Chaotic Behavior of the Folding Map on the Equilateral Triangle

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Abstract

In recent years, it becomes important to understand chaotic behaviors in order to analyze nonlinear dynamics because chaotic behavior can be observed in many models in the field of physics, biology, and so on. To understand chaotic behaviors, investigating mechanisms of chaos is necessary and it is meaningful to study simple models that shows chaotic behaviors. In this paper, we propose an extremely simple triangle folding map and show that the map has k-periodic points for any integer k, and show the map has sensitivity to initial conditions. Finally, we discuss the connection with the Sierpinski gasket and construct similar types of fractal geometry.

1. Introduction

Chaotic behavior embedded in dynamical systems has been attracting huge attention in the field of nonlinear dynamical systems theory since 1960s. Wide variety of results have also been reported concerning chaotic systems in many areas such as fundamental field of physics and biology as well.

One of the major objectives of investigating chaos is to elucidate the mechanism of generating chaotic behavior. A mathematical approach to address analysis problem of chaotic systems is to observe simple nonlinear dynamics and find key factors that give rise to chaos. The simplicity of the nonlinear models to observe is central in obtaining better understanding of complicated behaviors. Notable examples of such relatively simple dynamic models are the logistic map, the tent map, the Horseshoe map [1, 2], to cite but a few (see also [3–5] and the references therein).

In this paper, we propose a *simple* folding map for equilateral triangles that has sensitivity with respect to the initial conditions. Specifically, due to the symmetries that the equilateral triangle possess, we show that restricting the domain of the triangle through an equivalence relation reveals essential relationship of the folding map. Furthermore, we provide fixed point analysis and periodic point analysis associated with this mapping operation by sequentially partitioning the restricted domain. Finally, we discuss some connections of the folding map to the Sierpinski gasket, which is well known to be composed of self-homothetic triangles, and propose a scheme to construct other interesting fractals by removing certain regions of the equilateral triangle.

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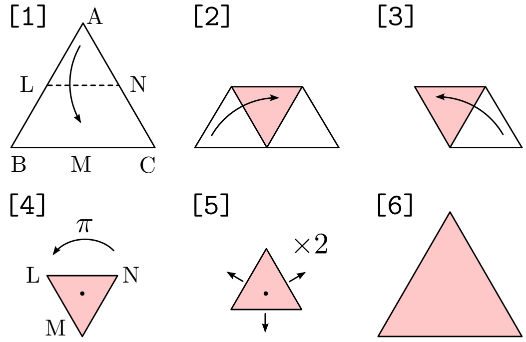

Figure 2.1: Operation of the folding for the equilateral triangle

2. Folding Map on the Equilateral Triangle

Consider the equilateral triangle ABC. Let its domain be denoted by $\mathcal{T}$ and let the midpoints of the edges AB, BC, CA be denoted by L, M, N, respectively. Furthermore, consider the following simple folding operation (also see Figure 2.1):

[1] Fold along NL and bring A to M.
[2] Fold along LM and bring B to N.
[3] Fold along MN and bring C to L.
[4] Rotate LMN around its center by π radian.
[5] Enlarge LMN by double so that MNL coincides with ABC.

Note that the triangle LMN in [4] is also equilateral and homothetic to the triangle ABC with the scale factor $\frac{1}{2}$. After the folding operation above, the resulting triangle in [6] becomes identical to the equilateral triangle in [1]. We denote this folding operation [1]–[6] by $F : \mathcal{T} \to \mathcal{T}$. For example, the point $\mathrm{P} \in \mathcal{T}$ shown in [1] of Figure 2.2 is mapped by the function F to another point $F(\mathrm{P})$ in [6]. It is immediate that the map F is surjective and injectivity is discussed below.

Now, it is important to note that there are several variations of prescribing the operation from [4] to [5] in Figure 2.1 in order to define the surjective map from $\mathcal{T}$ to $\mathcal{T}$. For example, by rotating the triangle LMN by $\frac{\pi}{6}$ radian, instead of π radian, counterclockwise (or clockwise), we can obtain a similar map to the original map $\mathcal{T}$. Or, more easily, the triangle LMN in [4] can be flipped upsidedown to arrive at

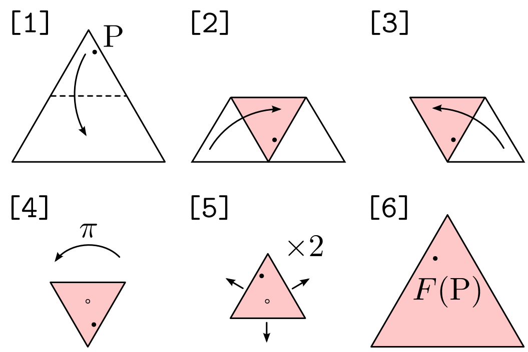

Figure 2.2: Visualization of the mapping scheme for a given point

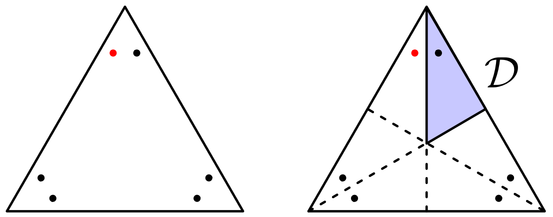

Figure 2.3: Equivalence relation

[5] through which we can construct a surjective map from $\mathcal{T}$ to $\mathcal{T}$. This commonality is due to the reflective and rotational symmetry that the equilateral triangles possess and it is preferable to characterize the map that describe the essential dynamics of the folding operation. The following notion precludes the ambiguity of the folding operation F.

Definition 2.1 Equivalence relation on $\mathcal{T}$. Consider the map $F : \mathcal{T} \to \mathcal{T}$ defined by Figure 2.1. Two points $\mathrm{P}, \mathrm{Q} \in \mathcal{T}$ are considered to be equivalent if P is transformed to Q via rotation by $2\pi/3$ or $4\pi/3$ radians, or reflection with respect to the symmetric axis of the equilateral triangle, or the combination of the rotation and the reflection. Specifically, we denote by

$$\mathrm{equiv}\ \mathcal{X} \triangleq \{t \in \mathcal{T} : t \text{ has the equivalence relation with a point in } \mathcal{X}\}, \quad (1)$$

the equivalence set associated with the set $\mathcal{X} \subset \mathcal{T}$.

Note that the center of the equilateral triangle has its equivalence relation with itself, and any point on the symmetric axis (except for the center) has 2 other points (on the other symmetric axes) that have equivalence relation with it. Otherwise, a point on $\mathcal{T}$ has 5 other points that have equivalence relation to each other (see Figure 2.3, left). In any case, it is important to note that any point $\mathrm{P} \in \mathcal{T}$ has a *unique* point in $\mathcal{D}$ that has the equivalence relation with P, where the closed subset $\mathcal{D} \subset \mathcal{T}$ is given by partitioning $\mathcal{T}$ with the three symmetric axes (Figure 2.3, right). Note that the 6 partitioned sets are right triangles and are all identical to each other in the shape and the size so that the choice of the partitioned set is not important.

Now, from the analysis above, we restrict the domain and the codomain into $\mathcal{D}$, instead of the original equilateral tri-

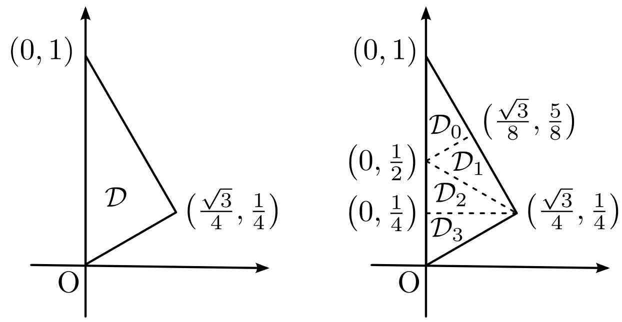

Figure 2.4: Domain $\mathcal{D}$ in the coordinate system (left) and its subdomains (right)

angle $\mathcal{T}$, and define a new map $f : \mathcal{D} \to \mathcal{D}$ associated with the folding map F under the equivalence relation given by Definition 2.1. Note that for a point $\mathrm{R} \in \mathcal{D}$, $f(\mathrm{R}) \in \mathcal{D}$ has the equivalence relation with $F(\mathrm{R}) \in \mathcal{T}$.

In order to describe the map f more clearly, we define the x-y coordinate system to the triangle. Specifically, let the length of the edges of $\mathcal{T}$ be $2\sqrt{3}$ and, as shown in Figure 2.4 (left), let the center of the equilateral triangle is placed at the origin and the bottom edge be parallel to the x-axis. In this case, the map f is described by a piecewise affine function given in Definition 2.2 below.

For the statement of the following results, let the domain $\mathcal{D}$ be further partitioned into the 4 identical closed subdomains $\mathcal{D}_i$, $i = 0, 1, 2, 3$, as given by Figure 2.4 (right), which are homothetic to the right triangle $\mathcal{D}$.

Definition 2.2 Folding map for the equilateral triangle. For the point $p = [x, y]^{\mathrm{T}}$ in the closed domain $\mathcal{D} \subset \mathbb{R}^2$, the folding map $f : \mathcal{D} \to \mathcal{D}$ for the equilateral triangle is given by

$$f(p) \triangleq f_i(p), \quad p \in \mathcal{D}_i, \quad i = 0, 1, 2, 3, \quad (2)$$

where

$$f_0(p) \triangleq \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad (3)$$

$$f_1(p) \triangleq \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}, \quad (4)$$

$$f_2(p) \triangleq \begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}, \quad (5)$$

$$f_3(p) \triangleq \begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \quad (6)$$

Since each subdomain $\mathcal{D}_i$ is defined as a closed set for all $i = 0, 1, 2, 3$, adjacent domains share the points on their boundaries. Note, however, that the map f defined in Definition 2.2 has no ambiguity in that when $p \in (\mathcal{D}_i \cap \mathcal{D}_j)$ it follows that $f_i(p) = f_j(p)$ so that the point p on the intersection of the domains is mapped to the same point in $\mathcal{D}$.

Henceforth, for a subset $S \subset \mathcal{D}$, $f(S)$ denotes the set of points $f(p)$, $p \in \mathcal{D}$, which is also a subset of $\mathcal{D}$.

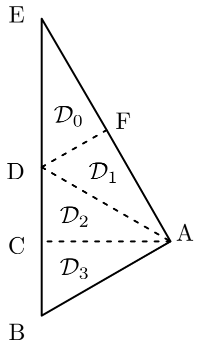

Figure 3.1: Definition of points A, B, ..., F on the boundary $\partial\mathcal{D}$ of the right triangle

3. Fixed Point and Periodic Point Analysis on the Boundary of $\mathcal{D}$

In this section, we restrict our attention on the boundary $\partial\mathcal{D}$ of the domain $\mathcal{D}$ and provide analysis in regards to the fixed points and the periodic points. Specifically, note that the folding map f maps every point on $\partial\mathcal{D}$ onto $\partial\mathcal{D}$. In other words, the set $\partial\mathcal{D}$ is a positively invariant set with respect to f. For the analysis presented in this section, we define the map $f_\partial : \partial\mathcal{D} \to \partial\mathcal{D}$ as

$$f_\partial(p) = f_i(p), \quad p \in (\mathcal{D}_i \cap \partial\mathcal{D}), \quad i = 0, 1, 2, 3, \tag{7}$$

where $f_i(\cdot)$, $i = 0, 1, 2, 3$, are given by (3)–(6).

Theorem 3.1. Consider the folding map f_∂ given by (7). For $\partial\mathcal{D}$, let $\mathrm{A}, \mathrm{B}, \ldots, \mathrm{F}$ denote the points shown in Figure 3.1. Then the map $f_\partial : \partial\mathcal{D} \to \partial\mathcal{D}$ satisfies the following properties.

i) The k-times composite map f_∂^k has 2^k fixed points on the edges A-B-E.

ii) The k-times composite map f_∂^k has 2^k fixed points on the edge E-A.

iii) The k-times composite map f_∂^k has $2^{k+1}-1$ fixed points on $\partial\mathcal{D}$.

iv) The Lyapunov exponent of f_∂ is $\ln\ 2$ almost everywhere.

4. Extended Fixed Point and Periodic Point Analysis over $\mathcal{D}$

4.1. Geometric Interpretation of the Triangle Folding Map

In this section, we provide characterization of the fixed and the periodic points of the folding map f over the domain $\mathcal{D}$ and compute the Lyapunov exponent of the map. As defined in Definition 2.2, the folding map f is composed of the operation such that any domain $\mathcal{D}_i$, $i \in \{0, 1, 2, 3\}$, is enlarged double and coincides with the domain $\mathcal{D}$ after certain operation of rotation and/or reflection. For example, $f(\mathcal{D}_1) = \mathcal{D}$ can be understood by looking at the operation shown in Figure 4.1. This is due to the fact that the piecewise map f is affine with its subdomains $\mathcal{D}_i$, $i = 0, 1, 2, 3$, are homothetic to $\mathcal{D}$ with the ratio $1/2$. In other words, the domain S that

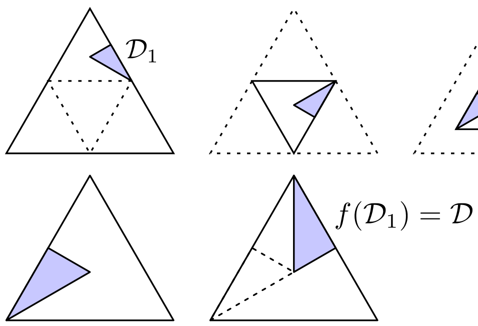

Figure 4.1: Mapping of the whole partitioned domain $\mathcal{D}_1$ by f

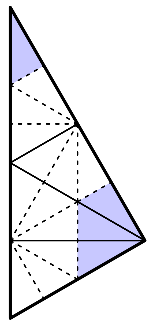

Figure 4.2: Four subdomains that are mapped by f to $\mathcal{D}_0$

satisfies $f(S) = \mathcal{D}$ is given by either $\mathcal{D}_0$, $\mathcal{D}_1$, $\mathcal{D}_2$, or $\mathcal{D}_3$. Furthermore, it can be observed that the finer domains each of which are mapped by f to $\mathcal{D}_0$ are shown in Figure 4.2. Each of these 4 finer domains is characterized as the "0th" domain where $\mathcal{D}_i$ is further partitioned into 4 smaller identical subdomains for $i = 0, 1, 2, 3$ as for the case of $\mathcal{D}$ shown in Figure 2.4. Since each of these 4 smaller subdomains is mapped by f to $\mathcal{D}_0$, it is mapped to $\mathcal{D}$ by the composite map $f \circ f$. Now, it follows from the above analysis that each of the small 16 right triangles in Figure 4.2 is mapped by $f^2 = f \circ f$ to $\mathcal{D}$ and this operation is understood as applying rotation and/or reflection, and magnification in an appropriate manner to the small right triangles.

The above interpretation of the mapping operation turns out to be useful in understanding the successive mapping $f \circ \cdots \circ f$. In the following, we give mathematical representation of the partitioned domains and characterized the relationship between the partitioned domains and the map f.

Definition 4.1 Partitioning operation Π. For a closed right triangle $\mathcal{E}$ that is homothetic to $\mathcal{D}$, the partitioning operation Π_i, $i = 0, 1, 2, 3$, over $\mathcal{E}$ is defined as in Figure 4.3.

Definition 4.2 Sequentially partitioned set by a sequence. For a given $n \in \mathbb{N}$, consider the collection of finite sequences $\mathcal{S}_n$ given by

$$\mathcal{S}_n \triangleq \left\{ s = \{s_i\}_{i=0}^{n-1} : s_i \in \{0, 1, 2, 3\},\ i = 0, 1, \ldots, n-1 \right\}. \tag{8}$$

Then the subset $\mathcal{D}_s$, where $s = \{s_0, s_1, \ldots, s_{n-1}\} \in \mathcal{S}_n$, is

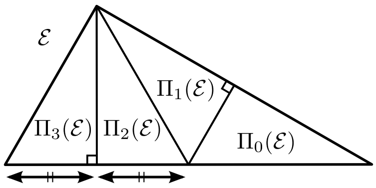

Figure 4.3: Partitioning operation Π_i over a right triangle $\mathcal{E}$

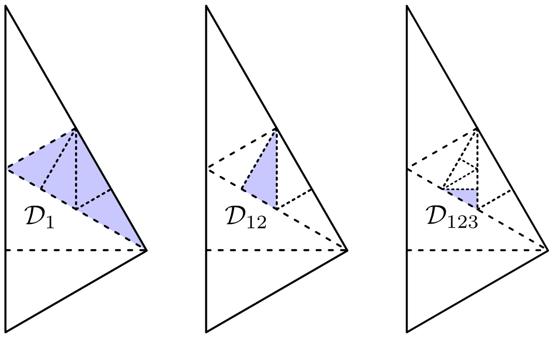

Figure 4.4: Examples for $\mathcal{D}_1$, $\mathcal{D}_{12}$, $\mathcal{D}_{123}$

defined as

$$\mathcal{D}_s \triangleq \Pi_{s_{n-1}} \circ \Pi_{s_{n-2}} \circ \cdots \circ \Pi_{s_0}(\mathcal{D}). \tag{9}$$

For simplicity of exposition, we write $\mathcal{D}_{ij\ldots k}$ to denote $\mathcal{D}_{\{i,j,\ldots,k\}}$.

Remark 4.1. According to this definition, the notation $\mathcal{D}_i$ used in the previous sections stands for $\mathcal{D}_{\{i\}}$.

Figure 4.4 shows several examples of the domains that are represented by the notation given by (9).

Next, we define a left shift operation for the sequence s.

Definition 4.3 Left shift operation of sequences.
For a given $n \in \mathbb{N}$ and the finite sequence $s = \{s_0, s_1, \ldots, s_{n-1}\} \in \mathcal{S}$, let $k \in \mathbb{N}$ be $k < n$. The left shift operation for s is defined as the binary operation $\ll: \mathcal{S} \times \mathbb{N} \to \mathcal{S}$ given by

$$s \ll i \triangleq \{s_i, s_{i+1}, \ldots, s_{n-1}\}. \tag{10}$$

This definition makes it possible to simply represent the folding map for the equilateral triangle.

Theorem 4.1. For a given $n \in \mathbb{N}$ and the finite sequence $s = \{s_0, s_1, \ldots, s_{n-1}\} \in \mathcal{S}$, let $k \in \mathbb{N}$ be $k < n$. Then it follows that

i) $f^k(D_s) = D_{s \ll k}$,

ii) $f^k(p) \in \mathcal{D}_{s \ll k}, p \in \mathcal{D}_s$.

The result *i*) in Theorem 4.1 indicates the fact that applying the folding map f is equivalent to shifting left the subscript s of $\mathcal{D}_s$ by 1, while *ii*) suggests that the mapped point $f^k(p)$ of $p \in \mathcal{D}$ by f^k may be estimated.

4.2. Periodic Points of the Triangle Folding Map

The main result of this section is shown in Theorem 4.2 below.

Theorem 4.2. The k-times folding map $f^k : \mathcal{D} \to \mathcal{D}$ for the equilateral triangle has 4^k fixed points.

Corollary 4.1. The number of k-periodic points associated with the folding map f is given by

$$N_k = 4^k - \sum_{m \in M_k} N_m, \;\; N_1 = 4, \tag{11}$$

where M_k is the collection of divisors of k except for k itself. Equivalently, the number of k-periodic solutions is given by N_k/k.

Theorem 4.3. For every fixed point $p \in \mathcal{D}$ and every $j \in \mathbb{N}$, there exists a finite sequence $s \in \mathcal{S}_k$ such that $p \in \mathcal{D}_{\mathrm{rep}(s,j)}$, where $\mathrm{rep}(s,j)$ represents

It is important to note that from the definition of the map given in Definition 2.2 every fixed point of f^k is an unstable periodic points of f, where its inverse does not exist, so that it is in general difficult to determine these points. Using Theorem 4.3, however, it turns out that those fixed points can be easily identified with arbitrary accuracy. Specifically, when $k = 3$, for example, it follows that each of the domains $\mathcal{D}_{000}, \mathcal{D}_{001}, \ldots, \mathcal{D}_{333}$ possesses 1 fixed point in its domain. Furthermore, it follows from Theorem 4.3 that the fixed point in $\mathcal{D}_{ijk}$ lies in a smaller domain $\mathcal{D}_{ijkijk\cdots}$. The domain $\mathcal{D}_{ijkijk\cdots}$, by definition, is a connected domain and the area of $\mathcal{D}_{ijkijk\cdots}$ can be made arbitrarily small by repeating the sequence ijk in the subscript. Thereby, the domain in which the fixed point is contained can be identified with arbitrary accuracy.

Figure 4.5 shows the fixed points of the maps $f, f^2, \ldots, f^6$. It is interesting to point out that the fix points of f^k are likely to be placed on circles, especially for the case of large k. Finally, Figure 4.6 shows (some of) the periodic points for periods 2, 3, and 4. The groups of periodic points are differentiated by color so that the points in the same color belong to the same set of periodic points, among which the order of the periodic points is designated by numeric numbers. Note that only 3 sets of periodic points are visualized for the case of f^3 and f^4.

4.3. Lyapunov exponent of the Folding Map

The following theorem provides the Lyapunov exponent of the folding map for the equilateral triangle.

Theorem 4.4. The Lyapunov exponent of the folding map f for the equilateral triangle is $\ln\ 2$ almost everywhere.

Theorem 4.5. First, note that the eigenvalues of the Jacobian matrices for f_0, f_1, f_2, f_3 defined in Definition 2.2 are 2 in their magnitude. Therefore, for every vector v on the unit circle as a initial point, the Lyapunov exponent $\lambda(p, v)$

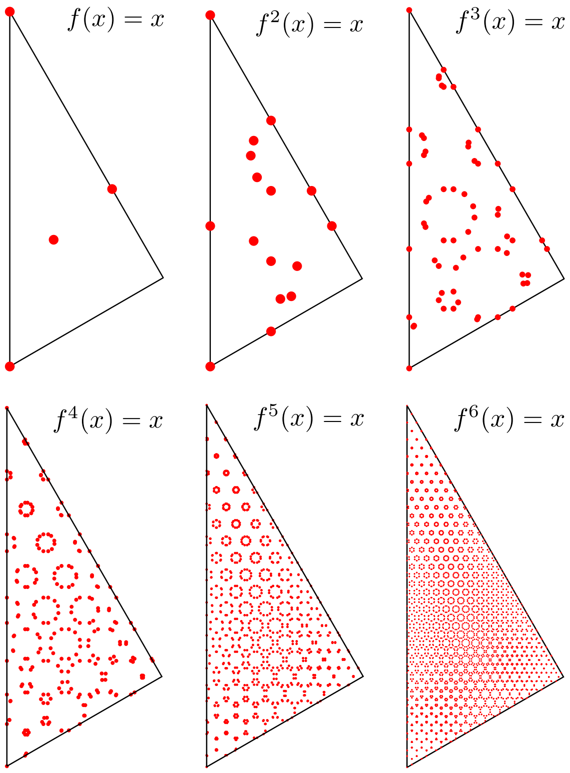

Figure 4.5: fixed points of the maps $f, f^2, \ldots, f^6$

is given by

$$\begin{aligned}\lambda(p, v) &\triangleq \limsup_{N\to\infty} \frac{1}{N} \ln \left\| Df^N(x)v \right\| \\ &= \limsup_{N\to\infty} \frac{1}{N} \ln \left\| 2^N v \right\| = \ln 2, \quad \text{a.e.,}\end{aligned}$$

where $Df^N(x)$ denotes the Jacobian matrix f^N at x. □

5. Triangle Folding Map and Sierpinski Gasket

In the triangle folding scheme defined in Figure 2.1, assume that the new operation that the small equilateral triangle LMN is removed is added in the scheme before the operation [1].

[0] Remove the triangle LMN from ABC.

Figure 5.1 shows the collection of initial points that are not taken off after the infinitely many repeated operation of [0]–[5]. This pictorial figure is widely known as the Sierpinski gasket which contain a fractal structure. In fact, Figure 5.2 depicts the sequence of removing the triangle areas and this evolution of geometry describes the way of construct the Sierpinski gasket.

Using the triangle folding map being analyzed in the paper, the representation of Sierpinski gasket can be explained as follows: Since Sierpinski gasket has the rotational and the reflective symmetries, we restrict our attention on the domain $\mathcal{D}$ as indicated in Figure 2.3. In this case, the removing operation [0] of the triangle LMN corresponds to removing the

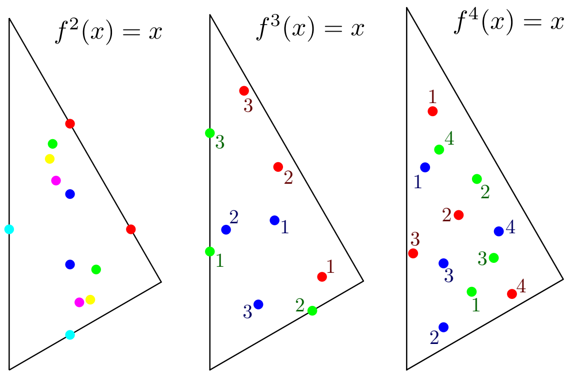

Figure 4.6: 2, 3, 4-periodic points

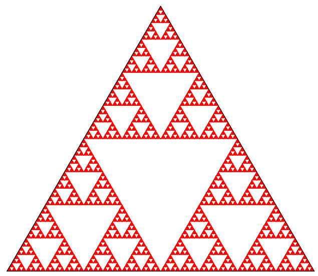

Figure 5.1: Collection of initial points that are not taken off after the infinitely many repeated mappings of f

domain $\mathcal{D}_3$ from $\mathcal{D}$ (recall Figure 2.4). Now, define

$$\mathcal{D}_{\mathrm{rm}}^{j,m} \triangleq \left\{ x \in \mathcal{D} \mid f^m(x) \notin \mathcal{D}_j \right\}, \tag{12}$$

which indicates the domain from which a point is not mapped onto the domain $\mathcal{D}_j$ after the mth operation of [0]–[5]. With this notation, the domain from which a point is never mapped onto the domain $\mathcal{D}_j$ after arbitrary number of the operation is characterized by

$$\mathcal{D}_{\mathrm{rm}}^{j} \triangleq \bigcap_{m=0}^{\infty} \mathcal{D}_{\mathrm{rm}}^{j,m}, \tag{13}$$

and hence the Sierpinski gasket is represented as $\mathrm{eqv}\ \mathcal{D}_{\mathrm{rm}}^{3}$.

Note that we obtain the Sierpinski gasket from $\mathrm{eqv}\ \mathcal{D}_{\mathrm{rm}}^{j}$ with $j = 3$. It is worth investigating the equivalence relation of $\mathrm{eqv}\ \mathcal{D}_{\mathrm{rm}}^{j}$ for $j = 0, 1, 2$. Figure 5.3 shows the pictorial figures for each of the cases.

6. Conclusion

We proposed a simple folding map associated with the equilateral triangle and provide analysis in terms of the self-homothetic partitioning of the domain. Future works include the fixed point and periodic point analysis on the Sierpinski gasket. It is also worth investigating the connections and the differences between the folding map and the well-known horseshoe map, which also has the notion of 'folding' in its operation.

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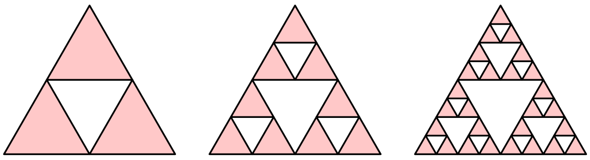

Figure 5.2: Resulting pictorial figure after the operation of `[0]–[5]` once (left), twice (center), and three times (right)

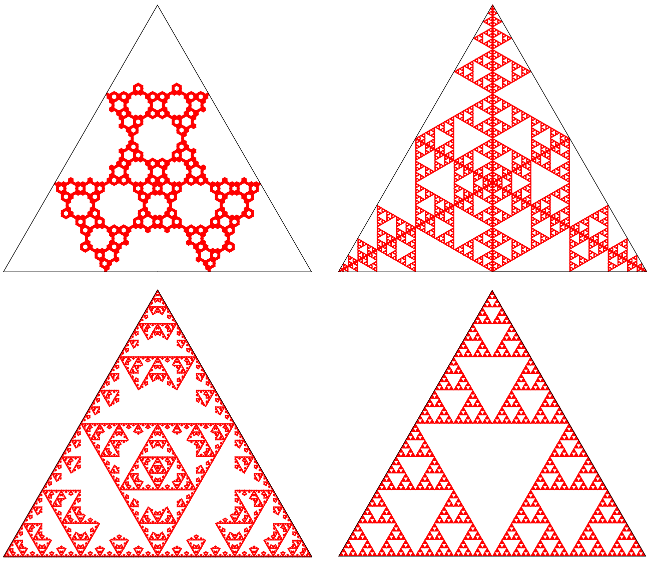

Figure 5.3: $\mathrm{eqv}\ \mathcal{D}^0_{\mathrm{rm}}$ (upper left), $\mathrm{eqv}\ \mathcal{D}^1_{\mathrm{rm}}$ (upper right), $\mathrm{eqv}\ \mathcal{D}^2_{\mathrm{rm}}$ (lower left), $\mathrm{eqv}\ \mathcal{D}^3_{\mathrm{rm}}$ (lower right)

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