

## Stability of NMPC with cyclic horizons

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**Abstract:** In this paper we present stability conditions for nonlinear model predictive control with cyclically varying horizons. Starting from a maximum horizon length, the horizon is reduced by one at each sampling time until a minimum horizon length is reached, at which the horizon is increased to the maximum length. The approach allows to utilize shapes and structures in the terminal constraints, which can otherwise not be handled. Examples are terminal box-constraints, where the terminal set cannot be rendered invariant, or quadratic terminal regions and penalties of diagonal structure. Such constraints are for example of advantage for distributed predictive control problems. To underline the applicability, the approach is used to control a four tank system.

*Keywords:* nonlinear model predictive control, distributed model predictive control, stability, cyclically varying horizons.

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### 1. INTRODUCTION

In nonlinear model predictive control (NMPC) the feedback is generated by solving at each time instance a finite horizon optimal control problem and applying the first part of the optimal input sequence as input, c.f. Grüne and Pannek (2011); Findeisen et al. (2007); Mayne et al. (2000). This allows to generate an input sequence such that the predicted state trajectory satisfies the constraints on the states and inputs and allows to "optimize" a performance specification. However due to the finite horizon stability and recursive feasibility are not necessarily guaranteed. To guarantee stability and recursive feasibility one can exploit special terminal constraints and costs, or a sufficient large horizon length subject to specific controllability conditions c.f. Grüne and Pannek (2011); Mayne et al. (2000).

Recently there has been a strong interest to design distributed predictive control schemes for system consisting of dynamically coupled subsystems, compare Scattolini (2009). Example are irrigation channels, see Negenborn et al. (2009), building control (Ma et al. (2011)), or power systems, c.f. Venkat et al. (2008); Savorgnan et al. (2011).

Often so-called cooperative schemes (Scattolini (2009)) are considered, i.e. each subsystem is controlled by a local controller and the controllers cooperate to minimize the overall performance and guarantee constraint satisfaction. Many such controllers are based on ideas of distributed optimization (c.f. Bertsekas and Tsitsiklis (1989)) and exploit the fact that the cost function and constraints are separable, i. e. the overall cost is the sum of each subsystem cost and each subsystem has its own state and input constraints. Consequently, typically also the terminal constraint and cost need to be separable. We refer

for more details to Giselsson and Rantzer (2010); Doan et al. (2011); Conte et al. (2012a,b); Kögel and Findeisen (2012); Savorgnan et al. (2011); Scattolini (2009); Stewart et al. (2011) and the references provided therein.

In the literature there exist different approaches to guarantee stability for such setups. The works Stewart et al. (2010, 2011); Venkat et al. (2008) investigate stability for input-constrained systems. Giselsson and Rantzer (2010) present a distributed predictive control approach without special terminal constraints and cost, which can guarantee stability assuming that suboptimality estimates (see e.g. Grüne and Pannek (2011)) are available. Unfortunately, it is difficult to guarantee recursive feasibility. In Doan et al. (2011) stability is enforced by restricting the terminal state to the origin, which however might decrease the overall control performance. In Conte et al. (2012a) a stability criterion based on a distributed invariance condition is presented assuming that the subsystems are only coupled via the state.

This work presents stability and recursive feasibility conditions for an NMPC scheme with a cyclically varying horizon length. This allows to use separable terminal costs and constraints and avoids some of the outlined limitations.

The results are based on the ideas presented in Kögel and Findeisen (2012), where we sketched the basic idea for linear MPC using polytopic terminal constraints. Here we expand and generalize the ideas to nonlinear systems, consider more general terminal constraints and present a more detailed and general theoretical analysis.

The structure of the remainder of the paper is as follows. The next section presents the problem setup and the NMPC scheme with a cyclic horizon. Section 3 contains feasibility and stability conditions of cyclic horizon NMPC. In Section 4 we discuss the existence of nontrivial terminal sets. In Section 5 we apply the approach to control a four tank system.

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The notation is mainly standard. We use  $\mathbb{N}^0$  to denote the set containing the natural numbers ( $\mathbb{N}$ ) and 0.  $\text{rem}(a, b)$ , ( $a \in \mathbb{Z}$ ,  $b \in \mathbb{N}$ ) denotes the remainder function of Euclidean division.  $\mathcal{B}_a$  is a ball centered at the origin:  $\mathcal{B}_a = \{x \text{ s.t. } \|x\|_2 \leq a\}$ . A class  $\mathcal{K}$  function  $f: [0, a) \rightarrow [0, \infty)$  is a monotone increasing, continuous function with  $f(0) = 0$ .  $\star$  denotes optimal values of variables/functions.

## 2. PROBLEM SETUP

We consider systems with the nonlinear dynamics

$$x_{k+1} = f(x_k, u_k), \quad (1)$$

with  $f(0, 0) = 0$  and  $k \geq 0$ . The state  $x_k$  and the input  $u_k$  of the system are constrained to the closed sets

$$x_k \in \mathcal{X} \subseteq \mathbb{R}^n \quad u_k \in \mathcal{U} \subseteq \mathbb{R}^p. \quad (2a)$$

To enforce the constraints we use nonlinear predictive control with a cyclically varying horizon  $N_k$ , given by

$$N_k = \underline{N} + M - \text{rem}(k, M), \quad (3)$$

where  $\underline{N} \geq 1$  denotes the minimum horizon length and  $M \geq 1$  the cycle length. Note that  $M = 1$  corresponds to the usual fixed horizon length, whereas for  $M > 1$  the horizon varies cyclically between  $\underline{N}$  and the maximum length  $\underline{N} + M - 1$ : at  $k = iM$ ,  $i \in \mathbb{N}^0$  it has the maximum length  $(\underline{N} + M - 1)$  and shrinks the next time instances  $k + 1, \dots$  until the horizon is restored at  $k = (i + 1)M$  to its maximum length, see Figure 1.

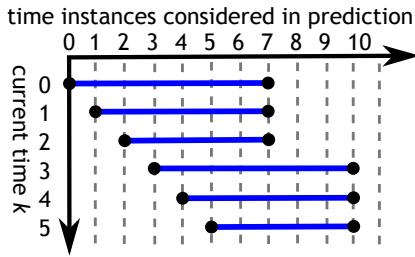


Fig. 1. Cyclic horizon (blue) for  $\underline{N} = 5$ ,  $M = 3$ .

Consequently, the optimal control problem determines a state trajectory  $\bar{x}_k$  and input sequence  $\bar{u}_k$

$$\bar{x}_k = \{\bar{x}_{k|k}, \dots, \bar{x}_{k+N_k|k}\} \quad (4a)$$

$$\bar{u}_k = \{\bar{u}_{k|k}, \dots, \bar{u}_{k+N_k-1|k}\} \quad (4b)$$

such that it is consistent with the dynamics (1), the current state  $x_k$  and that it satisfies the constraints (2) and minimizes the cost function

$$J_k(\bar{x}_k, \bar{u}_k) = S(\bar{x}_{k+N_k|k}) + \sum_{i=k}^{k+N_k-1} l(\bar{x}_{i|k}, \bar{u}_{i|k}), \quad (5)$$

where  $l(x, u) \geq 0$  is the stage cost and  $S(x) \geq 0$  the terminal cost function.

The optimal control problem solved at time  $k$  for the state  $x_k$  is denoted by  $\mathcal{M}_k(x_k)$  and given by

$$\mathcal{M}_k(x_k) : \inf_{\bar{x}_k, \bar{u}_k} J_k(\bar{x}_k, \bar{u}_k) \text{ s.t. } (\bar{x}_k, \bar{u}_k) \in \mathcal{F}_k(x_k), \quad (6)$$

where  $\mathcal{F}_k(x_k)$  denotes the set of constraints

$$\bar{x}_{i|k} \in \mathcal{X}, i = k, \dots, k + N_k - 1 \quad (7a)$$

$$\bar{u}_{i|k} \in \mathcal{U}, i = k, \dots, k + N_k - 1 \quad (7b)$$

$$\bar{x}_{k+N_k|k} \in \mathcal{T} \quad (7c)$$

$$\bar{x}_{i+1|k} = f(\bar{x}_{i|k}, \bar{u}_{i|k}), i = k, \dots, k + N_k - 1 \quad (7d)$$

$$\bar{x}_{k|k} = x_k. \quad (7e)$$

Note that we enforce the terminal state  $\bar{x}_{k+N_k|k}$  to be in the terminal set  $\mathcal{T}$ , which is typically different from  $\mathcal{X}$ . We assume that the terminal set  $\mathcal{T}$  is closed.

If this optimal control problem is feasible and a minimum exists, we denote by  $\mathcal{J}_k(x_k)$  the optimal value function and by  $\bar{x}_k^*, \bar{u}_k^*$  a (possibly non-unique) minimizer

$$\mathcal{J}_k(x_k) = \min_{(\bar{x}_k, \bar{u}_k) \in \mathcal{F}_k(x_k)} J_k(\bar{x}_k, \bar{u}_k) = J_k(\bar{x}_k^*, \bar{u}_k^*). \quad (8)$$

The resulting optimal feedbacks are given by

$$u_k = \bar{u}_{k|k}^*. \quad (9)$$

In the remainder of the work we present conditions, which guarantee recursive feasibility and stability of the predictive control scheme (6).

*Remark 1.* (Stability of varying/cyclic horizon NMPC) There exist different works, which consider stability of (N)MPC with a cyclic or varying horizon. For example in Mayne et al. (2000); Grüne and Pannek (2011) adaptive/varying horizon are discussed and in e.g. DeNicolao and Scattolini (1994); Grüne et al. (2010) a time varying control horizon. In move-blocking, see e.g. Cagienard et al. (2007); Gondhalekar and Imura (2010); Shekar and Maciejowski (2012) the number of optimization variables are reduced by fixing the input, its derivative or the offset from a control law over certain time-instances. We are interested in cyclic horizons and nonlinear system, which are not covered by the mentioned works. Also in Natarajan and Lee (2000); Lee et al. (2001) the input is fixed over a certain period and a lifting approach is used to deal with periodic operations, i.e. we do not operate the system repetitive or control systems with oscillatory behavior.

## 3. RECURSIVE FEASIBILITY AND STABILITY

As is well known (see e.g. Mayne et al. (2000)) optimality does not ensure stability, this also holds in the case of cyclic horizons. In this section we outline under which assumptions stability and recursive feasibility of the proposed NMPC scheme (6) can be guaranteed.

We first focus on recursive feasibility, in particular so-called strong feasibility, see Kerrigan (2000). In a nutshell strong feasibility means that any feasible solution of (6) will lead to a feasible problem at the next time (under nominal conditions), i.e. optimality is not required for recursive feasibility, cf. Scokaert et al. (1999).

*Definition 2.* (Strong feasibility)

The NMPC scheme (6) is called strongly feasible, if for each  $(\bar{x}_k, \bar{u}_k) \in \mathcal{F}_k(x_k)$ , there exists at least one  $(\bar{x}_{k+1}, \bar{u}_{k+1}) \in \mathcal{F}_{k+1}(x_{k+1})$ , where  $x_{k+1} = f(x_k, \bar{u}_{k|k})$ .

To guarantee strong feasibility we assume the following.

*Assumption 3.* (Conditions on cycle length  $M$ , terminal region  $\mathcal{T}$  and terminal control laws  $\{\kappa_i\}$ )

There exists a terminal set  $\mathcal{T}$ , so-called terminal control laws  $\kappa_0(x), \dots, \kappa_{M-1}(x)$  and a cycle length  $M$  such that

$$\hat{x}_{i+1} = f(\hat{x}_i, \kappa_i(\hat{x}_i)), i = 0, \dots, M - 1 \quad (10)$$

satisfies the constraints and  $\hat{x}_M \in \mathcal{T}$  for all  $\hat{x}_0 \in \mathcal{T}$ , i.e.  $\hat{x}_i \in \mathcal{X}$ ,  $\kappa_i(\hat{x}_i) \in \mathcal{U}$ ,  $\forall i = 0, \dots, M - 1$  and  $\hat{x}_M \in \mathcal{T}$ .

Basically, these assumptions require that for any state in the set  $\mathcal{T}$ , the closed loop dynamic (10) guarantees that the state is after  $M$  steps again in the set  $\mathcal{T}$  and the

sequences  $\{\kappa_i(\hat{x}_i)\}$  and  $\{\hat{x}_i\}$  satisfies the constraints (2). Note that  $\hat{x}_1, \dots, \hat{x}_{M-1}$  need not be in  $\mathcal{T}$ . In the special case  $M = 1$  we have the usual standard conditions, see Mayne et al. (2000).

This assumption allows us to guarantee strong feasibility. *Proposition 4. (Strong feasibility of cyclic horizon NMPC) If Assumption 3 holds, then the NMPC scheme (6) with a cyclically varying horizon is strongly feasible.*

**Proof.** We first focus on the cases  $k \neq jM$ ,  $j \in \mathbb{N}^0$ . Consider the input sequence  $\bar{\mathbf{u}}_{k+1}$  and the state trajectory  $\bar{\mathbf{x}}_{k+1}$  given by

$$\bar{u}_{i|k+1} := \bar{u}_{i|k}, \quad i = k+1, \dots, k+N_k-1 \quad (11a)$$

$$\bar{x}_{i|k+1} := \bar{x}_{i|k}, \quad i = k+1, \dots, k+N_k. \quad (11b)$$

Note that the sequences spans the horizon at  $k+1$ , since  $k+N_k = k+1+N_{k+1}$ . Since  $(\bar{\mathbf{x}}_k, \bar{\mathbf{u}}_k) \in \mathcal{F}_k(x_k)$  we have that  $\bar{u}_{i|k+1} \in \mathcal{U}$ ,  $\bar{x}_{i|k+1} \in \mathcal{X}$ ,  $\bar{x}_{i+1|k+1} = f(\bar{x}_{i|k+1}, \bar{u}_{i|k+1})$  for  $i = k+1, \dots, k+N_{k+1}$ ,  $\bar{x}_{k+1+N_{k+1}|k+1} \in \mathcal{T}$  and  $x_{k+1} = \bar{x}_{k+1|k+1}$ . So the constraints (7) are satisfied, i.e.  $(\bar{\mathbf{x}}_{k+1}, \bar{\mathbf{u}}_{k+1}) \in \mathcal{F}_{k+1}(x_{k+1})$ .

For the cases  $k = jM$ ,  $j \in \mathbb{N}^0$  let us choose the first part of  $\bar{\mathbf{u}}_{k+1}$  and  $\bar{\mathbf{x}}_{k+1}$  by (11) and the second part by

$$\bar{x}_{i+1|k+1} := f(\bar{x}_{i|k+1}, \kappa_i(\bar{x}_{i|k+1})) \quad (12a)$$

$$\bar{u}_{i|k+1} := \kappa_i(\bar{x}_{i|k+1}), \quad (12b)$$

where  $i = k+N_k, \dots, k+N_{k+1}$ . As above (7e) and (7a), (7b), (7d) for  $i = k, \dots, k+N_k-1$  are satisfied, since  $(\bar{\mathbf{x}}_k, \bar{\mathbf{u}}_k) \in \mathcal{F}_k(x_k)$ . Assumption 3 and  $\bar{x}_{k+N_k|k+1} \in \mathcal{T}$  guarantee that (7c) and also (7a), (7b), (7d) are satisfied for  $i = k+N_k, \dots, k+N_{k+1}$ . Therefore in this case also  $(\bar{\mathbf{x}}_{k+1}, \bar{\mathbf{u}}_{k+1}) \in \mathcal{F}_{k+1}(x_{k+1})$  holds.  $\square$

Besides feasibility we are interested in stability of the proposed scheme. To derive the stability conditions, we make the following assumptions on the stage cost  $l(x, u)$  and terminal cost  $S(x)$  to ensure that we can observe nonzero states and can guarantee decrease and convergence.

*Assumption 5. (Conditions on the costs  $l(x, u)$  and  $S(x)$ )*

- The stage cost  $l(x, u)$  is bounded below by a class  $\mathcal{K}$  function  $\alpha(\|x\|_2)$

$$l(x, u) \geq \alpha(\|x\|_2). \quad (13)$$

- The terminal cost  $S$  satisfies  $S(0) = 0$  and for the terminal constraint  $\mathcal{T}$ , terminal control laws  $\kappa_0(x), \dots, \kappa_{M-1}(x)$ , cycle length  $M$  from Assumption 3 and for all  $\hat{x}_0 \in \mathcal{T}$  and  $\hat{x}_i$  as in (10)

$$S(\hat{x}_0) \geq S(\hat{x}_M) + \sum_{i=0}^{M-1} l(\hat{x}_i, \kappa_i(\hat{x}_i)). \quad (14)$$

This assumption in combination with other conditions will guarantee that for  $(\bar{\mathbf{x}}_k, \bar{\mathbf{u}}_k) \in \mathcal{F}_k(x_k)$  the optimal cost is bounded from above by  $J_k$  (5), and that we can use a decreasing function argument to establish stability.

Next we present conditions on the (possibly) suboptimal state trajectories  $\bar{\mathbf{x}}_k$  and input sequences  $\bar{\mathbf{u}}_k$ , which guarantee stability. In a second step we establish that the conditions ensure stability for the optimal feedback (9).

*Proposition 6. (Existence, convergence and stability of suboptimal NMPC with cyclically varying horizon)*

*Let Assumptions 3 and 5 hold. If  $\mathcal{M}_k(x_k)$  is feasible, then there exist (suboptimal) feedbacks  $u_i = \bar{u}_{i|i}$  that satisfy*

$$(\bar{\mathbf{x}}_i, \bar{\mathbf{u}}_i) \in \mathcal{F}_i(x_i) \quad (15a)$$

$$J_j(\bar{\mathbf{x}}_j, \bar{\mathbf{u}}_j) \leq J_{j-1}(\bar{\mathbf{x}}_{j-1}, \bar{\mathbf{u}}_{j-1}) - \mu l(x_{j-1}, u_{j-1}) \quad (15b)$$

*for some  $\mu > 0$  and  $i \geq k$ ,  $j > k$ . Furthermore, for any such feedback the sequence  $x_{i+1} = f(x_i, \bar{u}_{i|i})$  converges to the origin. Moreover, if for a class  $\mathcal{K}$  function  $\beta$  and feedback satisfying (15)*

$$J_k(\bar{\mathbf{x}}_k, \bar{\mathbf{u}}_k) \leq \beta(\|x_k\|_2) \quad (16)$$

*holds in a neighborhood of the origin, then the closed loop is asymptotically stable.*

**Proof.** Let us first verify the existence of suboptimal feedbacks. We know that there is a  $(\bar{\mathbf{x}}_k, \bar{\mathbf{u}}_k) \in \mathcal{F}_k(x_k)$ . Let us show by induction that there is a feedback satisfying (15), by assuming that  $(\bar{\mathbf{x}}_i, \bar{\mathbf{u}}_i) \in \mathcal{F}_i(x_i)$  for some  $i \geq k$ .

If  $i \neq jM$ ,  $j \in \mathbb{N}^0$ , then choosing  $(\bar{\mathbf{x}}_{i+1}, \bar{\mathbf{u}}_{i+1})$  as in the previous proof (11) guarantees (15a) and since  $k+N_k = k+1+N_{k+1}$  (3)

$$J_i(\bar{\mathbf{x}}_i, \bar{\mathbf{u}}_i) = l(x_i, \bar{u}_{i|i}) + J_{i+1}(\bar{\mathbf{x}}_{i+1}, \bar{\mathbf{u}}_{i+1}), \quad (17)$$

cf. (5). Thus (15b) is satisfied with  $\mu = 1$ .

If  $i = jM$ ,  $j \in \mathbb{N}^0$ , then choosing  $(\bar{\mathbf{x}}_{i+1}, \bar{\mathbf{u}}_{i+1})$  as in (11), (12) guarantees that (15a) holds. Moreover,

$$\begin{aligned} J_i(\bar{\mathbf{x}}_i, \bar{\mathbf{u}}_i) &= l(x_i, \bar{u}_{i|i}) + J_{i+1}(\bar{\mathbf{x}}_{i+1}, \bar{\mathbf{u}}_{i+1}) \\ &\quad + S(\bar{x}_{i+N_i|i+1}) - S(\bar{x}_{i+1+N_{i+1}|i+1}) \\ &\quad - \sum_{j=i+N_i}^{i+1+N_{i+1}} l(\bar{x}_{j|i+1}, \bar{u}_{j|i+1}). \end{aligned} \quad (18)$$

Using (14) and  $i+1+N_{i+1} = i+N_i+M$  we obtain

$$J_i(\bar{\mathbf{x}}_i, \bar{\mathbf{u}}_i) \geq l(x_i, \bar{u}_{i|i}) + J_{i+1}(\bar{\mathbf{x}}_{i+1}, \bar{\mathbf{u}}_{i+1}), \quad (19)$$

which implies that (15b) is satisfied with  $\mu = 1$ . Hence such a feedback exists.

Next we investigate the convergence. We have that

$$J_j(\bar{\mathbf{x}}_j, \bar{\mathbf{u}}_j) \leq J_{j-1}(\bar{\mathbf{x}}_{j-1}, \bar{\mathbf{u}}_{j-1}) - \mu \alpha(\|x_{j-1}\|_2), \quad (20)$$

due to (13) and (15b). So  $J_i$  decreases unless  $x_i = 0$ . Thus  $J \rightarrow 0$ . Due to (13)  $J \rightarrow 0$  implies that also  $\|x_i\| \rightarrow 0$ , i.e.  $x_i$  converges to the origin.

Finally, we proof asymptotic stability. In addition to convergence for every  $\delta > 0$  there need to exist an  $\epsilon > 0$  such that  $\forall x_k, x_k \in \mathcal{B}_\epsilon$ ,  $i \geq 0$ ,  $x_{k+i} \in \mathcal{B}_\delta$ , c.f. Khalil (2002).

Let  $\delta > 0$  be arbitrary. Choose  $\epsilon > 0, \gamma > 0$  such that

$$J_k(\bar{\mathbf{x}}_k, \bar{\mathbf{u}}_k) \leq \gamma, \quad \forall x_k \in \mathcal{B}_\epsilon \quad (21a)$$

$$x_i \in \mathcal{B}_\delta, \quad \text{if } J_i(\bar{\mathbf{x}}_i, \bar{\mathbf{u}}_i) \leq \gamma. \quad (21b)$$

Furthermore from (15b)  $\gamma \geq J_k \geq J_{k+1} \geq \dots$  we have that  $x_i \in \mathcal{B}_\delta, \forall i \geq k, \forall x_k \in \mathcal{B}_\epsilon$ . Together with the convergence this yields that the system is asymptotically stable.

The existence of  $\gamma$  such that (21b) holds is guaranteed since  $J_k(\bar{\mathbf{x}}_k, \bar{\mathbf{u}}_k)$  is lower bounded by  $l(\bar{x}_{k|k}, \bar{u}_{k|k}) \geq \alpha(\|x_k\|_2)$  cf. (5), (13). So one possible choice for  $\gamma$  is  $\gamma > 0$  such that the set  $\{x | \alpha(\|x_k\|_2) \leq \gamma\}$  is contained in  $\mathcal{B}_\delta$ .

An  $\epsilon$  as in (21a) exists. Let  $\rho > 0$  be s. t.  $\mathcal{B}_\rho$  is contained in a neighborhood satisfying (16). This allows to choose  $\epsilon > 0$  s. t.  $\mathcal{B}_\epsilon \subseteq \mathcal{B}_\rho$  and  $\beta(x) \leq \gamma, \forall x \in \mathcal{B}_\epsilon$ , cf. (16).  $\square$

It is clear that the optimal feedbacks leads to asymptotic stability using similar assumptions.

*Corollary 7. (Stability of optimal feedbacks)*

Let Assumptions 3 and 5 hold and assume that for any feasible problem  $\mathcal{M}_i(x_i)$  (6) the minimum exists. If  $\mathcal{M}_k(x_k)$  is feasible, then for any optimal feedback  $u_i = \bar{u}_i^*$  (9) the sequence  $x_{i+1} = f(x_i, u_i)$ ,  $u_i = \bar{u}_{i|i}^*$  converges to the origin. Moreover, if the optimal cost function  $\mathcal{J}_k(x_k)$  is continuous in a neighborhood of the origin, then using any optimal feedback the closed loop is asymptotically stable.

**Proof.** We will first discuss recursive feasibility, then the convergence and finally asymptotic stability.

Since for any feasible  $\mathcal{M}_k(x_k)$  the minimizer exists and is feasible, any  $\mathcal{M}_i(x_i)$  is feasible,  $i \geq k$ , c.f. Prop. 4.

Let the optimal solution at time instance  $i \geq k$  be given by  $(\bar{x}_i^*, \bar{u}_i^*)$  and  $\mathcal{J}_i(x_i) = J_i(\bar{x}_i^*, \bar{u}_i^*)$ . Choose  $(\bar{x}_{i+1}, \bar{u}_{i+1})$  similar as in the proof of Proposition 4 (by (11) and possibly (12) with  $\bar{x}_i^* = \bar{x}_i$  and  $\bar{u}_i^* = \bar{u}_i$ ). In combination with (13), (15b) and

$$J_{i+1}(\bar{x}_{i+1}, \bar{u}_{i+1}) \geq \mathcal{J}_{i+1}(x_{i+1}), \quad (22)$$

we have that (15b) is satisfied:

$$\mathcal{J}_i(x_i) \leq \mathcal{J}_{i-1}(x_{i-1}) - l(x_{i-1}, \bar{u}_{i-1|i-1}^*). \quad (23)$$

Hence the optimal feedback satisfies the assumptions (15).

If the optimal value function is continuous in a neighborhood of the origin, then (by definition) for any  $\gamma > 0$  there exists an  $\epsilon > 0$  such that  $\forall x \in \mathcal{B}_\epsilon$ ,  $\mathcal{J}_k(x) - \mathcal{J}_k(0) \leq \gamma$ , i.e.  $\mathcal{J}_k(x) \leq \gamma$ ,  $\forall x \in \mathcal{B}_\epsilon$ . Hence there exists a class  $\mathcal{K}$  function  $\chi$  such that  $\mathcal{J}_k(x) \leq \chi(\|x\|_2)$ , compare Khalil (2002). This allows to establish asymptotic stability using similar arguments as in the previous proof ( $\chi$  satisfies (21a)).  $\square$

#### 4. EXISTENCE FOR SPECIAL CASES

In the previous section we presented conditions for recursive feasibility, convergence and stability of the proposed cyclic horizon NMPC. In this section we discuss the existence of suitable terminal costs  $S(x)$ , terminal constraints  $\mathcal{T}$  and cyclic length  $M$  such that Assumptions 3 and 5 are satisfied.

In this section we focus on nonlinear systems of the form

$$f(x, u) = Ax + Bu + Cg(x, u), \quad (24)$$

where  $g(x, u)$  satisfies for all  $x \in \tilde{\mathcal{X}} \subseteq \mathcal{X}$ ,  $u \in \tilde{\mathcal{U}} \subseteq \mathcal{U}$

$$\|g(x, u)\|_2^2 \leq \|Dx\|_2^2 + \|Eu\|_2^2. \quad (25)$$

where  $(A, B)$  is stabilizable. Note that this system class includes linear systems or Lur'e systems.

The sets  $\tilde{\mathcal{X}}, \tilde{\mathcal{U}}$  are assumed to be given as convex polytopes

$$\tilde{\mathcal{X}} = \{x | Xx \leq 1\}, \quad \tilde{\mathcal{U}} = \{u | Uu \leq 1\}, \quad (26)$$

where  $\leq$  holds element-wise and  $X \in \mathbb{R}^{c_x \times n}$ ,  $U \in \mathbb{R}^{c_u \times p}$ .

With respect to the performance criterion (5) we assume a quadratic stage and terminal cost given by

$$l(x, u) = x^T Qx + u^T Ru, \quad Q = Q^T > 0, \quad R = R^T \geq 0 \quad (27a)$$

$$S(x) = x^T Px, \quad P = P^T \geq 0, \quad (27b)$$

As terminal feedback we consider only linear control laws

$$\kappa_i(\hat{x}_i) = G_i \hat{x}_i. \quad (28)$$

As outlined, guaranteeing certain structure on the terminal set and penalty might be of advantage for certain applications, such as distributed NMPC.

We outline that one can enforce certain structure on the terminal constraints and penalty. Clearly, we cannot use any set as terminal set, but we can to some extent fix the shape of the terminal set  $\mathcal{T}$ . In detail, the terminal set  $\mathcal{T}$  should be given by

$$\mathcal{T} = \{x \text{ s.t. } \psi^{-1}x \in \tilde{\mathcal{T}}\}, \quad (29)$$

where  $\psi > 0$ . We assume that  $\tilde{\mathcal{T}}$  is bounded and contains a neighborhood of the origin, but can otherwise be arbitrary. In detail, there are  $\omega$  and  $\phi$  such that  $0 < \omega \leq \phi$  and

$$\tilde{\mathcal{T}} \subseteq \{y \text{ s.t. } \|y\|_\infty \leq \phi\} \quad \{z \text{ s.t. } \|z\|_\infty \leq \omega\} \subseteq \tilde{\mathcal{T}}. \quad (30)$$

Note that with  $\tilde{\mathcal{T}}$  we can enforce a special shape on  $\mathcal{T}$ .

For any large enough terminal penalty  $P$  and a terminal set  $\mathcal{T}$  with a desired shape satisfying (30) and Assumption 3 and 5 are satisfied as outlined in the following.

*Proposition 8. (Existence of structured terminal sets/costs)* Let constant terminal control laws  $G = G_i$  (28) be given such that  $A + BG$  is asymptotic stable. Furthermore let  $P$  with  $P > Y$  be given, where  $Y$  satisfies

$$Y = (A + BG)^T Y (A + BG) + Q + G^T R G. \quad (31)$$

If a)  $D = 0$ ,  $E = 0$ , b)  $\|D\|_\infty$  and  $\|E\|_\infty$  are small enough or c)  $f$  is analytic, then there exist  $M, \phi > 0$  such that Assumptions 3 and 5 are satisfied with the terminal cost  $x^T Px$ , the terminal control laws  $\kappa_i(x) = Gx$  and the terminal constraints as in (30).

For the proof we refer to the Appendix A. Consequently one can choose the terminal set  $\tilde{\mathcal{T}}$  as box-constraints and a diagonal terminal penalty  $P$ . Note that  $\tilde{\mathcal{T}}, P$  and  $M$  cannot be arbitrary, e.g.  $P$  needs to be large enough.

*Remark 9. (Design of terminal sets, constraints)*

Note that one can derive computational design methods for the considered system class as well as tailored methods for special systems such as Lur'e system. Due to a lack of space they are not presented here.

#### 5. SIMULATION EXAMPLE

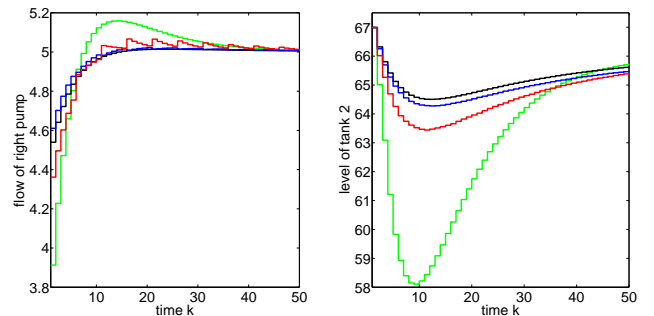


Fig. 2. Trajectories of the four tank system using cyclic horizon NMPC ( $N = 16$ ,  $M = 5$  with (33) (red), (34) (blue)). Standard NMPC ( $N_k = 20$ ) with  $\mathcal{T} = \{0\}$  (green) and (35) (black).

To illustrate the results we consider a four tank system described by the discrete-time, nonlinear model

$$x_{k+1}^I = x_k^I + T_s \rho \sqrt{x_k^{II}} - T_s \rho \sqrt{x_k^I} + T_s \xi u_k^{II} \quad (32a)$$

$$x_{k+1}^{II} = x_k^{II} - T_s \rho \sqrt{x_k^{II}} + T_s (1 - \xi) u_k^I \quad (32b)$$

$$x_{k+1}^{III} = x_k^{III} + T_s \rho \sqrt{x_k^{IV}} - T_s \rho \sqrt{x_k^{III}} + T_s \xi u_k^I \quad (32c)$$

$$x_{k+1}^{IV} = x_k^{IV} - T_s \rho \sqrt{x_k^{IV}} + T_s (1 - \xi) u_k^{II} \quad (32d)$$

with the parameters  $\rho = 0.4$ ,  $\xi = 0.35$ ,  $T_s = 5$ . We choose as steady state inputs  $u^{I,ss} = u^{II,ss} = 5$  resulting in the steady state  $x^{I,ss} = x^{III,ss} = 156$ ,  $x^{II,ss} = x^{IV,ss} = 66$ . We assume that the tank levels needs to be between 30 and 200 and the input flows between 2 and 10.

Now we want to outline the proposed NMPC scheme using a cyclic horizon and that it enables separable terminal sets and penalties. We assume that the system is split into two subsystems, where the first subsystem consists of the left tanks and the right pump. Moreover, we choose  $\underline{N} = 16$ ,  $M = 5$  and  $Q = 10I$  and  $R = I$ .

One can obtain as  $P$  and box constraints

$$P = \text{diag}(110.4, 110.6, 110.4, 110.6) \quad (33a)$$

$$T[1] = T[3] = 1.14 \cdot 10^{-4} \quad (33b)$$

$$T[2] = T[4] = 1.29 \cdot 10^{-4}. \quad (33c)$$

If we allow  $P$  and  $T$  to consist of  $2 \times 2$  blocks, we obtain

$$P = \begin{pmatrix} 83.7 & 33.1 & 0 & 0 \\ 33.1 & 63.2 & 0 & 0 \\ 0 & 0 & 83.7 & 33.1 \\ 0 & 0 & 33.1 & 63.2 \end{pmatrix} \quad (34a)$$

$$T[1] = T[2] = 10^{-3} \begin{pmatrix} 4.71 & 2.25 \\ 2.25 & 4.71 \end{pmatrix}. \quad (34b)$$

Note that the terminal penalty/cost (33), (34) are separable for the given partition, i.e. the terminal cost and constraints depend only on the states of each subsystem. Moreover they satisfy Assumptions 3 and 5.

In Figure 2 we illustrate the cyclic horizon NMPC with the terminal sets (33), (34). It is compared with standard NMPC with a horizon of  $N = 20$  and either the origin as terminal set ( $\mathcal{T} = \{0\}$ ) or an ellipsoidal terminal set and quadratic terminal penalty given by

$$P = \begin{pmatrix} 62.8 & 14.8 & -13.4 & -27.5 \\ 14.8 & 30.3 & -27.5 & -10.1 \\ -13.4 & -27.5 & 62.8 & 14.8 \\ -27.5 & -10.1 & 14.8 & 30.3 \end{pmatrix} \quad (35a)$$

$$T = 10^{-3} \begin{pmatrix} 2.49 & 2.40 & 1.92 & 1.71 \\ 2.40 & 6.60 & 1.71 & -2.27 \\ 1.92 & 1.71 & 2.49 & 2.40 \\ 1.71 & -2.27 & 2.40 & 6.60 \end{pmatrix}. \quad (35b)$$

We observe that restricting the terminal state to zero seems to results in poor performance. Moreover with cyclic horizon NMPC we can obtain results similar to standard NMPC using (35), especially if we use the less restrictive terminal constraints/penalty (34). However our approach features separable terminal sets/penalties, which enables a distributed solution using tailored algorithms. For example using an extension of the algorithm in Kögel and Findeisen (2012) to nonlinear systems based on sequential quadratic programming a distributed solution of (6) is possible.

## 6. SUMMARY

In this paper we proposed a nonlinear model predictive control scheme with a cyclically varying horizon. We presented nominal recursive feasibility and stability conditions. Furthermore we outlined that one can use structured terminal constraints and structured terminal penalties,

which is a key feature of the proposed approach and has applications in certain distributed control approaches. Moreover, we illustrated the approach by simulation examples.

In future work we will focus on improving and extending the approach as well as a more detailed evaluation. In detail, for certain system classes it is possible to consider computational design methods for the terminal constraints and cost. Furthermore, an extension to robust predictive control or control problems beyond stabilization, e.g. set-point tracking, seems to be possible.

Finally, note that the proposed approach decouples structure of the terminal set and cost from the dynamic, which could also useful for some NMPC problems beyond distributed NMPC, e.g. systems with switching dynamics.

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## Appendix A. PROOF OF PROPOSITION 9

First we discuss the linear case:  $D = E = 0$ . Since  $A + BG = \tilde{A}$  is asymptotic stable its spectral radius  $\rho(\tilde{A})$  is less than 1. Thus there is an  $M_I$  such that  $\|\tilde{A}^M\|_\infty < \xi^M$ , where  $\xi = \frac{1+\rho(\tilde{A})}{2} < 1 \forall M > M_I$  due to Gelfand's formula, see e.g. Grasselli and Pelinovsky (2008).

If  $\hat{x}_0 \in \mathcal{T}$  we have  $\forall M > M_I$  that

$$\|\hat{x}_M\|_\infty \leq \|\tilde{A}^M\|_\infty \|\hat{x}_0\|_\infty < \gamma \mu \xi^M. \quad (\text{A.1})$$

Since we have linear dynamics, we have (14) for some matrix  $\Psi(M)$  and  $\tilde{Q} = Q + G^T R G$

$$\begin{aligned} S(\hat{x}_M) &+ \sum_{i=0}^{M-1} l(\hat{x}_i, \kappa_i(\hat{x}_i)) \\ &= \hat{x}_0^T ((\tilde{A}^M)^T P \tilde{A}^M + \sum_{i=0}^{M-1} (\tilde{A}^i)^T \tilde{Q} \tilde{A}^i) \hat{x}_0 \\ &= \hat{x}_0^T (Y + \Psi(M)) \hat{x}_0, \end{aligned}$$

where  $\tilde{Q} = Q + G^T R G$ . Since  $P > Y$  and  $\|\Psi(M)\|_2 \rightarrow 0$  as  $M \rightarrow \infty$ , see e.g. Kailath et al. (2000), we can choose  $M > M_I$  such that  $Y + \Psi(M) < P$  and  $\phi \xi^M < \omega$ . Thus Assumption 5 is satisfied, since also  $Q > 0$ , see (27a).

Let such an  $M$  be given and let  $\sigma \in \mathbb{R}$  be such that  $\|\tilde{A}^i\|_\infty < \sigma$  for  $i = 0, \dots, M-1$ . Then, if  $\hat{x}_0 \in \mathcal{T}$

$$\|X \hat{x}_i\|_\infty \leq \sigma \|X\|_\infty \|\hat{x}_0\|_\infty < \sigma \phi \psi \|X\|_\infty \quad (\text{A.2a})$$

$$\|UG \hat{x}_i\|_\infty \leq \sigma \|UG\|_\infty \|\hat{x}_0\|_\infty < \sigma \phi \psi \|UG\|_\infty, \quad (\text{A.2b})$$

i.e. Assumption 3 is satisfied for any  $\psi > 0$  such that  $\sigma \phi \psi \max(\|X\|_\infty, \|UG\|_\infty) < 1$  (also  $\phi \xi^M < \omega$  holds). Now we discuss the first nonlinear case:  $\|D\|_\infty$  and  $\|E\|_\infty$  are small enough. Assume that we have for  $D = E = 0$  determined a  $M$  and a  $\gamma > 0$  such that (A.1) and (A.2) are satisfied. Since  $g$  is continuous and  $f$  depends affine on  $g$  for this  $M$  and there are  $D, E$  with  $\|D\|_\infty, \|E\|_\infty$  small enough such that for  $i \leq M$   $\tilde{f}(x) = f(x, Gx)$  satisfies

$$\|\tilde{f}^i(x)\|_\infty - \|\tilde{A}^i x\|_\infty \leq \delta \quad (\text{A.3a})$$

$$\|Z_i \tilde{f}^i(x)\|_2 - \|Z_i \tilde{A}^i x\|_2 \leq \delta. \quad (\text{A.3b})$$

Then we have that  $\|\tilde{f}^i(x)\|_\infty \leq \sigma$ ,  $i = 0, \dots, M-1$  and  $\|\tilde{f}^M(x)\|_\infty \leq \xi^M$ , i.e. Assumption 3 is satisfied. Moreover, assuming that  $\delta$  is small enough we have (compare (A.3b)) that

$$\begin{aligned} S(\hat{x}_M) &+ \sum_{i=0}^{M-1} l(\hat{x}_i, \kappa_i(\hat{x}_i)) \\ &= \tilde{f}^M(\hat{x}_0)^T P \tilde{f}^M(\hat{x}_0) + \sum_{i=0}^{M-1} \tilde{f}^i(\hat{x}_0)^T \tilde{Q} \tilde{f}^i(\hat{x}_0) \\ &\leq M \delta + \hat{x}_0^T \left( (\tilde{A}^M)^T P \tilde{A}^M + \sum_{i=0}^{M-1} (\tilde{A}^i)^T \tilde{Q} \tilde{A}^i \right) \hat{x}_0 \leq P. \end{aligned}$$

Hence Assumption 5 holds. If the function  $f(x, u)$  is analytic, then one obtains  $D$  and  $E$  with arbitrary small norm by restricting  $x$  and  $u$  to small enough  $\tilde{X}, \tilde{U}$  (the Taylor series exists and converges). Therefore there exists always terminal set and terminal constraints satisfying Assumption 3 and 5: choose first  $M$ , and bound the norm of  $D$  and  $E$  such that (A.1) holds, then obtain  $\tilde{X}, \tilde{U}$  containing the origin such that (25) is satisfied, finally choose  $\gamma > 0$  such that (A.2) holds for  $\tilde{X}$  and  $\tilde{U}$ .  $\square$