Non-existence of minimizing trajectories for steer-braking systems

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Abstract: In this paper we investigate an optimal control problem in which the objective is to decelerate a simplified vehicle model, subject to input constraints, from a given initial velocity down to zero by minimizing a quadratic cost functional. The problem is of interest because, although it involves apparently simple drift-less dynamics, a minimizing trajectory does not exist. This problem is motivated by a minimum-time problem for a fairly complex car vehicle model on a race track. Numerical computations run on the car problem provide evidence of non-existence of a minimizing trajectory and of an apparently unmotivated ripple in the steer angle. We abstract this situation to a very simple dynamics/objective setting, show that no minimizing trajectory exists, and reproduce the oscillating behavior on the steer angle as a mean to reduce the cost functional.

Keywords: Nonlinear optimal control, non-existence, trajectory exploration, direct method.

1. INTRODUCTION

Trajectory optimization problems are of key importance, and thus subject of strong interest, in most of the engineering areas and in particular in vehicle control. In many problems, arising from reality, one expects a minimizer to exist as practical evidence suggests. For example, when studying the problem of computing minimum lap-time trajectories for car vehicles on a race track, intuition (based on experience) suggests that a minimizing trajectory should exist. Numerical computations provide evidence that such an apparently well posed problem (arising from a practical scenario) in fact does not to have a minimizing solution when a simplified, but sufficiently realistic, vehicle model is used. In this paper, inspired by such numerical evidence, we study the (non-)existence of minimizing trajectories for a simplified dynamics and objective setting that includes some features of minimum lap-time trajectories in a braking phase.

Problems on existence of optimal controls have been extensively studied under various hypothesxes. See Cesari (1983), Fattorini (1999), Gelfand and Fomin (2000) (and references therein) as early general references. Next we mention a non-exhaustive set of references which are more closely related to the problem set up and the mathematical tools investigated in this paper. The existence of fuel-optimal solution for space-travel problems is discussed in Oberle and Taubert (1997). In Borisov (2000), the Fuller's phenomenon is presented, and the existence of extremals having an infinite number of switchings on finite time intervals is discussed. In Chang et al. (2006), the authors address the existence and uniqueness of an optimal trajectory for a particle in a dielectrophoretic system. In Automotive several numerical methods to solve vehicle trajectory optimization problems are available in the literature, see, e.g., Hendrikx et al. (1996), Casanova (2000), Velenis and Tsiotras (2005), Rucco et al. (2012). These numerical techniques allow to incorporate highly-complex dynamical models in the optimization process, thus producing quite realistic results. However, theoretical results characterizing the optimal trajectories are not so common.

The contributions of the paper are as follows. As a first contribution, we abstract the situation of decelerating a vehicle at maximum braking acceleration to a very simple dynamics/objective setting where it can be shown that no minimizing trajectory exists. We call the simplified model *steer-braking system* to highlight the braking by steering phenomenon appearing in the system trajectories. As main contribution of the paper, for such optimal control problem (*steer-braking problem*) we compute the infimum of the cost functional over the admissible system trajectories. Then, we prove that there does not exist a minimizing trajectory. We show it by constructing an infimizing sequence of trajectories that is not convergent to any admissible trajectory. We start the analysis with an unconstrained version of the optimal control problem and then we extend the results to the constrained version. Finally, we compute steer-braking trajectories by using an optimal control technique introduced in Hauser (2002) for unconstrained optimal control problems and extended to problems with constraints in Hauser and Saccon (2006).

The rest of the paper is organized as follows. In Section 2 we give the main motivation of this paper and we formulate the optimal control problem that we want to study. In Section 3 we address the non-existence of an optimal trajectory for the steer-braking system. In Section 4 we provide numerical computations validating the theoretical results.

2. PROBLEM SETTING

In this section we introduce the motivating scenario of this paper, namely the minimum-time problem for a car vehicle including steering forces. Then we introduce the simplified optimal control problem capturing the steer-braking phenomenon.

2.1 Motivating example: minimum lap-time problem

In Rucco et al. (2012), we have addressed the minimum lap-time problem for a single-track rigid car which includes tire models and load transfer (LT-CAR). In particular, we consider the problem of finding an LT-CAR trajectory that minimizes the time T to complete a given track with: fixed initial point, track boundary constraints, and input control constraints,

$$\begin{aligned} &\min \int_0^T 1 \; d\tau \\ &\text{subj. to } \; \dot{q}_r(t) = f_{q_r}(q_r(t), q_v(t)) \quad q_r(0) = q_{r0} \\ &\qquad\qquad \dot{q}_v(t) = f_{q_v}(q_v(t), u(t)) \quad q_v(0) = q_{v0} \\ &\qquad\qquad c_j(t, q_r(t), q_v(t), u(t)) \leq 0 \quad j = 1, \ldots, 4 \end{aligned}$$

where $\dot{q}_v(t) = f_{q_v}(q_v(t), u(t))$ is the dynamics of the vehicle, $\dot{q}_r(t) = f_{q_r}(q_r(t), q_v(t))$ is its kinematics, and $c_j(t, q_r(t), q_v(t), u(t))$, for $j = 1, \ldots, 4$, represent the track boundary, steering and tire point-wise constraints. Due to space limitations, we refer the reader to Rucco et al. (2012) (see also Rucco et al. (2010) for data and further details).

We have developed a minimum lap-time strategy to effectively solve the optimal control problem and compute lap-time trajectories. Numerical computations were performed. Here we show numerical computations for a 90deg maneuver, Figure 1. Confirming our intuition, the trajectory computed by the strategy, see Figures 1a and 1b, has the following features. First, the car accelerates in the straight portion of the track. Then it moves toward the outside edge of the corner, brakes, and turns into the corner through the apex point. Finally, the car starts the exit from the corner and accelerates with maximum traction force.

However, an "unexpected" feature appears when looking at the steer angle. Indeed, relaxing the constraint on the steer angle rate, the steer angle has a sharp swing with respect to its mean value right before the turn. In Figures 1c, 1d, 1e, and 1f, the steer angle is shown for different computations by considering the steer angle rate, $\dot{\delta}$, as control input and setting the constraint $|\dot{\delta}| \leq 20$deg/s, $|\dot{\delta}| \leq 40$deg/s, $|\dot{\delta}| \leq 60$deg/s, and $|\dot{\delta}| \leq 80$deg/s. This behavior can be explained as follows.

When the front wheel is turned so that there is a nonzero angle between the tire and the direction of motion, the tire force has two components. First, there is a lateral component that is used for steering the vehicle (e.g., going around in circles). Second, there is a longitudinal component that is actually providing a braking force (the tire is plowing). Roughly speaking, the lateral component is linear (in the steer angle) and the longitudinal component is quadratic.

Now, in order to minimize the time, the vehicle should decelerate at the maximum rate going into the turn, and at a point as late as possible. To accomplish this, the expert driver (the optimization routine) discovers that the front tires have a secret way to provide a larger braking force: steer braking! Indeed, by using an oscillatory steering motion, additional braking can be achieved with little change in the lateral motion.

In the rest of the paper we abstract this situation to a simple dynamics and objective setting. Then we show that such behavior can be explained in terms of non-existence of a minimizing trajectory.

2.2 Steer-braking problem

We consider the control problem of decelerating a "simplified" vehicle model from an initial velocity to the rest configuration. To keep the problem as simple as possible, all the parameters (e.g., mass, inertia, CG to front/rear axle distance, frictional coefficient) are considered equal to one. The vehicle can be accelerated by using the thrust, decelerated by using the brake, and can be steered by using the steer angle. Thus, the control $u_1(\cdot)$ represents the force on the vehicle due to either accelerating or decelerating and the control $u_2(\cdot)$ represents the steer angle. We choose the linear, $v(\cdot)$, and angular, $\omega(\cdot)$, velocities measured in the body coordinate system to describe the vehicle. Assuming that the tire produces an unit force laterally and for small steer angles, the dynamics is given by

$$\begin{aligned} \dot{v}(t) &= u_1(t) - u_2^2(t) \,, \\ \dot{\omega}(t) &= u_2(t) \,. \end{aligned}$$

We assume that the vehicle starts from a straight configuration, $\omega(0) = 0$, and with initial velocity $v(0) = v_0$. Based on physical limitations, the thrust and steer angle are bounded by u_{1M} and u_{2M}, respectively.

It can be shown that the thrust control does not affect the steer-braking behavior that we wish to highlight. For this reason we neglect the thrust $u_1(\cdot)$. Thus, in the rest of the paper, we will use the control $u(\cdot)$ to denote the steer angle $u_2(\cdot)$ and work with the model

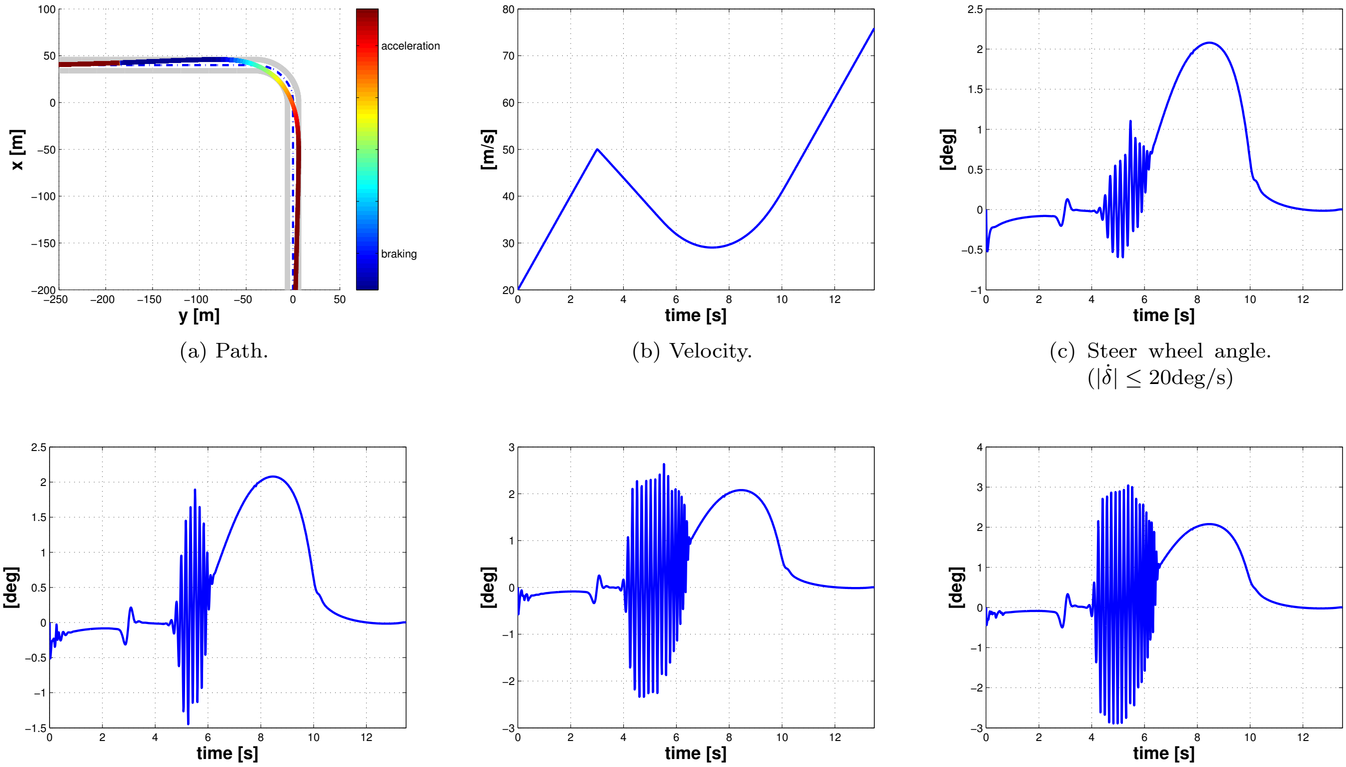
$$\begin{aligned} \dot{v}(t) &= -u^2(t) \,, \\ \dot{\omega}(t) &= u(t) \,. \end{aligned} \tag{1}$$

We call this model "steer-braking" system since the only way to reach the rest configuration is to use the steer angle as a braking control.

Given the steer-braking system we need to define the cost functional to be optimized. We recall that in the cost function we want to capture the objective of decelerating as much as possible while not changing the direction of motion ($\omega = 0$ for the simplified system). Intuitively, this behavior can be captured by minimizing both the linear and angular velocity norms. Thus, we choose the following infinite time horizon cost functional

$$J(\xi(\cdot)) = \int_0^\infty v^2(\tau) + \omega^2(\tau) + u^2(\tau) d\tau \,.$$

where $\xi(\cdot) = (v(\cdot), \omega(\cdot), u(\cdot))$.



(a) Path. (b) Velocity. (c) Steer wheel angle. ($|\dot{\delta}| \leq 20$deg/s)

(d) Steer wheel angle. ($|\dot{\delta}| \leq 40$deg/s) (e) Steer wheel angle. ($|\dot{\delta}| \leq 60$deg/s) (f) Steer wheel angle. ($|\dot{\delta}| \leq 80$deg/s)

Fig. 1. Minimum lap-time trajectory for a 90deg manoeuvre.

3. NON-EXISTENCE OF MINIMIZING TRAJECTORIES

In this section, we address the non-existence of the optimal solution for the problem stated above. To set the basis of the analysis for the constrained version, we start by considering the unconstrained version of the optimal control problem (i.e., we neglect the constraint $|u(\cdot)| \leq u_M$).

Let us consider the following optimal control problem

$$\begin{aligned} \min_{\xi(\cdot)\in L_2} \quad & J(\xi(\cdot)) = \int_0^\infty v^2(\tau) + \omega^2(\tau) + u^2(\tau) d\tau \\ \text{subj. to} \quad & \dot{v}(t) = -u^2(t), \qquad v(0) = v_0, \\ & \dot{\omega}(t) = u(t), \qquad \omega(0) = 0. \end{aligned} \tag{2}$$

In the next theorem we claim that there is no minimizing trajectory for problem (2).

The main idea is the following. We show that the cost functional is bounded below, and there is no trajectory such that the infimum is attained.

Theorem 1. Given the optimal control problem (2), the following holds true:

(i) v_0 is the infimum (or the greatest lower bound) for $J(\xi(\cdot))$, that is $v_0 = \inf_{\xi(\cdot)\in L_2} J(\xi(\cdot))$;
(ii) there does not exist $\xi(\cdot) \in L_2$ such that $J(\xi(\cdot)) = v_0$.

Proof. For any admissible $\xi(\cdot) \in L_2$ such that $J(\xi(\cdot))$ is finite, then $v(\cdot)$ is an absolute decreasing continuous function and $v(t) \geq 0 \; \forall t$. Therefore, we have

$$v(t) = \left(v_0 - \int_0^t u^2(\tau) d\tau \right) \to 0 \,,$$

so that $\int_0^\infty u^2(\tau)d\tau = v_0$. Hence

$$J(\xi(\cdot)) \geq v_0 \,, \forall \xi(\cdot) \in L_2 \,.$$

We can construct a family of trajectories $\xi_T(\cdot) \in L_2$, such that $(v_T(t), \omega_T(t)) = (0, 0)$, $\forall t \geq T > 0$, where T is a parameter. That is, let us consider

$$u_T(t) = \sqrt{\frac{v_0}{T}} \left(\mathbf{1}(t) - 2 \cdot \mathbf{1}\left(t - \frac{T}{2}\right) + \mathbf{1}(t - T) \right) ,$$

where $\mathbf{1}(t)$ denotes the unit step function,

$$\mathbf{1}(t) = \begin{cases} 1 \text{ for } t \geq 0 \\ 0 \text{ for } t < 0 \end{cases} .$$

By construction, we have, see Figure 2,

$$v_T(t) = v_0 - \frac{v_0}{T} t \,,$$

$$\omega_T(t) = \begin{cases} \sqrt{\dfrac{v_0}{T}} t & \text{for } t \in \left[0, \dfrac{T}{2}\right] \\ \sqrt{v_0 T} - \sqrt{\dfrac{v_0}{T}} t & \text{for } t \in \left[\dfrac{T}{2}, T\right] \end{cases} .$$

Then, we have $\|u_T(\cdot)\|^2_{L_2} = v_0$, $\|v_T(\cdot)\|^2_{L_2} = \frac{v_0^2 T}{3}$, $\|\omega_T(\cdot)\|^2_{L_2} = \frac{v_0 T^2}{12}$, and the cost function turns to be

$$\begin{aligned} J(\xi_T(\cdot)) &= \frac{v_0^2 T}{3} + \frac{v_0 T^2}{12} + v_0 \\ &= v_0 \left(\frac{T^2}{12} + \frac{v_0 T}{3} + 1 \right) . \end{aligned}$$

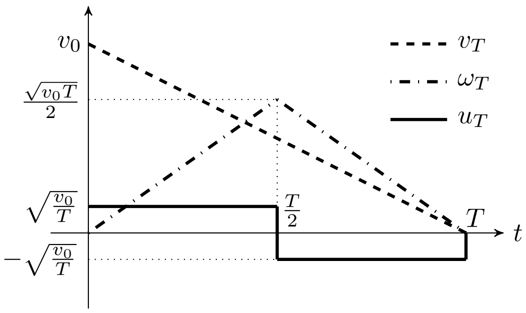


Fig. 2. Infimizing sequence of the (unconstrained) optimal control problem.

Thus, by taking $T = \frac{1}{k}$, the continuous sequence $\{\xi_k(\cdot)\} = \{v_k(\cdot), \omega_k(\cdot), u_k(\cdot)\}$ is such that

$$\lim_{k\to\infty} J(\xi_k(\cdot)) = v_0 .$$

In order to prove statement (ii) assume, by contradiction, that $\{\xi_k(\cdot)\}$ is an infimizing sequence and that $u_k(\cdot) \to \bar{u}(\cdot) \in L_2$. Then

$$\bar{v}(t) = v_0 - \int_0^t \bar{u}^2(\tau)d\tau$$

so that $\bar{v}(\cdot)$ is an absolutely continuous function, with $\bar{v}(0) = v_0$, that is non-increasing. Since $J(\xi(\cdot)) < +\infty$ implies that $v(t) \to 0$, for continuity, there exists a $t_0 > 0$ such that $\bar{v}(t_0) = \frac{v_0}{2}$ so that $\int_0^\infty \bar{v}^2(\tau)d\tau \geq t_0 \frac{v_0^2}{4} > 0$. This contradicts the fact that $\int_0^\infty v_k^2(\tau)d\tau \searrow 0$ for an infimizing sequence. □

Next, we extend the theorem to the constrained optimal control problem.

Let us consider the control input constraint $|u(t)| \leq u_M$, $\forall t$. We address the following constrained optimal control problem:

$$\begin{aligned} \min_{\xi(\cdot)\in L_2} \quad & J(\xi(\cdot)) = \int_0^\infty v^2(\tau) + \omega^2(\tau) + u^2(\tau)d\tau \\ \text{subj. to} \quad & \dot{v}(t) = -u^2(t), \qquad v(0) = v_0, \\ & \dot{\omega}(t) = u(t), \qquad \omega(0) = 0, \\ & |u(t)| \leq u_M, \qquad \forall t \end{aligned} \tag{3}$$

First, we give the main idea for constructing candidate trajectories. We focus our attention on control trajectories assuming the boundary values u_M and $-u_M$. If $u(t) = u_M$ on some time interval $[t_0, t_1]$, then we have

$$\begin{aligned} \dot{v}(t) &= -u_M^2 \\ \dot{\omega}(t) &= u_M, \end{aligned} \tag{4}$$

so that $\dot{v}(t) = -u_M\dot{\omega}(t)$ and

$$v(t) = -u_M\omega(t) + (u_M\omega(t_0) + v(t_0)) .$$

In other words, as long as the control is set to $u(t) \equiv u_M$, the state trajectory $(v(t), \omega(t))$ lies on a line $v(t) = -u_M\omega(t) + b$ for some constant b depending on the initial state at the beginning of the time interval. Consistently, as long as the control is set to $u(t) \equiv -u_M$, the trajectory stays on a line $v(t) = u_M\omega(t) + c$ for some constant c. Thus, applying a switching input causes the state trajectory to follow the two different families of lines as shown in Figure 3.

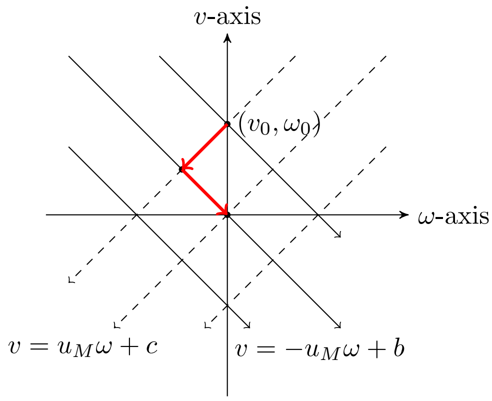


Fig. 3. Phase-plane trajectory: geometric interpretation.

In the next theorem we show that a sequence of trajectories of this sort reduces the cost functional, but does not converge to any minimizing trajectory. Thus, we prove the non-existence of an optimal control.

Theorem 2. Given the optimal control problem (3), the following holds true:

(i) $v_0\left(\frac{v_0^2}{3u_M^2} + 1\right) = \inf_{\xi(\cdot)\in L_2} J(\xi(\cdot))$;
(ii) there does not exist $\xi(\cdot) \in L_2$ such that $J(\xi(\cdot)) = v_0\left(\frac{v_0^2}{3u_M^2} + 1\right)$.

Proof. For the sake of space, we provide just a sketch of the proof. For any admissible $\xi(\cdot) \in L_2$ such that $J(\xi(\cdot))$ is finite, $v(\cdot)$ is an absolute decreasing continuous function and $v(t) \geq 0$ $\forall t$. Then we can show that $J(\xi(\cdot)) \geq v_0\left(\frac{v_0^2}{3u_M^2} + 1\right)$ for all $\xi(\cdot)$ admissible. Following a similar idea as in the unconstrained case, we can construct a sequence of piece-wise constant input trajectories with compact support $[0, T]$, $T > 0$. Since the input is bounded, the interval $[0, T]$ cannot be reduced arbitrarily (to reduce the cost functional) as in the unconstrained case. Therefore, in order to reach the infimum defined in (i), an increasing number of switches of the control input are needed.

The statement (ii) follows the same line of *Theorem 1* (it can be proven by contradiction). □

4. NUMERICAL APPROACH FOR COMPUTING STEER-BRAKING TRAJECTORIES

In this section we provide some numerical computations that highlight the behavior studied in the paper. For practical purposes, we are interested in finite horizon approximations of the infinite horizon optimization problem. In particular, let us define the equivalent finite horizon optimization problem

$$\begin{aligned} \min_{\xi(\cdot)} \quad & \int_0^T v^2(\tau) + \omega^2(\tau) + u^2(\tau)d\tau + m(v(T), \omega(T)) \\ \text{subj. to} \quad & \dot{v}(t) = -u^2(t), \qquad v(0) = v_0, \\ & \dot{\omega}(t) = u(t), \qquad \omega(0) = 0, \\ & |u(t)| \leq u_M, \qquad \forall t \end{aligned} \tag{5}$$

where the terminal cost $m(v(T), \omega(T))$ allows us to retain desirable features of the infinite-horizon problem, see Jadbabaie et al. (2001). The problem (5) has been addressed numerically by using a nonlinear least square method for the optimization of trajectory functionals with constraints, see Hauser (2002) and Hauser and Saccon (2006).

4.1 Optimization of trajectory functionals with constraints

We recall that a trajectory is a (state-input) curve $\xi = (x(\cdot), u(\cdot))$ defined on $L_\infty[0,T]$ such that

$$\dot{x}(t) = f(x(t), u(t)) \,.$$

for all $t \in [0,T]$, where $f : \mathbb{R}^2 \times \mathbb{R}^1 \longrightarrow \mathbb{R}^2$ and $f \in \mathcal{C}^r$ with $0 \le r \le \infty$. Let us define the cost functional

$$h(\xi) = \int_0^T v^2(\tau) + \omega^2(\tau) + u^2(\tau) d\tau + m(v(T), \omega(T)) \,,$$

and denote $\mathcal{T}$ the manifold of bounded trajectories $(x(\cdot), u(\cdot))$ on $[0,T]$.

In order to handle the constraint $|u(\cdot)| \le u_M$, we use a barrier function relaxation, developed in Hauser and Saccon (2006). Formally, let

$$c(u(t)) = \left(\frac{u(t)}{u_M}\right)^2 - 1 \le 0 \,, \quad \forall t \in [0,T]$$

denote the input constraint. For a given (state-input) curve $\xi = (x(\cdot), u(\cdot))$, a barrier functional can be defined as

$$b_\delta(\xi) = \int_0^T \beta_\delta(-c(u(\tau))) d\tau$$

where

$$\beta_\delta(x) = \begin{cases} -\log x, & x > \delta \\ \frac{1}{2}\left[\left(\frac{x-2\delta}{\delta}\right)^2 - 1\right] - \log \delta, & x \le \delta \end{cases} .$$

Using the barrier functional defined above, the relaxed version of problem (5) is given by

$$\min_{\xi \in \mathcal{T}} h(\xi) + \epsilon b_\delta(\xi). \qquad (6)$$

Using the projection operator defined in Hauser (2002) to locally parametrize the trajectory manifold, we may convert the constrained optimization problem (6) into one of minimizing the unconstrained functional

$$g_{\epsilon,\delta}(\xi) = h(\mathcal{P}(\xi)) + \epsilon b_\delta(\mathcal{P}(\xi)) \,. \qquad (7)$$

The PRojection Operator based Newton method for Trajectory Optimization (PRONTO) is used to optimize the functional (7), as part of a continuation method to seek an approximate solution to (6). The strategy is to start with a reasonably large ϵ and δ. Then, for the current ϵ and δ, the problem $\min g_{\epsilon,\delta}(\xi)$ is solved using the Newton method starting from the current trajectory.

Notice that this method has been effectively applied also to compute aggressive maneuvers of aerial vehicles, Notarstefano et al. (2005) Notarstefano and Hauser (2010).

4.2 Numerical computations

In Figure 4, we show the steer-braking trajectory obtained by applying the optimization algorithm described above with $v(0) = 5$, $\omega(0) = 0$, and $u_M = 1$. The time horizon is $T = 10$s and the sampling period is 0.01s. The initial trajectory $\xi_0(t) = [v_0(t), \omega_0(t), u_0(t)]^T$, $t \in [0,T]$, is chosen as follow. We set the control input as $u_0(t) = 0.1\cos(2\pi\tilde{f}t)$, with $\tilde{f} = \frac{1}{v(0)}$, and the state curves obtained through integration of the steer-braking system, namely

$$v_0(t) = v(0) - \int_0^t u_0^2(\tau) d\tau \,,$$

$$\omega_0(t) = \omega(0) + \int_0^t u_0(\tau) d\tau \,.$$

The algorithm terminates at the set maximum number of steps (30 steps) without reaching the desired descent termination condition, see Figure 5. At the first iterations, the algorithm shows a quadratic convergence rate. However, starting from the seventh iteration, we have observed that the second order functional approximations is not positive definite and the algorithm shows a linear convergence rate. This fact provides strong evidence for the non-existence of a minimizing trajectory.

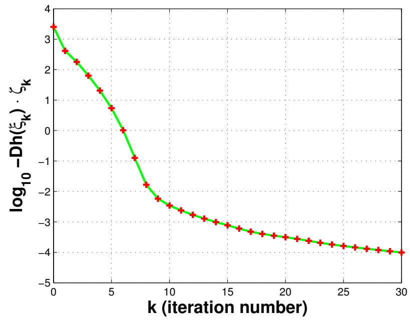


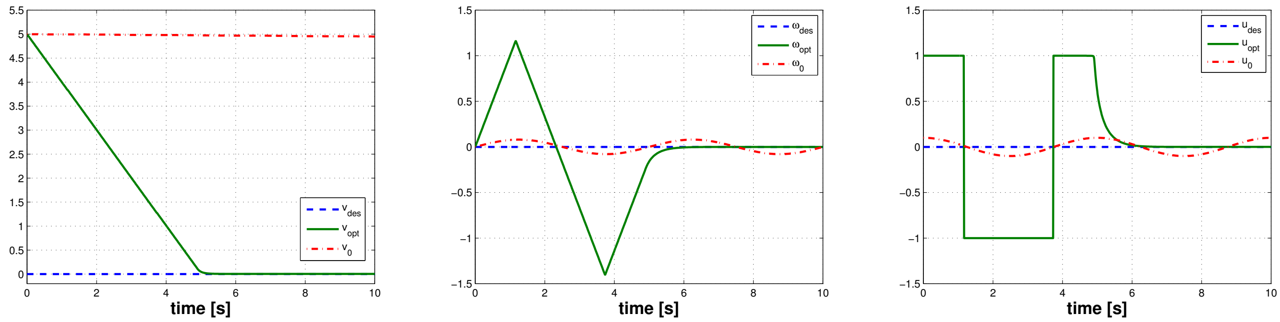
Fig. 5. Convergence rate. We show $\log_{10}(-Dh(\xi_k) \cdot \zeta_k)$ as a function of the number of iterations for a given couple of ϵ and δ. ξ_k and ζ_k are respectively the trajectory and the (optimal) descent direction at the k-th iteration.

Figure 6 shows the trajectory of the steer-braking system by considering the thrust as an additional control input. In particular, we show the steer-braking trajectory obtained by choosing as initial thrust input, $u_{10}(\cdot)$, the zero function and as initial steer input the cosine function $u_{20}(t) = 0.1\cos(2\pi\tilde{f}t)$, with frequency $\tilde{f} = 10v(0)$ (once again, the state curves are obtained through integration of the system).

As we can see in Figure 6c, the thrust u_1 is saturated and, in order to obtain a greater deceleration, the steer u_2 is used thus showing the steer-braking behavior. This fact confirms our assumption on the model in Section 2.2: the thrust control does not affect the steer-braking behavior. It is worth noting that the steer-braking trajectories obtained throughout the optimization are affected by the frequency $\tilde{f}$ of the initial trajectory. In other words, by choosing the frequency of the initial trajectory, we converge to a trajectory with the same frequency.

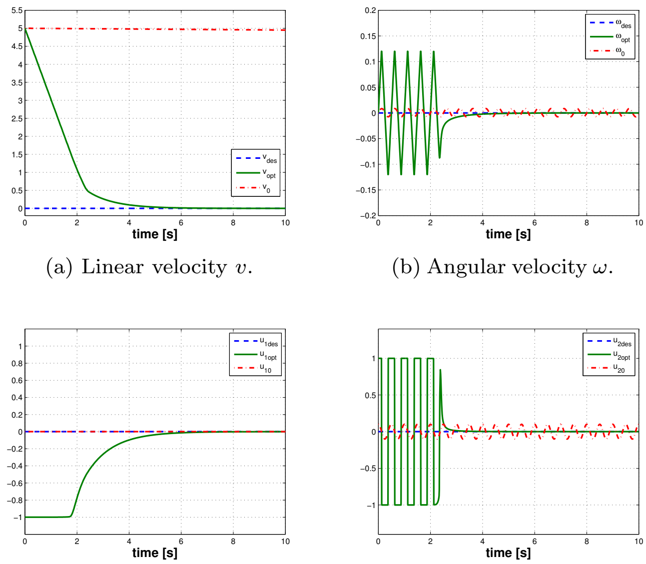
5. CONCLUSIONS

In this paper we have addressed the optimal control problem for a simplified dynamics and objective setting which includes some features of the minimum lap-time trajectory in a braking phase. A detailed theoretical analysis on the non-existence of an optimal trajectory is given. Numerical computations were performed highlighting the expected



(a) Linear velocity v.
(b) Angular velocity ω.
(c) Control input u.

Fig. 4. Steer-braking trajectory. In (a)-(b)-(c), the solid, dash, and dash-dot lines are the optimal, desired, and initial trajectories, respectively.



(a) Linear velocity v.
(b) Angular velocity ω.
(c) Thrust control u_1.
(d) Steer control u_2.

Fig. 6. Steer-braking trajectory by including the thrust as an additional control input.

steer-braking behavior. This analysis is carried out as a preliminary step to investigate the steer-braking feature in the minimum lap-time trajectory for a racing car.

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