On the Relation between Dwell-Time and Small-Gain Conditions for Interconnected Impulsive Systems

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Abstract: For interconnection of impulsive systems a relation between dwell-time and small-gain conditions is considered in this paper. In particular we show how the choice of gains or supply rates affects the restriction on time intervals between impulses to assure stability properties.

Keywords: Hybrid Systems, Impulsive Systems, Small-Gain Conditions, Dwell-Time Conditions, Input-to-State Stability.

1. INTRODUCTION

An impulsive system combines continuous and discontinuous behavior in one model. Such systems have many real-world applications and are very interesting from theoretical point of view. They can be considered as a subclass of hybrid systems. Since stability properties are of great importance for applications a lot of research was devoted to investigation of stability for such systems, see Samoilenko and Perestyuk [1995], Hespanha et al. [2008], and Chen and Zheng [2009]. Since impulsive systems combine two types of behavior it can happen that one of them stabilizes the system while another one destabilizes it. In this case one needs to restrict the number of impulses per time unite. Such restrictions are called dwell-time conditions. Several types of dwell-time conditions were developed in the literature, see Dashkovskiy and Mironchenko [2012].

Many applications lead to consideration of interconnected systems. It is known that even if each subsystem of an interconnection is stable the whole system can be unstable. One of the possible frameworks to study stability of interconnections is input-to-state stability, see Sontag [1989]. Small-gain condition that guarantee stability of large-scale interconnected impulsive systems were developed recently in Dashkovskiy et al. [2012]. This conditions are used in a combination with a dwell-time condition.

In the current paper we will study the interplay between these two conditions. First we provide a linear example to illustrate this interplay. However the most interesting and essentially more complicate is the case of nonlinear systems. Several preliminary results will be shown and illustrated for this case. We will also briefly show the open problems and explain the direction for further research. In the following section we introduce notation and necessary notions as well we recall some known related results. Section 3 explains the motivation, where for simplicity an interconnection of linear systems is considered. In section 4 we give our main results for the case of linear supply rates and indicate in section 5 what is expected in case of nonlinear supply rates. A nonlinear example is also provided there. Section 6 concludes the paper.

2. PRELIMINARY

First of all, let us review some important background. Denote by $\mathbb{R}$ the set of real numbers, $\mathbb{R}_{\geq 0} = [0, \infty)$, $\mathbb{R}^n$ denotes n-dimensional Euclidean space, $\mathbb{N} = \{0, 1, 2, 3, ...\}$, and $\varnothing$ denotes the empty set. For any vectors $a, b \in \mathbb{R}^n$, the relation $a > b$ is defined by $a_i > b_i$ for all $i = 1, 2, \dots, n$ and its logical negation is denoted by $a \not> b$ meaning that there exists $i \in \{1, 2, \dots, n\}$ such that $a_i \leq b_i$. The relations $\geq, <, \leq$ and their negation are also defined in the same manner. Denote by a^{T} the transposition of a vector $a \in \mathbb{R}^n$. By both of $\langle a, b\rangle$ and $a \cdot b$ denote the vector scalar product of a and b. By ∇ we denote the standard vector gradient. The Lebesgue spaces are denoted by $\mathcal{L}_p$, and $\|\cdot\|_p$ denotes the $\mathcal{L}_p$-norm, $|\cdot|$ denotes the Euclidean norm . A continuous function α is called a class $\mathcal{K}$-function, $\alpha \in \mathcal{K}$, if $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is strictly increasing, and $\alpha(0) = 0$. In addition, $\alpha \in \mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ is unbounded. A continuous function is called positive definite $\alpha \in \mathcal{P}$, if $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfies $\alpha(x) = 0$ iff $x = 0$. A continuous function $\alpha \in \mathcal{L}$ if $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is strictly decreasing, and $\lim_{t\to\infty} \alpha(t) = 0$. A function $\beta \in \mathcal{KL}$ if $\beta : \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}$, $\beta(\cdot, t) \in \mathcal{K}$ for all $t > 0$, and $\beta(r, \cdot) \in \mathcal{L}$ for all $r > 0$.

2.1 Impulsive Systems

Consider a system Σ consisting of n subsystems called Σ_i for $i = 1, 2, \dots, n$ where $n \geq 2$. Let $x = \left[x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, \dots, x_n^{\mathrm{T}}\right]^{\mathrm{T}} \in$

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$\mathbb{R}^N$ be the state vector of Σ where $x_i \in \mathbb{R}^{N_i}$ denotes the state vector of Σ_i, and $\sum N_i = N$. Let $T = \{t_1, t_2, \ldots,\}$ be a nondecreasing sequence of impulse times without finite accumulation points. Suppose that the dynamics of Σ_i are governed by

$$\Sigma_i : \begin{array}{ll} \dot{x}_i(t) = f_i\left(x_1(t), \ldots, x_n(t), u(t)\right) & , t \in [t_0, \infty) \setminus T, \\ x_i^+(t) = g_i\left(x_1(t), \ldots, x_n(t), u(t)\right) & , t \in T, \end{array} \tag{1}$$

where $f, g : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N$ and $t_0 \in \mathbb{R}_{\geq 0}$ denotes an initial time.

The first equation of (1) exhibits the continuous dynamics of Σ_i. Together with the second equation, the jumps of a state at impulse times, discrete dynamics, of Σ_i are described. The equations (1) with impulse times T define an impulsive system.

Our assumptions of the subsystem Σ_i are listed as follows: The external inputs $u \in \mathcal{L}_\infty\left([t_0, \infty), \mathbb{R}^M\right)$ and $x_j \in \mathcal{L}_\infty\left([t_0, \infty), \mathbb{R}^{N_j}\right)$ where $i \neq j$ are right-continuous and possesses left limit. The functions f_i and g_i are assumed to be such that for any initial condition there exists a unique solution for each subsystem Σ_i, $i = 1, \ldots, n$.

An interconnection of impulsive systems (1) can be written as one impulsive system as follows

$$\Sigma : \begin{array}{ll} \dot{x}(t) = f\left(x(t), u(t)\right), & t \in [t_0, \infty) \setminus T, \\ x^+(t) = g\left(x(t), u(t)\right), & t \in T, \end{array} \tag{2}$$

where $f = \left[f_1^{\mathrm{T}}, f_2^{\mathrm{T}}, \ldots, f_n^{\mathrm{T}}\right]^{\mathrm{T}} : \mathbb{R}^{N+M} \to \mathbb{R}^N$ and $g = \left[g_1^{\mathrm{T}}, g_2^{\mathrm{T}}, \ldots, g_n^{\mathrm{T}}\right]^{\mathrm{T}} : \mathbb{R}^{N+M} \to \mathbb{R}^N$. Note that to write (1) as one impulsive system (2) it is necessary to require that the set T of the impulse times is the same for each Σ_i. Next we introduce the stability notion that we will use throughout the paper.

2.2 Input-to-State Stability

Definition 1. (ISS). The impulsive system Σ_i is input-to-state stable if there exist $\beta_i \in \mathcal{KL}$, $\gamma_{ij} \in \mathcal{K}_\infty$ with $\gamma_{ii} := 0$, $\gamma_i \in \mathcal{K}_\infty$ such that for all initial conditions $x_i(t_0)$, for all inputs u, x_j where $i \neq j$ it holds

$$|x_i(t)| \leq \beta_i\left(|x_i(t_0)|, t - t_0\right) + \sum_{j=1}^{n} \gamma_{ij}\left(|x_j|\right) + \gamma_i\left(\|u\|_\infty\right). \tag{3}$$

The functions γ_{ij} and γ_i are called gains.

An important tool to investigate this kind of stability prvide ISS-Lyapunov functions that can be defined as follows

Definition 2. (ISS-Lyapunov Function). A smooth function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_{\geq 0}$ is an ISS-Lyapunov function for the impulsive system Σ_i if there exist $\psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$ such that for all $x_i \in \mathbb{R}^{N_i}$

$$\psi_{i1}(|x_i|) \leq V_i(x_i) \leq \psi_{i2}(|x_i|), \tag{4}$$

and there exist $\gamma_{ij} \in \mathcal{K}_\infty$ with $\gamma_{ii} := 0$, $\gamma_i \in \mathcal{K}_\infty$, $\alpha_i \in \mathcal{P}$ and $\varphi_i \in \mathcal{P}$ such that for all $x \in \mathbb{R}^N$ and for all $u \in \mathbb{R}^M$

$$V_i(x_i) \geq \sum_{j=1}^{n} \gamma_{ij}\left(V_j(x_j)\right) + \gamma_i\left(\|u\|_\infty\right) \tag{5}$$

implies

$$\dot{V}_i(x_i) := \langle \nabla V_i(x_i), f_i(x, u) \rangle \leq -\varphi_i(V_i(x_i)), \tag{6}$$

and

$$V_i(g_i(x, u)) \leq \alpha_i(V_i(x_i)) + \sum_{j=1}^{n} \gamma_{ij}\left(V_j(x_j)\right) + \gamma_i\left(\|u\|_\infty\right). \tag{7}$$

Sontag and Wang [1995] showed that for systems without impulses ($T = \varnothing$) the existence of an ISS-Lyapunov function for Σ_i is equivalent to its ISS property. However in case of $T \neq \varnothing$ one needs additionally to apply restrictions on the sequence T to assure ISS. One of them is called fixed dwell-time condition and was used in Dashkovskiy and Mironchenko [2012]:

Theorem 3. (Fixed Dwell-Time Condition). Let V_i be an ISS-Lyapunov function for Σ_i, and φ_i, α_i be as in the Definition 2. If there exists $\theta, \delta_i > 0$ such that for all $a > 0$

$$\int_a^{\alpha_i(a)} \frac{\mathrm{d}s}{\varphi_i(s)} \leq \theta - \delta_i, \tag{8}$$

then Σ_i is ISS for all impulse time sequences

$$T \in S_\theta := \left\{\{t_k\}_{k=1}^\infty \subset \mathbb{R}_{\geq 0} : \theta \leq t_{k+1} - t_k\right\}. \tag{9}$$

Definition 4. (Gain Operator). Let V_i be an ISS-Lyapunov function of the impulsive system Σ_i with corresponding gains $\gamma_{ij} \in \mathcal{K}_\infty$. The gain operator $\Gamma : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ is defined by

$$\Gamma(s) := \left(\sum_{j=1}^{n} \gamma_{1j}(s_j), \ldots, \sum_{j=1}^{n} \gamma_{nj}(s_j)\right)^T, \tag{10}$$

where $s = (s_1, \ldots, s_n) \in \mathbb{R}^n$.

Let us recall the notion of the gain operator and the small-gain condition for interconnected systems. Recall how the small-gain condition can be used Dashkovskiy et al. [2010]

Theorem 5. (The Small-Gain Condition). Let V_i be an ISS-Lyapunov function for Σ_i with $T = \varnothing$ and the corresponding gains be $\gamma_{ij} \in \mathcal{K}_\infty$. If there exists some $\rho \in \mathcal{K}_\infty$ such that the gain operator Γ defined in (10) satisfies

$$D \circ \Gamma(s) \not\geq s, \quad \forall s \in \mathbb{R}^n, s \neq 0, \quad D := \mathrm{diag}(\rho, \ldots, \rho) \tag{11}$$

then there exists an ISS-Lyapunov function for the interconnected system Σ implying that Σ is ISS.

For $T \neq \varnothing$ a combination of the dwell-time condition and the small-gain condition can be used to guarantee stability of the interconnection of impulsive systems.

Recall that, in case of linear gains, the condition (11) is equivalent to

$$\Gamma_{\max}(s) \not\geq s, \quad \forall s \in \mathbb{R}^n, s \neq 0, \tag{12}$$

where

$$\Gamma_{\max}(s) := \left(\max_{j=1}^{n} \gamma_{1j}(s_j), \ldots, \max_{j=1}^{n} \gamma_{nj}(s_j)\right), \tag{13}$$

Small-gain conditions (11) and (12) require that the gains of subsystems are small enough. Another way to equivalently state (12) is to require that all the gain cycle compositions are less than the identity, that is for any $p > 1$ it holds

$$\gamma_{k_1 k_2} \circ \gamma_{k_2 k_3} \circ \dots \gamma_{k_{p-1} k_p}(s) < s, \ \forall s > 0 \qquad (14)$$

where $(k_1, k_2, \dots, k_p) \in \{1, 2, \dots, n\}^p$ and $k_1 = k_p$. It is known that if all of gains γ_{ij} in Γ are linear functions, then the small gain condition (11) is equivalent to

$$\rho(\Gamma) < 1 \qquad (15)$$

where $\rho(\Gamma)$ denotes the spectral radius of the linear operator Γ, see Dashkovskiy et al. [2007]. To see the relation between a small-gain condition and a dwell time condition, a simple motivating example is provided in the next section.

3. MOTIVATION

Next we consider the case of linear systems to illustrate the interplay between the small-gain and the dwell-time condition that motivates the more general problem of this interplay for nonlinear systems. Let us consider the following interconnected linear impulsive systems

$$\Sigma_i : \begin{array}{ll} \dot{x}_i = -x_i + \sum\limits_{\substack{j=1 \\ i \neq j}}^{n} \gamma_{ij} x_j + u_i, & t \in [0, \infty) \setminus T \\ x_i^+ = e^{-d} x_i, & t \in T, \end{array} \qquad (16)$$

where $2 \leq n \in \mathbb{N}$, $\gamma_{ij} > 0$, $u_i \in \mathcal{L}_\infty[0, \infty)$, $d \in \mathbb{R}$. For simplicity we consider $d < 0$, i.e., the case where jumps at the impulse times destabilize the system.

Let us show that Σ_i possesses an ISS-Lyapunov function and calculate the corresponding gains. Let

$$V_i(x_i) := |x_i|.$$

For any $0 < \alpha_i < 1$ it follows that

$$V_i(x_i) \geq \sum_{\substack{j=1 \\ i \neq j}}^{n} \left(\frac{\gamma_{ij}}{1 - \alpha_i} \right) |x_i| + \frac{|u_i|}{1 - \alpha_i}$$

implies

$$\begin{aligned} \dot{V}_i(x_i) &= \operatorname{sign}(x_i) \left(-x_i + \sum_{\substack{j=1 \\ i \neq j}}^{n} \gamma_{ij} x_j + u_i \right) \\ &\leq - |x_i| + \sum_{\substack{j=1 \\ i \neq j}}^{n} \gamma_{ij} |x_j| + |u_i| \\ &\leq -\alpha_i |x_i| = -\alpha_i V_i(x_i). \end{aligned}$$

This shows that V_i is an ISS-Lyapunov function for Σ_i with $T = \varnothing$ and that the gains can be taken as

$$\Gamma_{ij} := \begin{cases} 0 & \text{if } i = j \\ \dfrac{\gamma_{ij}}{1 - \alpha_i} & \text{if } i \neq j \end{cases}$$

so that the linear gain operator is given by $\Gamma := [\Gamma_{ij}]_{n \times n}$.

Note that the gains are not unique and that in view of the application of the small-gain condition it is desired to have possibly small gains Γ_{ij} by adjusting $0 < \alpha_i < 1$, i.e., taking α_i close to 0. However in any case the gains are bounded from below by $\gamma_{ij} < \Gamma_{ij}$. As well it is important to notice that α_i characterizes the decay rate for Σ_i. Due to the dwell-time condition taking α_i close to 0 will lead to the conclusion that the time intervals between jumps have to be close to ∞, see below. This shows the trade-off between the choice of gains to be small and decay rates to be large.

To guarantee the ISS property for Σ_i we require that the set T satisfies $T \in S_\theta$, i.e., a dwell time condition

$$\frac{-d}{\alpha_i} < \theta \leq t_{k+1} - t_k.$$

is satisfied. Since at any impulse holds

$$V_i\left(x_i^+\right) = V_i\left(e^{-d} x_i\right) = \left|e^{-d} x_i\right| = e^{-d} V_i\left(x_i\right),$$

there exists $\delta_i > 0$ for all $a > 0$ such that

$$\int_a^{e^{-d} a} \frac{\mathrm{d}s}{\alpha_i s} = \frac{-d}{\alpha_i} \leq \theta - \delta_i.$$

Therefore, the subsystem Σ_i is ISS since a dwell-time condition is satisfied.

To guarantee the ISS property for the interconnection of $\Sigma_1, \Sigma_2, \dots, \Sigma_n$, where $n \geq 2$ and finite we firstly require that Γ satisfies the small-gain condition $\rho(\Gamma) < 1$. Secondly, we are going to show that Σ is ISS if it holds

$$\frac{-d}{\min_i \alpha_i} < \theta \leq t_{k+1} - t_k. \qquad (17)$$

To this end we construct an ISS-Lyapunov function for the interconnected impulsive system Σ. Since $\rho(\Gamma) < 1$, there exist $s = (s_1, s_2, \dots, s_n) \in \mathbb{R}_+^n$ such that

$$s_i > \sum_{\substack{j=1 \\ i \neq j}}^{n} \left(\frac{\gamma_{ij}}{1 - \alpha_i} \right) s_j.$$

Let $x := (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $u := (u_1, u_2, \dots, u_n) \in \mathbb{R}^n$ and

$$V(x) := \max_i \frac{V_i(x_i)}{s_i}, \text{and } \gamma(|u|) := \max_i \frac{|u|}{\kappa_i} \qquad (18)$$

where

$$\kappa_i := (1 - \alpha_i) \left(s_i - \sum_{\substack{j=1 \\ i \neq j}}^{n} \left(\frac{\gamma_{ij}}{1 - \alpha_i} \right) s_j \right).$$

Then $V(x) \geq \gamma(|u|)$ implies

$$\dot{V}(x) \leq -\alpha V(x), \text{ with } \alpha := \min_i \alpha_i.$$

At any impulsive time it holds

$$V(x^+) = \max_i \frac{V_i(x_i^+)}{s_i} = \max_i \frac{e^{-d} |x_i|}{s_i} = e^{-d} V(x).$$

Therefore by (17), for all $a > 0$, there exists $\delta > 0$ such that

$$\int_a^{e^{-d} a} \frac{\mathrm{d}s}{\alpha s} = \frac{-d}{\alpha} \leq \theta - \delta, \qquad (19)$$

and we can conclude that the interconnection is ISS.

Note that if some α_i is close to zero then α is close to zero. This implies that the the distance θ between any two impulse times needs to be close to infinity.

Now let $\rho\left([\gamma_{ij}]_{n\times n}\right) = 1-\varepsilon$ for some $0<\varepsilon<1$. Then since $\gamma_{ij} > \Gamma_{ij} \geq 0$ we have $\rho(\Gamma) < \rho\left([\gamma_{ij}]_{n\times n}\right) = 1-\varepsilon$

If in particular the spectral radius of the interconnecting matrix approaches 1, i.e., $\varepsilon \to 0$ then $\alpha \to 0$. This can be seen for example from the small-gain condition stated in the cycle from:

$\rho(\Gamma)$ approaches 1 means that

$$\frac{\gamma_{ij_1}}{1-\alpha_i} \cdot \frac{\gamma_{j_1 j_2}}{1-\alpha_{j_1}} \cdot \ldots \cdot \frac{\gamma_{j_c i}}{1-\alpha_{j_c}} \to 1.$$

Let $\gamma_{ij_1}\gamma_{j_1j_2}\ldots\gamma_{j_ci}$ approach 1, then it follows

$$(1-\alpha_i)(1-\alpha_{j_1})\ldots(1-\alpha_{j_c}) \to 1$$

and in particular $\alpha_i \to 0$ and hence $\alpha \to 0$. From (19) it follows that $\theta \to \infty$.

As a result for this example we obtain the following:

Proposition 6. Let the interconnection matrix of (16) satisfy $\rho\left([\gamma_{ij}]_{n\times n}\right) = 1-\varepsilon$ for some $0<\varepsilon<1$ then there exists some finite θ so that if the time distance between any two impulse times satisfies $t_{k+1}-t_k \geq \theta$, the interconnection Σ is ISS. Moreover it holds $\theta \to \infty$ for $\varepsilon \to 0$.

4. MAIN RESULTS

In the above example we have seen that the choice of ISS gains affects the choice of theta in the dwell-time condition. In a simple example we have seen how these gains can be adjusted to cope with the dwell-time condition. Coming back to the case of nonlinear systems given a general form we cannot calculate the gains explicitly and hence we have no possibility to adjust the gains. For this reason we require that each subsystem is equipped with an ISS-Lyapunov function satisfying an ISS dissipative inequality with given supply rates. Recall that existence of such an ISS-Lyapunov function is equivalent to the existence of Lyapunov function defined above.

Theorem 7. Given an impulsive system Σ, defined in (2), consisting of subsystems Σ_i defined in (1) for $i=1,2,\ldots,n$ where $2 \leq n < \infty$. Let each subsystem Σ_i admit a smooth function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_{\geq 0}$ satisfying the following:

(1) There exist $\psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$ such that for all $x_i \in \mathbb{R}^{N_i}$

$$\psi_{i1}(|x_i|) \leq V_i(x_i) \leq \psi_{i2}(|x_i|). \tag{20}$$

(2) There exist $\sigma_{ij} > 0$ with $\sigma_{ii} := 0$, $\sigma_i > 0$, and $d_i \in \mathbb{R}$ such that it holds

(a) For all $x \in \mathbb{R}^N$ and for all $u \in \mathbb{R}^M$

$$\begin{aligned}\dot{V}_i(x_i) &:= \langle \nabla V_i(x_i), f_i(x,u)\rangle \\ &\leq -V_i(x_i) + \sum_{i=1}^{j} \sigma_{ij} V_j(x_j) + \sigma_i \|u\|_\infty.\end{aligned} \tag{21}$$

(b) For all $x \in \mathbb{R}^N$ and for all $u \in \mathbb{R}^M$

$$V_i(g_i(x,u)) \leq e^{-d_i} V_i(x_i). \tag{22}$$

(c) There exists $0<\varphi_i<1$ such that for some $\varepsilon \in (0,1)$ and for any $p>1$ it holds

$$\rho\left([\sigma_{ij}]_{n\times n}\right) := 1-\varepsilon < \prod_{k=k_1}^{k_p}(1-\varphi_k) \tag{23}$$

where $(k_1,k_2,\ldots,k_p) \in \{1,2,\ldots,n\}^p$ and $k_1 = k_p$.

Then, there exists $\theta > 0$ such that Σ is ISS for all impulsive time sequences $T \in S_\theta$ defined in (9). Moreover, θ approaches to infinity as ε approaches to zero.

Proof. For any $0<\varphi_i<1$, we have that

$$V_i(x_i) \geq \sum_{j=1}^{n}\left(\frac{\sigma_{ij}}{1-\varphi_i}\right)V_j(x_j) + \left(\frac{\sigma_i}{1-\varphi_i}\right)\|u\|_\infty$$

implies

$$\begin{aligned}\dot{V}_i(x_i) &\leq -c_i V_i(x_i) + \sum_{i=1}^{j}\sigma_{ij}V_j(x_j) + \sigma_i\|u\|_\infty \\ &\leq -\varphi_i V_i(x_i).\end{aligned}$$

From (23) it follows that there exists $s=(s_1,s_2,\ldots,s_n) \in \mathbb{R}^n_+$ such that

$$s_i > \sum_{\substack{j=1\\ i\neq j}}^{n}\left(\frac{\sigma_{ij}}{1-\varphi_i}\right)s_j,$$

Therefore, define

$$V(x) := \max_i \frac{V_i(x_i)}{s_i}, \quad \forall x = (x_1,x_2,\ldots,x_n) \in \mathbb{R}^N$$

and

$$\gamma(r) := \max_i \frac{r}{\kappa_i}, \quad \forall r \in \mathbb{R}_{\geq 0}$$

where

$$\kappa_i := (1-\varphi_i)\left(s_i - \sum_{j=1}^{n}\left(\frac{\sigma_{ij}}{1-\varphi_i}\right)s_j\right).$$

It follows that

$$V(x) \geq \gamma(\|u\|_\infty) \Rightarrow \dot{V}(x) \leq -\varphi V(x),$$

and

$$\begin{aligned}V(g(x,u)) &= \max_i \frac{V_i(g_i(x,u))}{s_i} \\ &\leq \max_i \frac{e^{-d_i}V_i(x_i)}{s_i} \leq e^{-d}V(x,u)\end{aligned}$$

where

$$\varphi = \min_i\{\varphi_i\}, \text{ and } d = -\max_i\{-d_i\}.$$

Hence the interconnected impulsive system Σ is ISS since the dwell time condition (8) holds, i.e., there exist $\theta, \delta > 0$ for all $a>0$ such that

$$\int_a^{e^{-d}a}\frac{\mathrm{d}s}{\varphi s} = \frac{-d}{\varphi} \leq \theta - \delta. \tag{24}$$

Finally, suppose $\varepsilon \to 0^+$. From (23) it means that

$$\prod_{k=k_1}^{k_p}(1-\varphi_k) \to 1.$$

Please note that

$$\begin{aligned}\prod_{k=k_1}^{k_p}(1-\varphi_k) = 1-(&\varphi_{k_1}\prod_{k=k_2}^{k_p}(1-\varphi_k) + \varphi_{k_2}\prod_{k=k_3}^{p}(1-\varphi_k) \\ &+\ldots+\varphi_{k_{p-1}}(1-\varphi_{k_p}) + \varphi_{k_p}).\end{aligned}$$

We can conclude that φ_k approaches to zero. Therefore, φ also approaches to zero. To hold the dwell-time condition (24), θ eventually approaches to infinity.

5. NONLINEAR SUPPLY RATES

In the previous section we have considered the case of linear supply rates. If they are nonlinear functions then more research have to be done to obtain a counterpart of the above results. Here we consider an example with two interconnected nonlinear impulsive systems and show that similar effects are expected to happen in case of nonlinear supply rates.

Example 8. Let $\Sigma_i,\ i = 1, 2$ be given by

$$\begin{aligned} \dot{x}_i &= -(1+\varepsilon)x_i + \min\left\{\sqrt{x_j}, x_j^2\right\} + u_i \quad &, t \in [t_0,\infty) \setminus T, \\ x_i^+ &= e^{-d}x_i &, t \in T, \end{aligned} \tag{25}$$

where $j = 1, 2,\ j \neq i,\ d < 0 < \varepsilon$.

Choose any $\alpha \in (0, \varepsilon)$. Consider $V_i(x_i) := |x_i|$ as a candidate for a Lyapunov function and let the gains be given by

$$\gamma_{ij}(r) := \frac{\min\left\{\sqrt{r}, r^2\right\}}{1+\varepsilon-\alpha} \quad , \forall r \in \mathbb{R}_{\geq 0}, i \neq j \tag{26}$$

$$\gamma_i(r) := \frac{r}{1+\varepsilon-\alpha} \quad , \forall r \in \mathbb{R}_{\geq 0} \tag{27}$$

It is easy to check that

$$V_i(x_i) \geq \gamma_{ij}(|x_j|) + \gamma_i(\|u_i\|_\infty)$$

implies

$$\dot{V}_i(x_i) \leq -(1+\varepsilon)|x_i| + \min\left\{\sqrt{x_j}, x_j^2\right\} + \|u_i\|_\infty \tag{28}$$

$$\leq -\alpha|x_i|, \tag{29}$$

and

$$V_i(x_i^+) = e^{-d}|x_i|. \tag{30}$$

This shows that V_i is an ISS-Lyapunov function for Σ_i. As well it is easy to check that $\gamma_{12} \circ \gamma_{21}(s) < s,\ \forall s > 0$, i.e., the small-gain condition is satisfied. This implies that there exists a Lyapunov function V for the interconnection of Σ_1 and Σ_2 such that

$$V(x) \geq \gamma(\|u_i\|_\infty) \Rightarrow \dot{V}(x) \leq -\alpha V(x) \quad , \exists \gamma \in \mathcal{K}_\infty \tag{31}$$

and

$$V(x^+) \leq e^{-d}V(x). \tag{32}$$

Therefore, the interconnection is ISS, provided the impulse times satisfy the dwell-time condition: there exist $\theta, \delta > 0$ such that

$$\int_a^{e^{-d}a} \frac{\mathrm{d}s}{\alpha s} = \frac{-d}{\alpha} \leq \theta - \delta. \tag{33}$$

If the distance between impulse times is less then θ then the behaviour of the interconnection can be unstable.

Figures 1 and 2 illustrate simulations of this example with parameters $\varepsilon = 0.2, \alpha = 0.1, d = -0.2$, initial conditions $x_1(t_0) = 1, x_2(t_0) = 2, t_0 = 0$, external inputs

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{cases} \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \text{if } 0 < t \leq 2, \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \text{otherwise}, \end{cases} \tag{34}$$

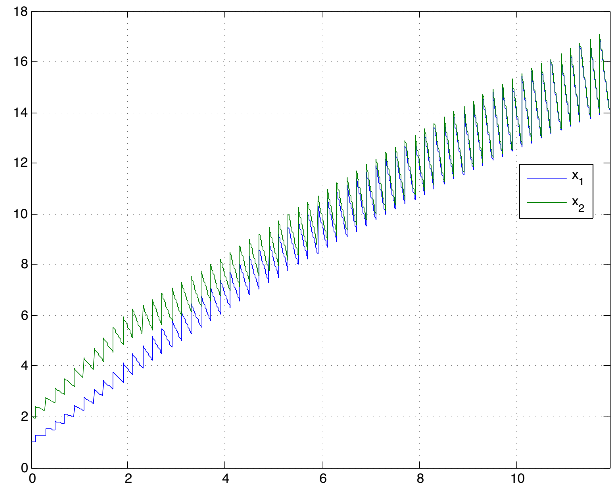

Fig. 1. A simulation of *Example 8* with impulse times $T = \{0.1, 0.3, 0.5, 0.7, \ldots\}$

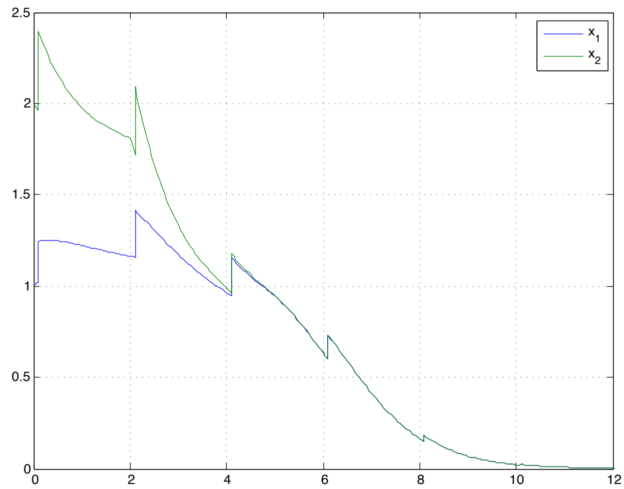

Fig. 2. A simulation of *Example 8* with impulse times $T = \{0.1, 2.1, 2.2, 2.3, \ldots\}$

and different impulse times T indicated below each figures: the first figure corresponds to rather frequent jumps, where the dwell-time condition is not satisfied and hence the behaviour is unstable. On the second figure the time distance between impulses satisfies the required dwell-time condition and the simulation shows a stable behaviour.

We want to investigate in case the compositions of γ_{ij} are extremely closed to identity. There are ways to make an investigation. First of all, in a case of an extremely small ε, namely $\varepsilon \to 0^+$, it follows that θ becomes extremely large to hold (33). On the another route, we just choose α to be very closed to ε, i.e., $\alpha \to \varepsilon$. Eventually, θ is finite to hold (33), i.e., choose any $\theta \in \left(-d\varepsilon^{-1} + \delta, \infty\right)$. Since gains are not unique, we avoid to choose the second route. Therefore, the following can be obtained.

Proposition 9. There exists finite θ such that if the time distance between any two impulse times satisfies $t_{k+1} - t_k \geq \theta$, the interconnection of (25) is ISS. Moreover it holds $\theta \to \infty$ as $\varepsilon \to 0$.

6. CONCLUSIONS

In this work we have shown that in case of interconnected impulsive systems one have to take care of the ISS gains to be small enough and the dependence of the dwell time on these gains. The relations between them is shown. In case of nonlinear gains or supply rates more investigation needs to be done and this is the currently under investigation.

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