ell}(\bar{\xi}_k) + f(q_k, p) \\ &= [A_k(\hat{\mathcal{X}}_k - q_k) + B_k(H_k(\hat{\mathcal{X}}_k - q_k) + d_k - p)] \dots \\ &\quad \oplus G\mathcal{V} \oplus \mathcal{L}_{ell}(\bar{\xi}_k) + f(q_k, p) \\ &= (A_k + B_k H_k) \hat{\mathcal{X}}_k \oplus G\mathcal{V} \oplus \mathcal{L}_{ell}(\bar{\xi}_k) \dots \\ &\quad + f(q_k, p) - (A_k + B_k H_k) q_k + B_k d_k - B_k p \end{aligned} \quad (22)$$

Remark 1. Note that the Minkowski addition in (22) is replaced by an elementwise addition of the reach set $\hat{\mathcal{X}}_k$ and the input ellipsoid $\tilde{\mathcal{U}}_k$. Since (20) is a state feedback control law, one control input has to be applied for one given state x_k .

The components of the considered controller can be interpreted as follows. The gain H_k should lead to a contraction of the ellipsoid $\hat{\mathcal{X}}_k$ in step k . The affine component d_k results in a convergence to the center point q_k of the reach set to the origin. To make this obvious, (22) can be split as follows. First, the dynamics of the center point q_k under the influence of d_k is considered.

$$\begin{aligned} q_{k+1} &= A_k(x_k - q_k)|_{x_k=q_k} + B_k(u_k - p)|_{u_k=d_k} \dots \\ &\quad + f(q_k, p) + l_k \\ &= B_k(d_k - p) + f(q_k, p) + l_k \end{aligned} \quad (23)$$

Note that l_k is the center of the ellipsoid of the over-approximating linearization error $\mathcal{L}_{ell}(\bar{\xi}_k)$. d_k has to be chosen such that the center point q_k converges to the origin.

Second, the dynamics of an arbitrary point $x_k \in \hat{\mathcal{X}}_k$ becomes:

$$\begin{aligned} x_{k+1} &= A_k(x_k - q_k) + B_k(u_k - p) + Gv_k \dots \\ &\quad + f(q_k, p) + \mathcal{L}(\bar{\xi}_k, z) \\ &= A_k(x_k - q_k) + B_k(H_k(x_k - q_k) + d_k - p) \dots \\ &\quad + Gv_k + f(q_k, p) + \mathcal{L}(\bar{\xi}_k, z) + l_k - l_k \\ &= (A_k + B_k H_k)(x_k - q_k) + Gv_k \dots \\ &\quad + \underbrace{\mathcal{L}(\bar{\xi}_k, z) + B_k(d_k - p) + f(q_k, p) + l_k}_{q_{k+1}} - l_k \end{aligned} \quad (24)$$

$$x_{k+1} - q_{k+1} = (A_k + B_k H_k)(x_k - q_k) + Gv_k \dots + \mathcal{L}(\bar{\xi}_k, z) - l_k \quad (25)$$

$$e_{k+1} = (A_k + B_k H_k)e_k + Gv_k + \mathcal{L}(\bar{\xi}_k, z) - l_k \quad (26)$$

Since e_k describes the difference between an arbitrary x_k and the center point q_k , the volume of the reach set would decrease to zero for a stabilizing H_k , if there were no affine terms $Gv_k + \mathcal{L}(\bar{\xi}_k, z) - l_k$. The set valued mapping corresponding to (26) is:

$$\underbrace{\hat{\mathcal{X}}_{k+1} - q_{k+1}}_{\mathcal{E}_{k+1}} = (A_k + B_k H_k) \underbrace{(\hat{\mathcal{X}}_k - q_k)}_{\mathcal{E}_k} \oplus G\mathcal{V} \oplus \mathcal{L}_{ell}(\bar{\xi}_k) - l_k, \quad (27)$$

and $\check{\mathcal{E}}_{k+1} := (A_k + B_k H_k)\mathcal{E}_k$ defines the difference ellipsoid before the Minkowski addition. If the volume of $\check{\mathcal{E}}_{k+1}$ is smaller than \mathcal{E}_k with the existing affine terms, the system (6) is stabilized and the reach set converges to an ellipsoid $\hat{\mathcal{X}}_\infty$ with constant volume and shape. If \mathcal{E}_k does not decrease due to the affine terms, the nonlinear system would not be stabilized under the given conditions.

The combination of the two components H_k and d_k thus results in a possibly stabilizing behavior of the original nonlinear system (6). In steady state, the resulting over-approximating reach set $\hat{\mathcal{X}}_\infty$ is either an ellipsoid centered in the origin with constant volume or a constantly increasing set. The next section shows how to rewrite the task into a series of optimization problems.

5. SOLUTION BASED ON LMIS

The Problem 1 is recast into an iterative LMI problem by applying the steps presented in the previous section, namely:

- considering the linearized system (12) with over-approximated Lagrange remainder and reach sets instead of the original nonlinear dynamics (6),
- and restricting the control law κ to affine form (20).

The problem is solved by iteratively applying the linearization procedure, computing a control law, and determining the reach set in any time step k . The LMI to be solved in time k is:

$$\min_{S_k, H_k, d_k, \alpha_k} \text{trace}(S_k) \quad (28)$$

$$\text{s. t. } q_{k+1}^T M q_{k+1} - \rho q_k^T M q_k \leq \alpha_k \quad (29)$$

$$q_{k+1} = B_k(d_k - p) + f(q_k, p) + l_k \quad (30)$$

$$\alpha_k \leq \max_{i \in \{1, \dots, k\}} \omega^i \alpha_{k-i} \quad (31)$$

$$\text{trace}(S_k) \leq \text{trace}(Q_k) \quad (32)$$

$$\begin{bmatrix} S_k & (A_k + B_k H_k)^T \\ (A_k + B_k H_k) & Q_k^{-1} \end{bmatrix} \geq 0 \quad (33)$$

$$\begin{bmatrix} (b - r_i d_k) I_n & r_i H_k Q_k^{\frac{1}{2}} \\ Q_k^{\frac{1}{2}} H_k^T r_i^T & b - r_i d_k \end{bmatrix} \geq 0, \quad \forall i = \{1, \dots, n_c\}. \quad (34)$$

This problem is coupled to the problems of the previous time steps $k - i$ by the variables α_k , q_k and the difference ellipsoid $\mathcal{E}_k = \varepsilon(0, Q_k)$. To relax the Lyapunov equation (29), $\rho \in [0, 1)$ and $\omega \in [0, 1)$ are chosen. For each time step, the solution of the LMI problem determines a control law (20), which ensures convergence of the center point q_k of the difference ellipsoid \mathcal{E}_k , minimizes the reach ellipsoid \mathcal{E}_k in an appropriate sense, and satisfies the input constraints.

In general, different optimality criteria $J(Q)$ parametrized by an ellipsoid Q may be used. Here, we assume that $J(Q_1) \geq J(Q_2)$, if $Q_1 - Q_2$ is a nonnegative definite matrix. Consider two important cases of the general criterion $J(Q)$. First, it models the volume of an ellipsoid which scales with the determinant of the shape matrix: $J(Q) = \det(Q)$. Second, it is chosen as the sum of the squared semi-axis, i.e. it equals the trace of the shape matrix: $J(Q) = \text{trace}(Q)$, [2]. Using the volume of an ellipsoid as the cost function, the reduction of only one semi-axes to zero leads to an optimal value for the considered cost function (without regarding the remaining semi-axes). This results in a degenerate ellipsoid, since the determinant of the shape matrix becomes zero. Thus, the sum of squared semi-axes must be minimized by taking into account all semi-axes, and $\text{trace}(Q)$ is used in problem (28).

The shape matrix Q_{k+1} of the ellipsoid $\tilde{\mathcal{E}}_{k+1}$ is given by $(A_k + B_k H_k) Q_k (A_k + B_k H_k)^T$ (see (4)). To formulate the

cost function $J(\tilde{\mathcal{E}}_{k+1}) = \text{trace}(\tilde{\mathcal{E}}_{k+1})$ as LMI problem, the shape matrix is over-approximated with a new matrix S_k :

$$\begin{aligned} S_k &\geq (A_k + B_k H_k) Q_k (A_k + B_k H_k)^T \\ S_k - (A_k + B_k H_k) Q_k (A_k + B_k H_k)^T &\geq 0 \end{aligned} \quad (35)$$

By applying the Schur complement [5], (35) is transferred into (33), and the matrix S_k over-approximates the shape matrix of $\tilde{\mathcal{E}}_{k+1}$. Thus, the constraint (32) ensures that the sum of the squared semi-axes of $\tilde{\mathcal{E}}_{k+1}$ is minimized by H_k in (35).

Because the linearized dynamics (A_k, B_k) may change in every time step k , finding a stabilizing H_k for (A_k, B_k) is not sufficient to enforce convergence of the center point q_k to the origin. Thus, a time-invariant Lyapunov function $V(q_k) = q_k^T M q_k$ with positive definite matrix M is employed. However, it may be impossible to find a quadratic Lyapunov function which monotonically decreases (i.e. $V(q_{k+1}) \leq \rho V(q_k)$, $\rho \in [0, 1)$) for the nonlinear dynamics. To relax this condition, the concept of flexible Lyapunov functions [1] is used, which introduces slack variables α_k . As a result, the Lyapunov function is flexible in the sense that it may be locally non-monotone, in contrast to a monotone decrease in standard Lyapunov functions. Nonetheless, asymptotic convergence is guaranteed if $\alpha_k \rightarrow 0$ for $k \rightarrow \infty$, which is ensured by the constraint (31) and $\omega \in [0, 1)$ [1]. This concept is used here to couple the LMI problem at time step k to the problems at $k - i$ to enforce convergence of the center point q_k of $\tilde{\mathcal{X}}_k$ over the iterations of Algorithm 1. The LMI constraints (34) enforce the input constraint (21) at each time step, as stated in the following Proposition.

Proposition 2. The input constraint $H_k e_k + d_k \in \mathcal{U}$ holds for H_k , d_k and all $e_k \in \mathcal{E}_k$ if (34) holds.

Proof 2. By substituting (20) into the input constraint in (6) it can be seen that the input constraint holds for all $e_k \in \mathcal{E}_k$ if:

$$R(H_k e_k + d_k) \leq b, \quad \forall e_k \in \mathcal{E}_k \quad (46)$$

By row-wise maximization of the linear inequality in (46), the universal quantifier can be eliminated and the following condition is obtained:

$$\max_{w \in \mathcal{W}} r_i w \leq b - r_i d_k, \quad \forall i = \{1, \dots, n_c\} \quad (47)$$

$$\mathcal{W} = \{w \in \mathbb{R}^m \mid w = H_k e_k, e_k \in \mathcal{E}_k\} \quad (48)$$

The ellipsoid \mathcal{E}_k can be mapped into a unit ball $\|z_k\|_2 \leq 1$ by the change of variables according to $z_k = Q_k^{-\frac{1}{2}} e_k$. This yields:

$$\mathcal{W} = \left\{ w \in \mathbb{R}^m \mid w = H_k Q_k^{\frac{1}{2}} z_k, \|z_k\|_2 \leq 1 \right\} \quad (49)$$

The maximization problem (47) subject to (48) can be recast as follows [8]:

$$\max_{w \in \mathcal{W}} r_i w = \|r_i H_k Q_k^{\frac{1}{2}}\|_2 \leq b - r_i d_k \quad (50)$$

Finally, the Euclidean norm in (50) can be expressed as LMI [10], resulting in:

$$\begin{bmatrix} (b - r_i d_k) I_n & r_i H_k Q_k^{\frac{1}{2}} \\ Q_k^{\frac{1}{2}} H_k^T r_i^T & b - r_i d_k \end{bmatrix} \geq 0 \quad \forall i = \{1, \dots, n_c\} \quad (51)$$

Thus, (46) and (51) are identical and the proposition follows. Note that this results in n_c LMI constraints. \square

The convex problem (28) must be solved in every time step k and the synthesis computation is successful if the reach set $\hat{\mathcal{X}}_k$ is contained in the target set \mathbb{T} . To ensure a termination of Algorithm 1, a maximum of desired iteration k_{max} must be defined and if the reach set does not enter the target set within the maximal iteration, the algorithm terminates without success. At time $k = 0$, the matrix M has to be specified. In Algorithm 1, this matrix is computed by solving the Lyapunov equation:

$$M - (A_0 + B_0K)^T M (A_0 + B_0K) \geq 0, \quad (42)$$

where K is a stabilizing feedback for (A_0, B_0) . Note that K is an auxiliary control law used only to obtain a candidate Lyapunov function $V(x_k) = x_k^T M x_k$.

Algorithm 1. Ellipsoidal Control Algorithm

Given: $f(x_0, u_0)$, $F(\mathcal{X}_0, \mathcal{U})$, $\mathcal{X}_0, \mathcal{U}, \mathcal{V}$ as well as $\mathbb{T}, \rho, \omega, \alpha_0, k_{max}$ and M according to (42).

Define: $k := 0$

while $\hat{\mathcal{X}}_k \not\subseteq \mathbb{T}$ and $k < k_{max}$ **do**

- Compute hyperbox $[x_k]$ using the function *intval*.
- Apply linearization procedure according to section ?? to get $A_k, B_k, \mathcal{L}_{ell}(\xi_k)$.
- Solve the optimization problem (28).
if no feasible H_k, d_k is found **do**
stop algorithm (synthesis failed).
end if
- Evaluate system dynamic (22) and compute the over-approximating ellipsoid $\hat{\mathcal{X}}_{k+1}$.
- $k := k + 1$

end while

Lemma 3. If Algorithm 1 terminates with $\hat{\mathcal{X}}_k \subseteq \mathbb{T}$, $k \leq k_{max}$, Problem 1 is successfully solved and a control law (20) exists which steers the initial state $x_0 \in \mathcal{X}_0$ into the target set \mathbb{T} in N steps for all possible disturbances $v_k \in \mathcal{V}$. Furthermore, the input constraint $u_k \in \mathcal{U}$ holds for all $0 < k < N$, and the center point q_k of the reach set $\hat{\mathcal{X}}_k$ asymptotically converges to the origin.

Proof 3. According to Proposition 1 it holds that $\mathcal{X}_k \subseteq \hat{\mathcal{X}}_k$, i.e. the true reach set at each time step is over-approximated by the ellipsoid $\hat{\mathcal{X}}_k$. The reach set $\hat{\mathcal{X}}_{k+1}$ at time $k + 1$ is computed in Algorithm 1 for all $x_k \in \hat{\mathcal{X}}_k \supseteq \mathcal{X}_k$ and all disturbances $v_k \in \mathcal{V}$. Thus, it follows that $\mathcal{X}_{k+1} \subseteq \hat{\mathcal{X}}_{k+1}$ and by induction $x_k \in \mathcal{X}_k \subseteq \hat{\mathcal{X}}_k$ for all $k > 0$. Successful termination of Algorithm 1 implies that $\hat{\mathcal{X}}_N \in \mathcal{T}$ and consequently $x_N \in \mathcal{X}_N \subseteq \mathcal{T}$ holds for all initial states $x_0 \in \mathcal{X}_0$ and all disturbances $v_k \in \mathcal{V}$.

By construction $x_k \in \hat{\mathcal{X}}_k$ implies that $e_k \in \mathcal{E}_k$. It follows, that the input constraint $u_k \in \mathcal{U}$ holds at each time step, if $u_k = H_k e_k + d_k \in \mathcal{U}$ holds for all $e_k \in \mathcal{E}_k$, which is established in Proposition 2. Finally, (29) and (31) imply that $V(x_k) = q_k^T M q_k$ can be used to ensure asymptotic convergence of the center point q_k of $\hat{\mathcal{X}}_k$ (cf. Lemma III.4 in [1]) over the considered horizon N . \square

6. NUMERICAL EXAMPLE

In order to show the principle of the proposed algorithm, it is applied to a numerical example of a three-tank-system, in which there is a connecting pipe between tank one and two and another connecting pipe between tank two and three. Additionally, the second and third tank have an open outlet at the bottom, whereas the outlet of the second tank is controllable. The connecting pipes are located at the bottom of the tanks and have the same cross-section area as the outlets, which is denoted by a . The cross section of each tank is denoted by $A_i, i = 1, 2, 3$. The input is modeled as a percentage of fully opened valve, which effects the inflow into the first and third tank, and the outflow of the second tank. The characteristic curve of the valve is approximated by a quadratic function for $u_{i,k} \in [0, 1], i = 1, 2, 3$. The dynamic system of the three-tank-system in discrete-time form is given in (43) with $\tau = 2$. The state x_k denotes the liquid level in each tank according to a reference level x_{ref} : $\tilde{x}_{i,k} = x_{i,k} + x_{i,ref}, i = 1, 2, 3$. The target set is defined as:

$$\mathbb{T} = \varepsilon \left([0, 0, 0]^T, 1e^{-2} \begin{bmatrix} 0.84 & 3.3 & 0.9 \\ 3.3 & 18 & 3.84 \\ 0.9 & 3.84 & 1.08 \end{bmatrix} \right),$$

and the remaining parameters of Algorithm 1 are chosen as follows: $\alpha_0 = 1e-4, \omega = 0.98, \rho = 0.98$. Fig. 1 shows exemplarily some reachability sets $\hat{\mathcal{X}}_k$ computed by the algorithm. For ease of interpretation, one sample trajectory for the nonlinear system is included (green). It can be seen that for every step k the control law steers the state x_k of the system to the center point q_{k+1} of the ellipsoidal reach set $\hat{\mathcal{X}}_{k+1}$. The closer the linearized system gets to the center point q_{k+1} of $\hat{\mathcal{X}}_{k+1}$, the smaller is the linearization error. The over-approximated reachable set is contained in the target set \mathbb{T} after 280 time steps and Algorithm 1 terminates in 916s. The average solution time for one single LMI problem is 0.539s.

The optimization problem (28) subject to (29) - (32) was solved with Matlab 7.12.0 with YALMIP 3.0 and SeDuMi 1.3. The reachability computations were performed with the ellipsoidal toolbox ET [11].

$$x_{k+1} = \begin{bmatrix} x_{1,k} + \tau \cdot \left(\frac{1}{A_1} (v_{max} u_{1,k}^2 - a \cdot 0.5 \cdot g \cdot \tanh(\tilde{x}_{1,k} - \tilde{x}_{2,k})) \right) + v_{1,k} \\ x_{2,k} + \tau \cdot \left(\frac{a}{A_2} 0.5 \cdot g \cdot \tanh(\tilde{x}_{1,k} - \tilde{x}_{2,k}) - \frac{a}{A_2} 0.5 \cdot g \cdot \tanh(\tilde{x}_{2,k} - \tilde{x}_{3,k}) \cdot (1 + u_{2,k}^2) \right) + v_{2,k} \\ x_{3,k} + \tau \cdot \left(\frac{a}{A_3} 0.5 \cdot g \cdot \tanh(\tilde{x}_{2,k} - \tilde{x}_{3,k}) + \frac{1}{A_3} (v_{max} u_{3,k}^2 - a \cdot 0.5 \cdot g \cdot \tanh(\tilde{x}_{3,k})) \right) + v_{3,k} \end{bmatrix}, \mathcal{X}_0 = \varepsilon \left(-x_{ref}, 1e^{-3} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right), \quad (43)$$

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad v_k \in \mathcal{V} = \varepsilon \left([0, 0, 0]^T, 1e^{-6} \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.2 \end{bmatrix} \right), \quad \mathcal{U} = \left\{ u \in \mathbb{R}^m \mid \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} u \leq \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad x_{ref} = \begin{bmatrix} 0.4113 \\ 0.3091 \\ 0.2067 \end{bmatrix} \quad (44)$$

$$A_1 = A_3 = 4, \quad A_2 = 3, \quad a = 0.02, \quad v_{max} = 5 \quad (45)$$

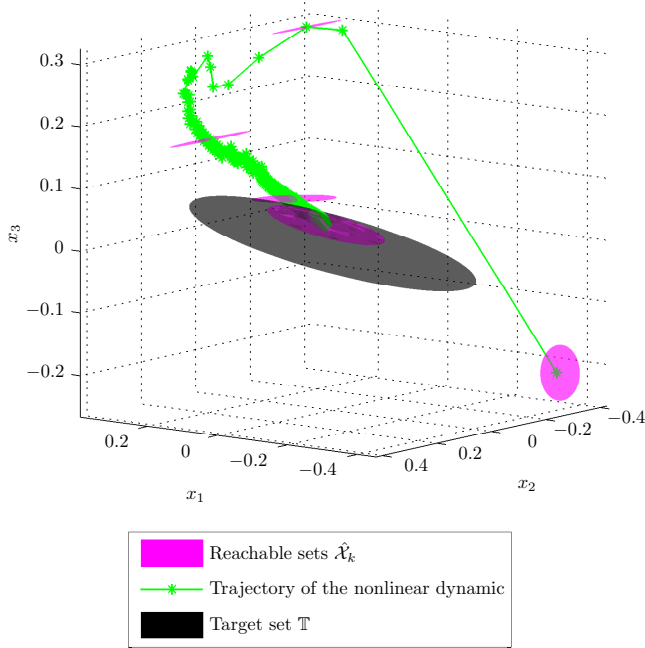


Fig. 1. Numerical example: Algorithm 1 terminates after 280 steps with $\hat{\mathcal{X}}_N$ contained in the target set T . For ease of interpretation only a few ellipsoid are shown.

7. CONCLUSION

This paper provides a method to algorithmically synthesize a control law for discrete-time nonlinear systems with ellipsoidal initial set and bounded disturbances. The major contribution of this paper is the combination of the well known ellipsoidal calculus with a nonlinear system dynamic to formulate a convex LMI problem by the use of a conservative pointwise linearization method. The pointwise conservative linearization is implemented by a conservative approximation of the Lagrange remainder. The linear set-valued closed loop equation obtained for an affine state feedback control structure is used to formulate the convex LMI problem. It is shown that the nonlinear uncertain system is stabilized into a target set, if a feasible solution of the LMI problem can be determined in any iteration of the synthesis algorithm.

Future research topics will include the direct approximation of the Lagrange remainder as an ellipsoid (without the intermediate step of computing a hyperbox). In addition, it seems interesting to replace the bounded uncertainty by probability distributions and to use a notion of stochastic reachable sets in the synthesis.

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