Flatness of Two-Input Control-Affine Systems Linearizable via One-Fold Prolongation

Florentina NICOLAU * **Witold RESPONDEK** **

* *INSA-Rouen, LMI, 76801 Saint-Etienne-du-Rouvray, France (e-mail: florentina.nicolau@insa-rouen.fr).*
** *INSA-Rouen, LMI, 76801 Saint-Etienne-du-Rouvray, France (e-mail: witold.respondek@insa-rouen.fr).*

Abstract: We study flatness of two-input control-affine systems, defined on an n-dimensional state-space. We give a complete geometric characterization of systems that become static feedback linearizable after a one-fold prolongation of a suitably chosen control. They form a particular class of flat systems: they are of differential weight $n+3$. We provide a system of first order PDE's to be solved in order to find all minimal flat outputs. We illustrate our results by two examples: the induction motor and the polymerization reactor.

Keywords: flatness, flat outputs, linearization.

1. INTRODUCTION

In this paper, we study flatness of nonlinear control systems of the form

$$\Xi : \quad \dot{x} = F(x, u),$$

where x is the state defined on a open subset M of $\mathbb{R}^n$ and u is the control taking values in an open subset U of $\mathbb{R}^m$ (more generally, an n-dimensional manifold M and an m-dimensional manifold U). The dynamics F are smooth and the word smooth will always mean $\mathcal{C}^\infty$-smooth.

The notion of flatness has been introduced in control theory in the 1990's by Fliess, Lévine, Martin and Rouchon (Fliess et al. [1992, 1995], see also Martin [1992], Jakubczyk [1993], Pomet [1995]) and has attracted a lot of attention because of its multiple applications in the problem of trajectory tracking and motion planning (Fliess et al. [1999], Pomet [1997], Pereira da Silva and Corrêa Filho [2001], Martin et al. [2003], Respondek [2003], Schlacher and Schoeberl [2007], Lévine [2009]). The fundamental property of flat systems is that all their solutions may be parametrized by m functions and their time-derivatives, m being the number of controls. More precisely, the system $\Xi : \dot{x} = F(x, u)$ is *flat* if we can find m functions, $\varphi_i(x, u, \ldots, u^{(r)})$, for some $r \geq 0$, called *flat outputs*, such that

$$x = \gamma(\varphi, \ldots, \varphi^{(s)}) \text{ and } u = \delta(\varphi, \ldots, \varphi^{(s)}), \qquad (1)$$

for a certain integer s, where $\varphi = (\varphi_1, \ldots, \varphi_m)$. Therefore all state and control variables can be determined from the flat outputs without integration and all trajectories of the system can be completely parameterized.

It is well known that systems linearizable via invertible static feedback are flat. Their description (1) uses the minimal possible, which is $n + m$, number of time-derivatives of the components of flat outputs φ_i. For any flat system, that is not static feedback linearizable, the minimal number of derivatives needed to express x and u (which will be called the differential weight) is thus bigger than $n + m$ and measures actually the smallest possible dimension of a precompensator linearizing dynamically the system. Any single input-system is flat if and only if it is static feedback linearizable (and thus of differential weight n+1), see Charlet et al. [1991], Pomet [1995]. Therefore the simplest systems for which the differential weight is bigger than $n + m$ are systems with two controls linearizable via one-dimensional precompensator, thus of differential weight $n + 3$. They form the class that we are studying in the paper: our goal is to give a geometric characterization of two-input control-affine systems that become static feedback linearizable after a one-fold prolongation of a suitably chosen control.

The paper is organized as follows. In Section 2, we recall the definition of flatness and define the notion of differential weight of a flat system. In Section 3, we give our main results. We characterize two-input control-affine systems linearizable via one-fold prolongation, that is, flat systems, of differential weight $n+3$. We give in Section 4 a system of first order PDE's to be solved in order to find all minimal flat outputs. We illustrate our results by two examples in Section 5 and provide sketches of the proofs in Section 6.

2. FLATNESS

Flat systems form a class of control systems, whose set of trajectories can be parameterized by a finite number of functions and their time-derivatives. Fix an integer $l \geq -1$ and denote $M^l = M \times U \times \mathbb{R}^{ml}$ and $\bar{u}^l = (u, \dot{u}, \ldots, u^{(l)})$. For $l = -1$, we put $M^{-1} = M$ and $\bar{u}^{-1}$ is empty.

Definition 2.1. The system $\Xi : \dot{x} = F(x, u)$ is *flat* at $(x_0, \bar{u}^l_0) \in M^l$, for $l \geq -1$, if there exists a neighborhood $\mathcal{O}^l$ of $(x_0, \bar{u}^l_0)$ and m smooth functions $\varphi_i = \varphi_i(x, u, \dot{u}, \ldots, u^{(l)})$, $1 \leq i \leq m$, defined in $\mathcal{O}^l$, having the following property : there exist an integer s and smooth functions γ_i, $1 \leq i \leq n$, and δ_j, $1 \leq j \leq m$, such that

$$x_i = \gamma_i(\varphi, \dot{\varphi}, \ldots, \varphi^{(s)}) \text{ and } u_j = \delta_j(\varphi, \dot{\varphi}, \ldots, \varphi^{(s)})$$

along any trajectory $x(t)$ given by a control $u(t)$ that satisfies $(x(t), u(t), \ldots, u^{(l)}(t)) \in \mathcal{O}^l$, where $\varphi = (\varphi_1, \ldots, \varphi_m)$ and is called a *flat output.*

When necessary to indicate the number of derivatives of u on which the flat outputs φ_i depend, we will say that the system Ξ is $(x, u, \cdots, u^{(r)})$-flat if $u^{(r)}$ is the highest derivative on which φ_i depend and in the particular case $\varphi_i = \varphi_i(x)$, we will say that the system is x-flat. In general, r is smaller than the integer l needed to define the neighborhood $\mathcal{O}^l$ which, in turn, is smaller than the number of derivatives of φ_i that are involved (in our study $r = -1$ and $l = -1$ or 0). The minimal number of derivatives of components of a flat output, needed to express x and u, will be called the differential weight of a flat output and will be formalized as follows. By definition, for any flat output φ of the flat system Ξ there exist integers $s_1, \ldots, s_m$ such that

$$\begin{aligned} x &= \gamma(\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(s_1)}, \ldots, \varphi_m, \dot{\varphi}_m, \ldots, \varphi_m^{(s_m)}) \\ u &= \delta(\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(s_1)}, \ldots, \varphi_m, \dot{\varphi}_m, \ldots, \varphi_m^{(s_m)}), \end{aligned}$$

and let $(\lambda_1, \ldots, \lambda_m)$ be the smallest m-tuple of integers verifying this property (which always exists, see Respondek [2003]). We will call $\sum_{i=1}^m (\lambda_i + 1)$ the differential weight of φ. A flat output φ of Ξ is called *minimal* if its differential weight is the lowest among all flat outputs of Ξ. We define the *differential weight* of a flat system to be the differential weight of a minimal flat output.

Consider a control-affine system

$$\Sigma \;:\; \dot{x} = f(x) + \sum_{i=1}^m u_i g_i(x),$$

where f and $g_1, \cdots, g_m$ are smooth vector fields on M. The system Σ is linearizable by static feedback if it is equivalent via a diffeomorphism $z = \phi(x)$ and an invertible feedback transformation, $u = \alpha(x) + \beta(x)v$, to a linear controllable system $\Lambda : \dot{z} = Az + Bv$. The problem of static feedback linearization was solved by Jakubczyk and Respondek [1980] and Hunt and Su [1981] who gave the following geometric necessary and sufficient conditions. Define the distributions $\mathcal{D}^{i+1} = \mathcal{D}^i + [f, \mathcal{D}^i]$, where $\mathcal{D}^0 = \text{span}\{g_1, \cdots, g_m\}$. Σ is locally static feedback linearizable if and only if for any $i \geq 0$, the distributions $\mathcal{D}^i$ are of constant rank, *involutive* and $\mathcal{D}^{n-1} = TM$. Therefore the geometry of static feedback linearizable systems is given by the following sequence of nested involutive distributions :

$$\mathcal{D}^0 \subset \mathcal{D}^1 \subset \cdots \subset \mathcal{D}^{n-1} = TM.$$

A feedback linearizable system is static feedback equivalent to the Brunovsky canonical form

$$(Br) \quad \begin{aligned} \dot{z}_{ij} &= z_{ij+1} \\ \dot{z}_{i\rho_i} &= v_i \end{aligned}$$

where $1 \leq i \leq m$, $1 \leq j \leq \rho_i - 1$, and $\sum_{i=1}^m \rho_i = n$, see Brunovsky [1970], and is clearly flat with $\varphi = (z_{11}, \cdots, z_{m1})$ being a minimal flat output (of differential weight $n + m$). In fact, an equivalent way of describing static feedback linearizable systems is that they are flat systems of differential weight $n + m$.

In general, a flat system is not linearizable by invertible static feedback, with the exception of the single-input case where flatness reduces to static feedback linearization. Flat systems can be seen as a generalization of linear systems. Namely they are linearizable via dynamic, invertible and endogenous feedback, see Fliess et al. [1992, 1995], Martin [1992], Pomet [1995, 1997]. Our goal is thus to describe the simplest flat systems that are not static feedback linearizable: two-inputs control-affine systems that become static feedback linearizable after one-fold prolongation, which is the simplest dynamic feedback. They are flat of differential weight $n + 3$. In this paper, we will completely characterize them and show how their geometry differs but also how it reminds that given by the involutive distributions $\mathcal{D}^i$ for static feedback linearizable systems.

3. MAIN RESULTS

Throughout, we will consider two-input control-affine systems of the form

$$\Sigma \;:\; \dot{x} = f(x) + u_1 g_1(x) + u_2 g_2(x), \qquad (2)$$

where $x \in M$, $u = (u_1, u_2)^t \in \mathbb{R}^2$ and f, g_1, and g_2 are smooth vector fields on M. We deal only with systems that are not static feedback linearizable. This occurs if there exists an integer k such that $\mathcal{D}^k$ is not involutive. Suppose that k is the smallest integer satisfying that property and assume $\text{rk}\,\mathcal{D}^k - \text{rk}\,\mathcal{D}^{k-1} = 2$ (see Proposition 7, in Section 6, asserting that the latter is necessary for dynamic linearizability and thus for flatness). From now on, unless stated otherwise, we assume that all ranks involved are constant in a neighborhood of a given $x_0 \in M$. All results presented here are valid on an open and dense subset of $M \times U \times \mathbb{R}^{ml}$ (the integer l being large enough) and hold locally, around a given point $(x_0, \bar{u}_0^l)$ of that set.

Proposition 1. The following conditions are equivalent:

(i) Σ is flat at $(x_0, \bar{u}_0^l)$, with the differential weight $n+3$;
(ii) Σ is x-flat at (x_0, u_0), with the differential weight $n + 3$;
(iii) There exists, around x_0, an invertible static feedback transformation $u = \alpha(x) + \beta(x)\tilde{u}$, bringing the system Σ into the form $\tilde{\Sigma} : \dot{x} = \tilde{f}(x) + \tilde{u}_1\tilde{g}_1(x) + \tilde{u}_2\tilde{g}_2(x)$, such that the prolongation

$$\tilde{\Sigma}^{(1,0)} : \begin{cases} \dot{x} &= \tilde{f}(x) + y_1\tilde{g}_1(x) + v_2\tilde{g}_2(x) \\ \dot{y}_1 &= v_1 \end{cases}$$

is locally static feedback linearizable, where $y_1 = \tilde{u}_1$, $v_2 = \tilde{u}_2$, $\tilde{f} = f + \alpha g$ and $\tilde{g} = g\beta$, where $g = (g_1, g_2)$ and $\tilde{g} = (\tilde{g}_1, \tilde{g}_2)$.

A system Σ satisfying (*iii*) will be called dynamically linearizable via invertible one-fold prolongation. Notice that $\tilde{\Sigma}^{(1,0)}$ is, indeed, obtained by prolonging the control $\tilde{u}_1$ as $v_1 = \dot{\tilde{u}}_1$ (which explains the notation). The above results asserts that for systems of weight $n + 3$, flatness and x-flatness coincide and that, moreover, they are equivalent to linearizability via the simplest dynamic feedback, namely one-fold preintegration.

Our main result describing flat systems of differential weight $n+3$ is given by two following theorems corresponding to the first noninvolutive distribution $\mathcal{D}^k$ being either $\mathcal{D}^0$, i.e., $k = 0$ (Theorem 3) or $\mathcal{D}^k$, for $k \geq 1$ (Theorem 2). For both theorems, we assume that $\bar{\mathcal{D}}^k \neq TM$, where $\bar{\mathcal{D}}^k$ is the involutive closure of $\mathcal{D}^k$. The particular case $\bar{\mathcal{D}}^k = TM$ (met in applications, see Example 5.1) will be discussed at the end of this section (Theorem 4).

Theorem 2. Assume $k \geq 1$ and $\bar{\mathcal{D}}^k \neq TM$. A control system Σ, given by (2), is x-flat at x_0, with the differential weight $n+3$, if and only if

(A1) $\mathrm{rk}\,\bar{\mathcal{D}}^k = 2k+3$;
(A2) $\mathrm{rk}\,(\bar{\mathcal{D}}^k + [f, \mathcal{D}^k]) = 2k+4$, implying the existence of a nonzero vector field $g_c \in \mathcal{D}^0$ such that $ad_f^{k+1} g_c \in \bar{\mathcal{D}}^k$;
(A3) The distributions B^i, for $i \geq k$, are involutive, where $B^k = \mathcal{D}^{k-1} + \mathrm{span}\,\{ad_f^k g_c\}$ and $B^{i+1} = B^i + [f, B^i]$;
(A4) There exists ρ such that $B^\rho = TM$.

The geometry of the systems described by Theorem 2 can be summarized by the following sequence of inclusions:
$$\mathcal{D}^0 \underset{2}{\subset} \cdots \underset{2}{\subset} \mathcal{D}^{k-1} \underset{1}{\subset} B^k \underset{1}{\subset} \mathcal{D}^k \underset{1}{\subset} \bar{\mathcal{D}}^k = B^{k+1} \underset{2}{\subset} \cdots \underset{2}{\subset} B^\mu \underset{1}{\subset} \cdots \underset{1}{\subset} B^\rho = TM$$
where all the distributions, except $\mathcal{D}^k$, are involutive and the integers beneath "$\subset$" indicate coranks. Notice the existence of a corank one involutive subdistribution B^k in $\mathcal{D}^k$ which plays an important role in our analysis. It is easy to check that $\bar{\mathcal{D}}^k = B^{k+1}$. Indeed, by definition, $B^{k+1} = \mathcal{D}^k + \mathrm{span}\,\{ad_f^{k+1} g_c\}$ and is involutive. Moreover, $\mathrm{rk}\,B^{k+1} = 2k+3$, otherwise we obtain $B^{k+1} = \mathcal{D}^k$ and $\mathcal{D}^k$ would be involutive. Since $\mathcal{D}^k \subset B^{k+1}$ and $\mathrm{rk}\,B^{k+1} = 2k+\ 3$, it follows that $\bar{\mathcal{D}}^k = B^{k+1}$. Thus the direction completing $\mathcal{D}^k$ to $\bar{\mathcal{D}}^k$ has to be colinear with $ad_f^{k+1} g_c$.

The previous theorem enables us to define, up to a multiplicative function, the *characteristic control*, i.e., the control to be prolonged in order to obtain a locally static feedback linearizable $\tilde{\Sigma}^{(1,0)}$. The vector field $g_c \in \mathcal{D}^0$ (see $(A2)$) can be expressed as $g_c = \beta_1 g_1 + \beta_2 g_2$, for some smooth functions (not vanishing simultaneously) on M. We define the characteristic control as $u_c(t) = \beta_2(x(t))u_1(t) - \beta_1(x(t))u_2(t)$ and it is the characteristic control that needs to be preintegrated in order to dynamically linearize the system, that is, we put $v_1 = \frac{d}{dt}(\beta_2 u_1 - \beta_1 u_2) = \frac{d}{dt}\tilde{u}_1$.

If $k = 0$, i.e., the first noninvolutive distribution is $\mathcal{D}^0$, then a similar result holds, but in the chain of involutive subdistributions $B^0 \subset B^1 \subset B^2 \subset ...$ (playing the role of $B^k \subset B^{k+1} \subset B^{k+2} \subset ...$), with $B^0 = \mathrm{span}\,\{g_c\}$, the distribution B^1 is not defined as B^{k+1} but as $\mathcal{G}^1 = \mathcal{D}^0 + [\mathcal{D}^0, \mathcal{D}^0]$ (compare $(A3)$ and $(A3)'$). Moreover, flat systems with $k = 0$ exhibit a singularity in the control space (created by one-fold prolongation of the characteristic control) which is defined by
$U_{sing}(x) = \{u \in \mathbb{R}^2 :\ (g_1 \wedge g_c \wedge [f + u_1 g_1 + u_2 g_c, g_c])(x) = 0\}$ and excluded by (CR).

Theorem 3. Assume $k = 0$ and $\bar{\mathcal{D}}^k \neq TM$. A system Σ, given by (2), is x-flat at (x_0, u_0), with the differential weight $n+3$, if and only if

(A1)' $\mathcal{G}^1$ is involutive;
(A2)' $\mathrm{rk}\,\mathcal{G}^1 + [f, \mathcal{D}^0] = 4$, implying the existence of a nonzero vector field $g_c \in \mathcal{D}^0$ such that $ad_f g_c \in \mathcal{G}^1$;
(A3)' The distributions B^i, for $i \geq 1$, are involutive, where $B^1 = \mathcal{G}^1$ and $B^{i+1} = B^i + [f, B^i]$, for $i \geq 1$;
(A4)' There exists ρ such that $B^\rho = TM$;
(CR) $u_0 \notin U_{sing}(x_0)$.

The conditions of both theorems are verifiable, i.e., given a two-input control-affine system, we can easily verify whether it is flat of weight $n+3$ and verification involves derivations and algebraic operations only, without solving PDE's or bringing the system into a normal form.

Let us now consider the case $\bar{\mathcal{D}}^k = TM$. It is immediate, by Proposition 7 (in Section 6), that $n = 2k+3$. The involutivity of $\mathcal{D}^k$ can be lost in two different ways: either $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$ and $[ad_f^k g_1, ad_f^k g_2] \notin \mathcal{D}^k$ or $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$. As asserts Theorem 4 below, in the first case, the system is flat of differential weight $n+3$ without any additional condition whereas in the second case, the system Σ has to verify some additional conditions analogous to those of Theorem 2. Since the condition $(A2)$, enabling us to compute the involutive subdistribution B^k, has no sense in that case, we have to define B^k in another way. To this end, we introduce the characteristic distribution of $\mathcal{D}^k$, defined as follows. For a distribution $\mathcal{D}$, a characteristic vector field c belongs to $\mathcal{D}$ and satisfies $[c, \mathcal{D}] \subset \mathcal{D}$. The characteristic distribution of $\mathcal{D}$ is the distribution spanned by all its characteristic vector fields. It follows directly from the Jacobi identity that the characteristic distribution is always involutive.

In the case $k = 0$ and $D^k = TM$, the singular controls are not defined by $U_{sing}(x)$ but as
$$U'_{sing}(x) = \{u \in \mathbb{R}^2 :\ \dim \mathrm{span}\,\{g_1, g_2, ad_f g_1 + u_2[g_2, g_1], ad_f g_2 + u_1[g_1, g_2]\}(x) = 3\}.$$

Theorem 4. Assume $k \geq 0$ and $\bar{\mathcal{D}}^k = TM$. Then

(i) either $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$ and then Σ is x-flat at any $x_0 \in M$ (x-flat at any $(x_0, u_0) \in M \times \mathbb{R}^2$, such that $u_0 \notin U'_{sing}(x_0)$, if $k = 0$). Moreover, if Σ is flat, it is flat of differential weight $n+3$.
(ii) or $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$, then $k \geq 1$ and Σ is x-flat of differential weight $n+3$ at $x_0 \in M$ if and only if, around x_0, Σ satisfies:
   (C1) $\mathrm{rk}\,\mathcal{C}^k = 2k$, where $\mathcal{C}^k$ is the characteristic distribution of $\mathcal{D}^k$;
   (C2) $\mathrm{rk}\,(\mathcal{C}^k \cap \mathcal{D}^{k-1}) = 2k-1$;
   (C3) The distribution $B^k = \mathcal{C}^k + \mathcal{D}^{k-1}$ is involutive;
   (C4) $B^{k+1} = TM$, where $B^{k+1} = B^k + [f, B^k]$.

It can be shown that in the case $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$ (no mater whether $\bar{\mathcal{D}}^k = TM$ or not), the involutive subdistribution B^k can always be defined as above, i.e., the definition of B^k given by item $(A3)$ of Theorem 2 and that provided by conditions $(C1) - (C3)$ of Theorem 4 are equivalent if $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$. This is not valid anymore if $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$; indeed, in that case $\mathcal{C}^k = \mathcal{D}^{k-1}$, $(C2)$ is not verified and $(C3)$ would give $B^k = \mathcal{D}^{k-1}$.

4. CALCULATING FLAT OUTPUTS

In this section, firstly, we answer the question whether a given pair of smooth functions on M forms a flat output and, secondly, provide a system of PDS's to be solved in order to find all minimal flat outputs. In particular, we will discus uniqueness of flat outputs for flat systems of differential weight $n+3$. Let μ be the largest integer such that corank $(B^{\mu-1} \subset B^\mu)$ is two and ρ is the smallest integer such that $B^\rho = TM$.

Proposition 5. Consider a control system Σ, given by (2), that is flat at x_0 (at (x_0, u_0), if $k = 0$), of weight $n+3$.

(i) Assume $\bar{\mathcal{D}}^k \neq TM$ or $\bar{\mathcal{D}}^k = TM$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$. Then a pair (φ_1, φ_2) of smooth functions on a neighborhood of x_0 is a minimal x-flat output at x_0 if and only if (after permuting φ_1 and φ_2, if necessary)

$$d\varphi_1 \perp B^{\rho-1}, \quad d\varphi_2 \perp B^{\mu-1},$$

and $d\varphi_2 \wedge d\varphi_1 \wedge dL_f\varphi_1 \wedge \cdots \wedge dL_f^{\rho-\mu}\varphi_1(x_0) \neq 0$. Moreover, the pair (φ_1, φ_2) is unique, up to a diffeomorphism, i.e., if $(\tilde{\varphi}_1, \tilde{\varphi}_2)$ is another minimal x-flat output, then there exist smooth maps h_1 and h_2, smoothly invertible (h_2 with respect to its first argument), such that $\tilde{\varphi}_1 = h_1(\varphi_1)$ and $\tilde{\varphi}_2 = h_2(\varphi_2, \varphi_1, L_f\varphi_1, \ldots, L_f^{\rho-\mu}\varphi_1)$; if $\rho = \mu$, then $\tilde{\varphi}_i = h_i(\varphi_1, \varphi_2)$, $1 \leq i \leq 2$, and $h = (h_1, h_2)$ is a diffeomorphism.

(ii) Assume $\bar{\mathcal{D}}^k = TM$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$. Then a pair (φ_1, φ_2) of smooth functions on a neighborhood of x_0 is a minimal x-flat output at x_0 if and only if $(d\varphi_1 \wedge d\varphi_2)(x_0) \neq 0$ and the involutive distribution $\mathcal{L} = (\mathrm{span}\,\{d\varphi_1, d\varphi_2\})^\perp$ satisfies

$$\mathcal{D}^{k-1} \subset \mathcal{L} \subset \mathcal{D}^k.$$

Moreover, for any function φ_1, satisfying $d\varphi_1 \perp \mathcal{D}^{k-1}$ and $(L_{ad_f^k g_1}\varphi_1, L_{ad_f^k g_2}\varphi_1)(x_0) \neq (0,0)$, there exists φ_2 such that the pair (φ_1, φ_2) is a minimal x-flat output; given any such φ_1, the choice of φ_2 is unique, up to a diffeomorphism, that is, if $(\varphi_1, \tilde{\varphi}_2)$ is another minimal x-flat output, then there exists a smooth map h, smoothly invertible with respect to the second argument such that $\tilde{\varphi}_2 = h(\varphi_1, \varphi_2)$.

In the case $\bar{\mathcal{D}}^k = TM$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$, there is as many flat outputs as functions of three variables. Indeed, the distribution $\mathcal{D}^{k-1}$ is involutive and of corank three. According to item (ii), φ_1 can be chosen as any function of three independent functions, whose differentials span $(\mathcal{D}^{k-1})^\perp$ and then there exists a unique φ_2 (up to a diffeomorphism) completing it to a minimal x-flat output. This reminds very much non-uniqueness of flat outputs of two-control driftless systems, Li and Respondek [2012].

As an immediate corollary of Proposition 5, we obtain a system of PDE's whose solutions give all minimal x-flat outputs. In the case $\bar{\mathcal{D}}^k \neq TM$ or $\bar{\mathcal{D}}^k = TM$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$, the vector field g_c is well defined, so denote $v_{2j-1} = ad_f^{j-1}g_c$, for $1 \leq j \leq \mu+1$, and $v_{2j} = ad_f^{j-1}g_1$, for $1 \leq j \leq \mu$, and complete them, for $1 \leq i \leq \rho - \mu$, by $v_{2\mu+1+i} = ad_f^{\mu+i-1}g_1$, if $ad_f^{\mu}g_1 \notin B^\mu$, or by $v_{2\mu+1+i} = ad_f^{\mu+i}g_c$, otherwise. We thus have defined $n-1$ vector fields $v_1, \ldots, v_{n-1}$ satisfying $B^{\mu-1} = \mathrm{span}\,\{v_1, \ldots, v_{2\mu-1}\}$ and $B^{\rho-1} = \mathrm{span}\,\{v_1, \ldots, v_{n-1}\}$. In this case the result follows immediately and is stated as item (i) of proposition below. If $\bar{\mathcal{D}}^k = TM$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$, then for $1 \leq j \leq k = \mu$, denote $w_j = ad_f^{j-1}g_1$ and $w_{\mu+j} = ad_f^{j-1}g_2$. Clearly, $\mathcal{D}^{k-1} = \mathrm{span}\,\{w_1, \ldots, w_{2k}\}$ but we have to construct one more vector field w, as described in item (ii).

Proposition 6. Consider a system Σ, given by (2), that is flat at x_0 (at (x_0, u_0), if $k = 0$), of differential weight $n+3$.

(i) Assume $\bar{\mathcal{D}}^k \neq TM$ or $\bar{\mathcal{D}}^k = TM$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$. Then a pair (φ_1, φ_2) of smooth functions on a neighborhood of x_0 is a minimal x-flat output at x_0 if and only if (after permuting φ_1 and φ_2, if necessary)

$$L_{v_j}\varphi_1 = 0, \quad 1 \leq j \leq n-1$$
$$L_{v_j}\varphi_2 = 0, \quad 1 \leq j \leq 2\mu - 1$$

and $d\varphi_2 \wedge d\varphi_1 \wedge dL_f\varphi_1 \wedge \cdots \wedge dL_f^{\rho-\mu}\varphi_1(x_0) \neq 0$.

(ii) Assume $\bar{\mathcal{D}}^k = TM$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$. Then a pair (φ_1, φ_2) of smooth functions on a neighborhood of x_0 is a minimal x-flat output at x_0 if and only if (after permuting φ_1 and φ_2, if necessary) φ_1 is any function satisfying

$$L_{w_j}\varphi_1 = 0, \quad 1 \leq j \leq 2k,$$

and $(L_{ad_f^k g_1}\varphi_1, L_{ad_f^k g_2}\varphi_1)(x_0) \neq (0,0)$ and, for any φ_1 as above, φ_2 is given by

$$L_{w_j}\varphi_2 = 0, \ 1 \leq j \leq 2k, \text{ and } L_w\varphi_2 = 0,$$

where $w = (L_{ad_f^k g_2}\varphi_1)ad_f^k g_1 - (L_{ad_f^k g_1}\varphi_1)ad_f^k g_2$ and $(d\varphi_1 \wedge d\varphi_2)(x_0) \neq 0$.

Clearly, the distribution $\mathcal{L}$ spanned by w and $\mathcal{D}^{k-1}$ is of corank two and, as can be proved, involutive thus implying that for any φ_1 we can solve the system of equations for φ_2. Different choices of φ_1 lead, in general, to different involutive distributions $\mathcal{L}$ and thus to different functions φ_2 and, as we have mentioned, there is as many choices as nondegenerate functions of three variables.

5. EXAMPLES

5.1 Induction motor

Consider the induction motor (called direct-quadrature model in Chiasson [1998], see also Martin and Rouchon [1996], Delaleau et al. [2001]):

$$\Sigma_{IM} \begin{cases} \dot{\omega} &= \mu\psi_d i_q - \dfrac{\tau_L}{J} \\ \dot{\psi}_d &= -\eta\psi_d + \eta M i_d \\ \dot{\rho} &= n_p\omega + \dfrac{\eta M i_q}{\psi_d} \\ \dot{i}_d &= -\gamma i_d + \dfrac{\eta M \psi_d}{\sigma L_R L_S} + n_p\omega i_q + \dfrac{\eta M i_q^2}{\psi_d} + \dfrac{u_d}{\sigma L_S} \\ \dot{i}_q &= -\gamma i_q - \dfrac{M n_p \omega \psi_d}{\sigma L_R L_S} - n_p\omega i_d - \dfrac{\eta M i_d i_q}{\psi_d} + \dfrac{u_q}{\sigma L_S} \end{cases}$$

where u_d, u_q are the inputs (the stator voltages), i_d and i_q are the stator currents, ψ_d and ρ are two well-chosen functions of the rotor fluxes (see Chiasson [1998] for their precise expression) and ω is the rotor speed. All other parameters of the motor (the inductances L_S and L_R, the load-torque τ_L, etc.) can be supposed constant and known. After applying a static feedback transformation (which has also a physical interpretation, see Chiasson [1998] for more details) the system is transformed into the form:

$$\tilde{\Sigma}_{IM} \begin{cases} \dot{\omega} = \mu\psi_d i_q - \dfrac{\tau_L}{J} & \dot{\rho} = n_p\omega + \dfrac{\eta M i_q}{\psi_d} \\ \dot{\psi}_d = -\eta\psi_d + \eta M i_d & \dot{i}_q = v_q \\ \dot{i}_d = v_d & \end{cases}$$

This system is not static feedback linearizable. Indeed, the distribution $\mathcal{D}^1 = \mathrm{span}\,\{\frac{\partial}{\partial i_d}, \frac{\partial}{\partial i_q}, \frac{\partial}{\partial \psi_d}, \frac{\partial}{\partial \omega} + \frac{\eta M}{\mu \psi_d^2}\frac{\partial}{\partial \rho}\}$ is not involutive, $\bar{\mathcal{D}}^1 = TM$ and $[\mathcal{D}^0, \mathcal{D}^1] \subset \mathcal{D}^1$. Here $k = 1$ and we are in the case of Theorem 4(i) and the system is flat without additional condition, a property that has been already observed and applied, Martin and Rouchon [1996], Delaleau et al. [2001].

According to Propositions 5(ii) and 6(ii), the system admits many flat outputs (the choice being parameterized by a function of three well defined variables) and let us calculate some of them. Recall that a pair of independent functions (φ_1, φ_2) is a minimal x-flat output if and only if the involutive distribution $\mathcal{L} = (\mathrm{span}\{d\varphi_1, d\varphi_2\})^\perp$ satisfies $\mathcal{D}^0 \subset \mathcal{L} \subset \mathcal{D}^1$. Hence $\mathcal{L}$ has to be of the form $\mathcal{L} = \mathrm{span}\{\frac{\partial}{\partial i_d}, \frac{\partial}{\partial i_q}, h\}$, where h is any vector field of the form $h = \alpha\frac{\partial}{\partial \psi_d} + \beta(\frac{\partial}{\partial \omega} + \frac{\eta M}{\mu\psi_d^2}\frac{\partial}{\partial \rho})$ such that $\mathcal{L}$ is involutive and $(\alpha, \beta) \neq (0, 0)$. Let us first take $\mathcal{L} = \mathrm{span}\{\frac{\partial}{\partial i_d}, \frac{\partial}{\partial i_q}, \frac{\partial}{\partial \psi_d}\}$. The associated flat outputs are independent functions of ω, ρ and we can take $(\varphi_1,\ varphi_2) = (\omega, \rho)$.

Let us now give some less intuitive minimal flat outputs. Choose $\mathcal{L} = \mathrm{span}\{\frac{\partial}{\partial i_d}, \frac{\partial}{\partial i_q}, \frac{\partial}{\partial \omega} + \frac{\eta M}{\mu\psi_d^2}\frac{\partial}{\partial \rho}\}$. Any two independent functions φ_1 and φ_2 depending on ω, ψ_d, ρ whose differentials annihilate $\mathcal{L}$, that is, satisfying $\frac{\partial\varphi_i}{\partial\omega} + \frac{\eta M}{\mu\psi_d^2}\frac{\partial\varphi_i}{\partial\rho} \equiv 0$, for $1 \le i \le 2$, can be taken as minimal flat outputs. Solving those equations, we get $\varphi_i = \varphi_i(\psi_d, \frac{\eta M}{\mu\psi_d^2}\omega - \rho)$. We can choose, for instance, $(\varphi_1, \varphi_2) = (\psi_d, \frac{\eta M}{\mu\psi_d^2}\omega - \rho)$.

Finally, let $\mathcal{L} = \mathrm{span}\{\frac{\partial}{\partial i_d}, \frac{\partial}{\partial i_q}, \frac{\partial}{\partial \psi_d} + \frac{\partial}{\partial \omega} + \frac{\eta M}{\mu\psi_d^2}\frac{\partial}{\partial \rho}\}$. The functions φ_1 and φ_2 depend on ω, ψ_d, ρ and satisfy $\frac{\partial\varphi_i}{\partial\psi_d} + \frac{\partial\varphi_i}{\partial\omega} + \frac{\eta M}{\mu\psi_d^2}\frac{\partial\varphi_i}{\partial\rho} \equiv 0$, for $1 \le i \le 2$. Solving those equations, we obtain $\varphi_i = \varphi_i(\rho + \frac{\eta M}{\mu\psi_d}, \psi_d - \omega)$. We can choose $(\varphi_1, \varphi_2) = (\rho + \frac{\eta M}{\mu\psi_d}, \psi_d - \omega)$.

5.2 Polymerization reactor

Consider the reactor, Martin et al. [2003], Rouchon [1995]:

$$\Sigma \begin{cases} \dot{C}_m &= \dfrac{C_{mm_s}}{\tau} - (1 + \bar{\epsilon}\dfrac{\mu}{\mu + M_m C_m})\dfrac{C_m}{\tau} + R_m(C_m, C_i, C_s, T) \\ \dot{C}_i &= -k_i(T)C_i + u_2\dfrac{C_{ii_s}}{V} - (1 + \bar{\epsilon}\dfrac{\mu}{\mu + M_m C_m})\dfrac{C_i}{\tau} \\ \dot{C}_s &= u_2\dfrac{C_{si_s}}{V} + \dfrac{C_{sm_s}}{\tau} - (1 + \bar{\epsilon}\dfrac{\mu}{\mu + M_m C_m})\dfrac{C_s}{\tau} \\ \dot{\mu} &= -M_m R_m(C_m, C_i, C_s, T) - (1 + \bar{\epsilon}\dfrac{\mu}{\mu + M_m C_m})\dfrac{\mu}{\tau} \\ \dot{T} &= \theta(C_m, C_i, C_s, \mu, T) + \alpha_1 T_j \\ \dot{T}_j &= f_6(T, T_j) + \alpha_4 u_1 \end{cases}$$

where u_1, u_2 are the control inputs and C_{mm_s}, C_{ii_s}, C_{si_s}, C_{sm_s}, M_m, $\bar{\epsilon}$, τ, V, α_1, α_4 are constant parameters. The functions R_m, k_i, θ and f_6 are not well-known and can be considered arbitrary: they derive from experimental data and involve kinetic laws, heat transfer coefficients and reaction enthalpies. After applying a change of coordinates and a suitable static feedback transformation, we obtain :

$$\tilde{\Sigma}_{PR} \begin{cases} \dot{\tilde{C}}_i = \tilde{C}_s & \dot{\tilde{C}}_m = \tilde{\mu} \\ \dot{\tilde{C}}_s = \tilde{u}_1 & \dot{\tilde{\mu}} = b(\tilde{C}_m, \tilde{C}_i, \tilde{C}_s, \tilde{\mu}, \tilde{T}) \\ & \dot{\tilde{T}} = \tilde{T}_j \\ & \dot{\tilde{T}}_j = \tilde{u}_2 \end{cases}$$

where b is a smooth function depending explicitly on $\tilde{T} = T$. If $(\frac{\partial^2 b}{\partial\tilde{T}\partial\tilde{C}_s}, \frac{\partial^2 b}{\partial\tilde{C}_s^2}) \neq (0, 0)$, then the distribution $\mathcal{D}^1 = \mathrm{span}\{\frac{\partial}{\partial\tilde{C}_s}, \frac{\partial}{\partial\tilde{C}_i} + \frac{\partial b}{\partial\tilde{C}_s}\frac{\partial}{\partial\tilde{\mu}}, \frac{\partial}{\partial\tilde{T}_j}, \frac{\partial}{\partial T}\}$ is noninvolutive, $\mathrm{rk}\,\bar{\mathcal{D}}^1 = 5$ and $\bar{\mathcal{D}}^1 \neq TM$. Consequently, we are in the case of Theorem 2, with $k = 1$. Let us suppose that $\frac{\partial^2 b}{\partial\tilde{C}_s^2} \neq 0$. Therefore, $[\mathcal{D}^0, \mathcal{D}^1] \not\subset \mathcal{D}^1$ and the corank one involutive subdistribution B^1 can be computed in two different ways (see condition $(A3)$ of Theorem 2 and the comment following Theorem 4). We will calculate B^1 by applying the procedure given by Theorem 2. The distribution

$$\bar{\mathcal{D}}^1 + [f, \mathcal{D}^1] = \mathrm{span}\{\frac{\partial}{\partial\tilde{C}_s}, \frac{\partial}{\partial\tilde{C}_i}, \frac{\partial}{\partial\tilde{T}_j}, \frac{\partial}{\partial T}, \frac{\partial}{\partial\tilde{\mu}}, \frac{\partial b}{\partial\tilde{C}_s}\frac{\partial}{\partial\tilde{C}_m}\}$$

is of rank 6 (provided that $\frac{\partial b}{\partial\tilde{C}_s}$ does not vanish) and $\tilde{g}_2 = \frac{\partial}{\partial\tilde{T}_j}$ is such that $ad_f\tilde{g}_2 \in \bar{\mathcal{D}}^1$. Therefore, item $(A2)$ of Theorem 2 is verified and $\tilde{g}_2$ plays the role of g_c. Thus the corank one subdistribution B^1 is given by

$$B^1 = \mathcal{D}^0 + \mathrm{span}\{ad_f\tilde{g}_2\} = \mathrm{span}\{\frac{\partial}{\partial\tilde{C}_s}, \frac{\partial}{\partial\tilde{T}_j}, \frac{\partial}{\partial T}\}$$

and is clearly involutive. We have $B^2 = B^1 + [f, B^1] = \mathrm{span}\{\frac{\partial}{\partial\tilde{C}_s}, \frac{\partial}{\partial\tilde{C}_i}, \frac{\partial}{\partial\tilde{T}_j}, \frac{\partial}{\partial T}, \frac{\partial}{\partial\mu}\}$ involutive and $B^3 = TM$. The system $\tilde{\Sigma}_{PR}$ satisfies all conditions of Theorem 2, hence the corresponding prolongation $\tilde{\Sigma}_{PR}^{(1,0)}$, obtained by prolonging $\tilde{u}_1$, is locally static feedback linearizable and can be brought into the Brunovsky canonical form with $\tilde{C}_m = M_m C_m + \mu$, $\tilde{C}_i = C_i - \frac{C_{ii_s}}{C_{si_s}}C_s$ playing the role of top variables. Let us now compute the minimal flat outputs (φ_1, φ_2) of $\tilde{\Sigma}_{PR}$. We are in the first case of Proposition 5, with $\rho = 3$ and $\mu = 2$. Since the differential of φ_1 annihilates B^2, it follows that $\varphi_1 = \varphi_1(\tilde{C}_m)$ with $\frac{\partial\varphi_1}{\partial\tilde{C}_m} \neq 0$. The differential of φ_2 annihilates B^1 and satisfies $d\varphi_2 \wedge d\varphi_1 \wedge dL_f\varphi_1 \neq 0$. This yields $\varphi_2 = \varphi_2(\tilde{C}_m, \tilde{C}_i, \tilde{\mu})$ with $\frac{\partial\varphi_2}{\partial\tilde{C}_i} \neq 0$. Hence, a choice of minimal flat outputs is $(\varphi_1, \varphi_2) = (\tilde{C}_m, \tilde{C}_i)$.

6. SKETCHES OF PROOFS

6.1 Notations and useful results

Consider a control system of the form $\Sigma : \dot{x} = f(x) + u_1 g_1(x) + u_2 g_2(x)$. By $\Sigma^{(1,0)}$ we will denote the system Σ with one-fold prolongation of the first control, that is

$$\Sigma^{(1,0)} : \begin{cases} \dot{x} &= f(x) + y_1 g_1(x) + +v_2 g_2(x) \\ \dot{y}_1 &= v_1 \end{cases}$$

with $y_1 = u_1$ and $v_2 = u_2$. Throughout this section,

$$F = \sum_{i=1}^{n}(f_i + y_1 g_{1i})\frac{\partial}{\partial x_i}$$

stands for the drift and

$$G_1 = \frac{\partial}{\partial y_1}, \quad H_2 = \sum_{i=1}^{n} g_{2i}\frac{\partial}{\partial x_i}$$

denote the control vector fields of the prolonged system. To $\Sigma^{(1,0)}$, we associate the distributions $D_p^0 = \mathrm{span}\{G_1, H_2\}$ and $D_p^{i+1} = D_p^i + [F, D_p^i]$, for $i \ge 0$, (the subindex p referring to the prolonged system $\Sigma^{(1,0)}$).

We start by stating two results needed in our proofs.

Proposition 7. Consider Σ given by (2), dynamically linearizable via invertible one-fold prolongation and let $\mathcal{D}^k$ be the first noninvolutive distribution. If $k \ge 1$, then $\mathrm{rk}\,\mathcal{D}^k - \mathrm{rk}\,\mathcal{D}^{k-1} = 2$.

Proposition 8. Consider Σ given by (2), and let $\mathcal{D}^k$ be the first noninvolutive distribution. Assume $k \ge 1$ and $\mathcal{D}^k$ satisfies the conditions $(A1) - (A2)$ of Theorem 2. If the distribution $B^k = \mathcal{D}^{k-1} + \mathrm{span}\{ad_f^k g_c\}$ is involutive, where g_c is defined by item $(A2)$, then all distributions $\mathcal{E}^i = \mathcal{D}^{i-1} + \mathrm{span}\{ad_f^i g_c\}$, for $1 \le i \le k-1$, are involutive.

6.2 Proof of Theorem 2

We give a sketch of the proof of Theorem 2, which is a general result whereas Theorems 3 and 4 deal with the particular cases $k = 0$ and $\bar{\mathcal{D}}^k = TM$.

Necessity. Consider an x-flat system $\Sigma : \dot{x} = f(x) + u_1 g_1(x) + u_2 g_2(x)$ of weight $n+3$. By Proposition 1, there exists an invertible feedback transformation $u = \alpha(x) + \beta(x)\tilde{u}$, bringing Σ into $\tilde{\Sigma} : \dot{x} = \tilde{f}(x) + \tilde{u}_1 \tilde{g}_1(x) + \tilde{u}_2 \tilde{h}_2(x)$, such that the prolongation $\tilde{\Sigma}^{(1,0)}$ is locally static feedback linearizable. For simplicity of notation, we drop the tilde, but we keep distinguishing g_1 from h_2 (which could also be denoted g_2) whose control is not preintegrated. Since $\Sigma^{(1,0)}$ is locally static feedback linearizable, for any $i \geq 0$ the distributions D_p^i are involutive, of constant rank, and there exists an integer ρ such that $\operatorname{rk} D_p^\rho = n+1$. It can be proved (by an induction argument) that, for $1 \leq i \leq k$,

$$D_p^i = \operatorname{span}\{\frac{\partial}{\partial y_1}, g_1, \cdots, ad_f^{i-1} g_1, h_2, \cdots, ad_f^i h_2\}.$$

Since the intersection of involutive distributions is an involutive distribution, it follows that

$$B^k = D_p^k \cap TM = \operatorname{span}\{g_1, \cdots, ad_f^{k-1} g_1, h_2, \cdots, ad_f^k h_2\}$$

is involutive. It is immediate that $\mathcal{D}^{k-1} \subset B^k \subset \mathcal{D}^k$, where both inclusions are of corank one, otherwise $B^k = \mathcal{D}^k$ and $\mathcal{D}^k$ would be involutive, which contradicts our hypotheses. The involutivity of $D_p^{k+1} = \operatorname{span}\{\frac{\partial}{\partial y_1}, g_1, \cdots, ad_f^k g_1, h_2, \cdots, ad_f^{k+1} h_2\}$ implies that of $\mathcal{D}^k + \operatorname{span}\{ad_f^{k+1} h_2\}$. It yields $\bar{\mathcal{D}}^k = \mathcal{D}^k + \operatorname{span}\{ad_f^{k+1} h_2\}$ and $\operatorname{rk} \bar{\mathcal{D}}^k = 2k+3$. This gives $(A1)$. Recall that $B^i = B^{i-1} + [f, B^{i-1}]$, for $i \geq k+1$. We have

$$D_p^{k+i} = \operatorname{span}\{\frac{\partial}{\partial y_1}\} + B^{k+i},\ i \geq 1.$$

Consequently, the involutivity of D_p^{k+i} implies that of B^{k+i}, for $i \geq 1$. Moreover, $\operatorname{rk} D_p^\rho = n+1$, implying that $\operatorname{rk} B^\rho = n$, i.e., $B^\rho = TM$, which proves $(A3)$ and $(A4)$. It remains to show that $\operatorname{rk}(\bar{\mathcal{D}}^k + [f, \mathcal{D}^k]) = 2k+4$. We have $D_p^{k+1} = \operatorname{span}\{\frac{\partial}{\partial y_1}\} + \bar{\mathcal{D}}^k$. Assume $ad_f^{k+1} g_1 \in \bar{\mathcal{D}}^k$. Hence for any vector field $\xi \in \mathcal{D}^k$, we have $[f, \xi] \in \bar{\mathcal{D}}^k$, implying that $\bar{\mathcal{D}}^k + [f, \bar{\mathcal{D}}^k] = \bar{\mathcal{D}}^k$. Therefore, for the prolonged system we obtain $D_p^{k+2} = \operatorname{span}\{\frac{\partial}{\partial y_1}\} + \bar{\mathcal{D}}^k + [f, \bar{\mathcal{D}}^k] = D_p^{k+1}$, thus contradicting the existence of ρ such that $\operatorname{rk} D_p^\rho = n+1$ (since $\bar{\mathcal{D}}^k \neq TM$) and implying that $\Sigma^{(1,0)}$ is not static feedback linearizable. By Proposition 1, the system Σ would not be x-flat of differential weight $n+3$ and thus $\operatorname{rk}(\bar{\mathcal{D}}^k + [f, \mathcal{D}^k]) = 2k+4$ and $(A2)$ holds.

Sufficiency: Consider a control system $\Sigma : \dot{x} = f(x) + u_1 g_1(x) + u_2 g_2(x)$ satisfying $(A1) - (A4)$ and transform it via a static feedback into $\tilde{\Sigma} : \dot{x} = \tilde{f}(x) + \tilde{u}_1 \tilde{g}_1(x) + \tilde{u}_2 g_c(x)$, where g_c is defined by $(A2)$. It is immediate to see that the prolongation $\tilde{\Sigma}^{(1,0)}$ is static feedback linearizable. Indeed, the linearizability distributions D_p^i of $\tilde{\Sigma}^{(1,0)}$ are of the form

$$D_p^i = \operatorname{span}\{\frac{\partial}{\partial y_1}\} + \mathcal{E}^i,\ 1 \leq i \leq k-1,$$

$$D_p^i = \operatorname{span}\{\frac{\partial}{\partial y_1}\} + B^i,\ i \geq k$$

where $\mathcal{E}^i = \mathcal{D}^{i-1} + \operatorname{span}\{ad_f^i g_c\}$ and by Proposition 8 are involutive, for $1 \leq i \leq k-1$. The involutivity of $\mathcal{E}^i$ and B^i implies that of D_p^i. Moreover, $\operatorname{rk} B^\rho = n$, thus $\operatorname{rk} D_p^\rho = n+1$ and $\Sigma^{(1,0)}$ is static feedback linearizable. According to Proposition 1, Σ is flat of weight $n+3$.

REFERENCES

Brunovsky, P. (1970). A classification of linear controllable systems. *Kybernetika*, 3(6), 173–188.

Charlet, B., Lévine, J., and Marino, R. (1991). Sufficient conditions for dynamic state feedback linearization. *SIAM J. Control Optim.*, 29(1), 38–57.

Chiasson, J. (1998). A new approach to dynamic feedback linearization control of an induction motor. *IEEE Trans. Automat. Control*, 43(3), 391–397.

Delaleau, E., Louis, J., and Ortega, R. (2001). Modeling and control of induction motors. *AMCS*, 11(1), 105–130.

Fliess, M., Levine, J., Martin, P., and Rouchon, P. (1992). Sur les systemes non linéaires différentiellement plats. *C. R. Acad. Sci. Paris Sér. I Math.*, 315(5), 619–624.

Fliess, M., Lévine, J., Martin, P., and Rouchon, P. (1995). Flatness and defect of non-linear systems: introductory theory and examples. *Internat. J. Control*, 61(6), 1327–1361.

Fliess, M., Lévine, J., Martin, P., and Rouchon, P. (1999). A Lie-Bäcklund approach equivalence and flatness of nonlinear systems. *IEEE Trans. Automat. Control*, 44(5), 922–937.

Hunt, L. and Su, R. (1981). Linear equivalents of nonlinear time varying systems. In *Proc. MTNS, Santa Monica, CA*, 119–123.

Jakubczyk, B. (1993). Invariants of dynamic feedback and free systems. In *Proc. ECC 1993*, 1510–1513.

Jakubczyk, B. and Respondek, W. (1980). On linearization of control systems. *Bull. Acad. Polonaise Sci. Ser. Sci. Math.*, 517–522.

Lévine, J. (2009). *Analysis and Control of Nonlinear Systems: A Flatness-Based Approach.* Springer.

Li, S. and Respondek, W. (2012). Flat outputs of two-input driftless control systems. *ESAIM Control Optim. Calc. Var.*, 18, 774–798.

Martin, P. (1992). *Contribution à l'étude des systèmes différentiellement plats.* Ph.D. thesis, l'Ecole Nationale Supérieure de Mines de Paris.

Martin, P. and Rouchon, P. (1996). Two remarks on induction motors. In *Proc. CESA*, volume 96, 76–79.

Martin, P., Rouchon, P., and Murray, R. (2003). Flat systems, equivalence and trajectory generation, CDS Technical Report, Caltech.

Pereira da Silva, P. and Corrêa Filho, C. (2001). Relative flatness and flatness of implicit systems. *SIAM J. Control Optim.*, 39(6), 1929–1951.

Pomet, J. (1995). A differential geometric setting for dynamic equivalence and dynamic linearization. *Banach Center Publ., Vol. 32*, 319–339.

Pomet, J. (1997). On dynamic feedback linearization of four-dimensional affine control systems with two inputs. *ESAIM Control Optim. Calc. Var*, 2, 151–230.

Respondek, W. (2003). Symmetries and minimal flat outputs of nonlinear control systems. In *New Trends in Nonlinear Dynamics and Control and their Applications*, volume LNCS 295, 65–86. Springer.

Rouchon, P. (1995). Necessary condition and genericity of dynamic feedback linearization. *J. Math. Systems Estim. Control*, 5(3), 345–358.

Schlacher, K. and Schoeberl, M. (2007). Construction of flat outputs by reduction and elimination. In *IFAC Symp. NOLCOS 2007*, volume 8, 666–671.

