Moment matching for nonlinear port Hamiltonian and gradient systems ⋆

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Abstract: The problem of moment matching with preservation of port Hamiltonian and gradient structure is studied. Based on the time-domain approach to linear moment matching, we characterize the (subset of) port Hamiltonian/gradient models from the set of parameterized models that match the moments of a given port Hamiltonian/gradient system, at a set of finite points.

1. INTRODUCTION

Port Hamiltonian and gradient systems represent an important class of systems used in modeling, analysis and control of physical systems, see e.g. van der Schaft [2000], Willems [1972]. Physical modeling often leads to systems of high dimension, usually difficult to analyze and simulate and unsuitable for control design. It is customary to perform model reduction on linearized versions of the models. However the linearised models might lose certain important features which should be retained by the reduced order model. Hence model reduction for nonlinear models is more suitable. There are many control oriented model reduction techniques. We mention here the balanced truncation method, which preserves stability and passivity, but is highly difficult to compute, especially in a nonlinear setting, see e.g., Ionescu et al. [2010, 2011].

In the problem of model reduction, moment matching techniques represent an efficient tool, see e.g. Antoulas [2005], van Dooren [1995], Feldman and Freund [1995], Jaimoukha and Kasenally [1997], Antoulas and Sorensen [1999] for a complete overview for linear systems. With such techniques, the (reduced order) model is obtained by constructing a lower degree rational function that approximates a given transfer function (assumed rational). The low degree rational function matches the given transfer function at various points in the complex plane. Krylov methods have been applied to linear port Hamiltonian systems, resulting in reduced order models that match the Markov parameters of the given port Hamiltonian system, see e.g., Polyuga and van der Schaft [2009, 2010], Gugercin et al. [2009], Wolf et al. [2010]. Recently in Ionescu and Astolfi [2011] the time-domain moment matching techniques have been applied to linear port-Hamiltonian systems, resulting in *subclasses of* reduced order models that achieve moment matching and preserve the port Hamiltonian structure. Therein the symmetric or gradient case has not been treated.

In this paper, we use the time-domain approach to nonlinear moment matching from the recent works Astolfi [2010], Ionescu and Astolfi [2010]. This approach yields a parametrization of a family of reduced order models achieving moment matching, in the sense defined in Astolfi [2010]. These models depend on a set of free parameters, useful for enforcing properties such as, e.g., passivity, stability Ionescu and Astolfi [2010], relative degree, etc. We characterize the reduced order models that preserve the port Hamiltonian or gradient structure and matches the moments of the given nonlinear port Hamiltonian system. In other words, from the family of models that achieve moment matching, we select the reduced order model that inherits the port Hamiltonian/gradient form, by picking a particular (subset of) member(s), i.e., we obtain a (family of) reduced order model(s) that matches (match) the moments and inherit (inherit) the structure of the given system.

The paper is organized as follows. In Section 2, we give a brief overview of the notions of nonlinear port Hamiltonian and gradient systems and present the definitions of moment and moment matching for nonlinear systems. In Section 3, we discuss the problem of moment matching with preservation of the port Hamiltonian and gradient structure. We give a brief overview of the linear results and then give the nonlinear extensions. Hence we compute the families of port Hamiltonian and gradient models that achieve moment matching. Furthermore, we give a necessary and sufficient condition for a reduced order model that achieves moment matching to be a port Hamiltonian or gradient model. The paper is completed by some conclusions.

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2. PRELIMINARIES

2.1 Definitions

The systems we study are defined as follows. Consider a nonlinear single-input, single-output, continuous-time system, described by equations of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \\ y &= h(x), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$ and f, g and h are smooth mappings. In the sequel we define two particular instanced of a nonlinear system (1). A nonlinear system (1) is a port Hamiltonian system if it satisfies the following equations for all u

$$\begin{aligned} \dot{x} &= (J(x) - R(x))\frac{\partial \mathcal{H}(x)}{\partial x} + g(x)u, \\ h(x) &= g^T(x)\frac{\partial \mathcal{H}(x)}{\partial x}, \end{aligned} \tag{2}$$

where $J(x) = -J^T(x) \in \mathbb{R}^{n\times n}$, $R(x) = R^T(x) \in \mathbb{R}^{n\times n}$ and $\mathcal{H} : \mathbb{R}^n \to \mathbb{R}$ is a smooth function such that $\mathcal{H}(x) > 0,\ \forall x \neq 0$, $\mathcal{H}(0) = 0$, called the Hamiltonian.

A nonlinear system (1) is a gradient system if there exists an invertible matrix $G(x) = G^T(x) \in \mathbb{R}^{n\times n}$, such that (1) can be written as

$$\begin{aligned} G(x)\dot{x} &= -\frac{\partial \mathcal{P}(x)}{\partial x} + \frac{\partial h(x)}{\partial x}u, \\ y &= h(x), \end{aligned} \tag{3}$$

where $\mathcal{P} : \mathbb{R}^n \to \mathbb{R}$ is a smooth potential function and $G(x)$ defines a pseudo-Riemannian metric on $\mathbb{R}^n$, for all u see Crouch [1981], Cortes et al. [2006].

2.2 Nonlinear moment matching

Consider the nonlinear system (1) and the signal generator

$$\begin{aligned} \dot{\omega} &= s(\omega), \\ \theta &= l(\omega), \end{aligned} \tag{4}$$

with $\omega(t) \in \mathbb{R}^\nu$, $\theta(t) \in \mathbb{R}$ and $s(\cdot)$ and $l(\cdot)$ smooth. Further assume that $f(0) = 0$, $h(0) = 0$, $s(0) = 0$ and $l(0) = 0$.

Assumption 1. The signal generator is zero-state observable and Poisson stable.

Let $\pi(\omega)$, $\pi : \mathbb{R}^\nu \to \mathbb{R}^n$ be a mapping, locally defined in the neighbourhood of $\omega = 0$, which solves the partial differential equation

$$\frac{\partial \pi(\omega)}{\partial \omega}s(\omega) = f(\pi(\omega)) + g(\pi(\omega))l(\omega). \tag{5}$$

Assumption 2. The mapping $\pi(\omega)$ is the unique solution of equation (5).

Definition 1. Astolfi [2008, 2010] Consider system (1) and the signal generator (4) such that Assumptions 1 and 2 hold. The function $h(\pi(\omega))$ is the moment of (1) at $\{s(\omega), l(\omega)\}$. □

Definition 2. (Moment matching). Astolfi [2010] The system $\dot{\xi} = \phi(\xi) + \delta(\xi)u$, $\eta = \kappa(\xi)$, with $\xi(t) \in \mathbb{R}^\nu$, matches the moment of (1) at $\{s(\omega), l(\omega)\}$ if it has the same moment at $\{s(\omega), l(\omega)\}$ as (1), i.e., the equation

$$h(\pi(\omega)) = \kappa(p(\omega)), \tag{6}$$

for a well defined diffeomorphishm $p(\omega)$. □

Theorem 1. (Nonlinear moment matching). Astolfi [2010] Consider a nonlinear system described by equations of the form

$$\begin{aligned} \dot{\xi} &= \phi(\xi) + \delta(\xi)u, \\ \psi &= \kappa(\xi), \end{aligned} \tag{7}$$

where $\xi(t) \in \mathbb{R}^\nu$, $\psi(t) \in \mathbb{R}$, $\delta(\xi) \in \mathbb{R}^n$ and κ is a smooth mapping, $\kappa(0) = 0$. Then the system (7) matches the moment of (1) at $\{s(\omega), l(\omega)\}$ if the equation

$$\phi(p(\omega)) + \delta(p(\omega))l(\omega) = \frac{\partial p(\omega)}{\partial \omega}s(\omega) \tag{8}$$

has a unique solution $p(\omega)$, $p(0) = 0$, such that

$$h(p(\omega)) = \kappa(p(\omega)). \tag{9}$$

□

Assumption 3. Assume $p(\omega)$ is a diffeomorphism and $\nu < n$.

Selecting $p(\omega) = \omega$, a class of reduced models of order ν, parameterized in δ, that achieve moment matching is described by

$$\Sigma_{\delta(\xi)} : \begin{cases} \dot{\xi} = s(\xi) - \delta(\xi)l(\xi) + \delta(\xi)u, \\ \psi = h(\pi(\xi)), \end{cases} \tag{10}$$

where δ is such that the equation

$$s(p(\omega)) - \delta(p(\omega))l(p(\omega)) + \delta(p(\omega))l(\omega) = \frac{\partial p(\omega)}{\partial \omega}s(\omega)$$

has the unique solution $p(\omega) = \omega$.

In the sequel, assuming that the nonlinear system (1) is port Hamiltonian/gradient , we compute the parameter δ which yield the port Hamiltonian/gradient reduced order models that approximate (1) by achieving moment matching at $\{s(\omega), l(\omega)\}$.

3. MOMENT MATCHING FOR PORT HAMILTONIAN AND GRADIENT SYSTEMS

3.1 Linear systems

Let $S \in \mathbb{C}^{\nu\times\nu}$ and $L \in \mathbb{C}^{1\times\nu}$ be such that the pair (L, S) is observable. Consider a linear port Hamiltonian system (2) with $J = -J^T$, $J \in \mathbb{R}^{n\times n}$, $R = R^T > 0$, $R \in \mathbb{R}^{n\times n}$ and $g(x) = B \in \mathbb{R}^n$ constant matrices and $\mathcal{H}(x) = \frac{1}{2}x^TQx$, where $Q \in \mathbb{R}^{n\times n}$ is a constant symmetric matrix. Let $\Pi \in \mathbb{R}^{n\times\nu}$ be the unique solution of the Sylvester equation

$$(J - R)Q\Pi + BL = \Pi S. \tag{11}$$

Similarly consider a gradient system (3) with $G(x) = T$, $T \in \mathbb{R}^{n\times n}$ an invertible, symmetric constant matrix, $h(x) = C \in \mathbb{R}^{1\times n}$ a constant matrix, and $P = P^T$, $P \in \mathbb{R}^{n\times n}$ be such that $\mathcal{P}(x) = \frac{1}{2}x^TPx$. Let $\bar{\Pi}$ be the unique solution of the Sylvester equation

$$T\bar{\Pi}S + P\bar{\Pi} = C^TL. \tag{12}$$

The following result yields a ν order port Hamiltonian/gradient system that matches the moments of the given port Hamiltonian/gradient system at a set of ν finite interpolation points given by the set $\sigma(S)$.

Proposition 1. Let L and S be two matrices such that (L, S) is an observable pair. Then, the following statements hold.

(1) (Ionescu and Astolfi [2011]) Consider a linear port Hamiltonian system and let Π be the unique solution of (11). A port Hamiltonian reduced order model achieving moment matching at $\sigma(S)$ is given by

$$\Sigma_{\Pi Ham}: \begin{cases} \dot{\xi} = (\widetilde{J} - \widetilde{R})\widetilde{Q}\xi + \widetilde{B}u, \\ \psi = \widetilde{B}^*\widetilde{Q}\xi, \end{cases} \tag{13}$$

with $\xi(t) \in \mathbb{R}^\nu$ and

$$\begin{aligned} &\widetilde{J} = \Pi^* Q J Q \Pi,\ \widetilde{R} = \Pi^* Q R Q \Pi, \\ &\widetilde{Q} = (\Pi^* Q \Pi)^{-1},\ \widetilde{B} = \Pi^* Q B. \end{aligned} \tag{14}$$

(2) Consider a linear gradient system and let $\bar{\Pi}$ be the unique solution of (12). A gradient reduced order model achieving moment matching at $\sigma(S)$ is given by

$$\Sigma_{\Pi Grad}: \begin{cases} \widetilde{T}\dot{\xi} = -\widetilde{P}\xi + \widetilde{C}u, \\ \psi = \widetilde{C}\xi, \end{cases} \tag{15}$$

with $\xi(t) \in \mathbb{R}^\nu$ and

$$\begin{aligned} &\widetilde{T} = \Pi^* T \Pi,\ \widetilde{P} = \Pi^* P \Pi, \\ &\widetilde{C} = C\Pi. \end{aligned} \tag{16}$$

□

Consider the class of ν order models

$$\Sigma_G: \begin{cases} \dot{\xi} = (S - GL)\xi + Gu, \\ \psi = C\Pi\xi, \end{cases} \tag{17}$$

with $\xi(t) \in \mathbb{R}^n$ and Π the unique solution of (11) or (12).

We now show that $\Sigma_{\Pi Grad}$ and $\Sigma_{\Pi Ham}$ are members of the family of reduced order models (17), obtained for particular instances of the parameter G.

Theorem 2. Consider the class of ν order systems Σ_G as in (17). Then the following statements hold.

(1) (Ionescu and Astolfi [2011]) Let Σ_G be a reduced order model of the system (2). Then Σ_G is equivalent [1] to a port Hamiltonian system $\Sigma_{\Pi Ham}$ as in (13), i.e. (17) preserves the port Hamiltonian structure of (2), if and only if $G = (\Pi^* Q \Pi)^{-1}\Pi^* Q B$.
(2) Let Σ_G be a reduced order model of the system (3). Then Σ_G is equivalent to a gradient system $\Sigma_{\Pi Grad}$ as in (15), i.e. Σ_G preserves the gradient structure of (3), if and only if $G = (\Pi^* T \Pi)^{-1}\Pi^* C^*$. □

Remark 1. A system Σ_G is gradient (symmetric) if and only if there exists an invertible matrix $\bar{T} \in \mathbb{R}^{\nu\times\nu}$ such that $\bar{T}(S - GL) = (S - GL)^*\bar{T}$ and $G^*\bar{T} = C\Pi$ (see, e.g., Scherpen and van der Schaft [2011]). Computations yield that $\bar{T} = \widetilde{T} = \Pi^* T \Pi$, as in Theorem 2 (note that $\bar{T}$ is not unique). □

3.2 Nonlinear systems

In this section we compute the reduced models of order $\nu < n$, that achieve moment matching in the sense of Theorem 1 and preserve the physical structure of a given port Hamiltonian or gradient system, respectively.

[1] Two systems described by state-space equations are equivalent if they have the same transfer functions, i.e., the same input-output behaviour.

Consider the mapping π which satisfies equation (5), such that Assumption 2 holds. For a port Hamiltonian system (2) equation (5) becomes

$$\frac{\partial \pi(\omega)}{\partial \omega} s(\omega) = [J(\pi(\omega)) - R(\pi(\omega))] \frac{\partial^T \mathcal{H}(x)}{\partial x}\bigg|_{x=\pi(\omega)} + g(\pi(\omega)) l(\omega). \tag{18}$$

By Definition 1, the moment of the port Hamiltonian system (2) at $\{s(\omega), l(\omega)\}$ is $g^T(\pi(\omega))\frac{\partial^T \mathcal{H}(x)}{\partial x}\Big|_{x=\pi(\omega)}$. Similarly, for a port Hamiltonian system (2) equation (5) becomes

$$G(\pi(\omega))\frac{\partial \pi(\omega)}{\partial \omega} s(\omega) = -\frac{\partial^T \mathcal{P}(x)}{\partial x}\bigg|_{x=\pi(\omega)} + \frac{\partial^T h(x)}{\partial x}\bigg|_{x=\pi(\omega)} l(\omega). \tag{19}$$

Throughout the rest of the paper, we make the following working assumptions on the Jacobian of π.

Assumption 4. The Jacobian of $\pi(\omega)$ has full row rank.

The following result yields the (a family of) reduced order port Hamiltonian model(s) which matches (match) the moment of the given port Hamiltonian system.

Theorem 3. Consider the pair of functions $\{s(\omega), l(\omega)\}$, $\omega \in \mathbb{R}^\nu$ satisfying equations (4) and the mapping π such that Assumptions 2 and 4 hold. The following statements hold.

(1) Consider the system (2). Let a port Hamiltonian reduced model of order ν be given by equations of the form

$$\Sigma_{\rho(x)Ham}: \begin{cases} \dot{\omega} = (\bar{J}(\omega) - \bar{R}(\omega))\dfrac{\partial^T \bar{\mathcal{H}}(\omega)}{\partial \omega} + \bar{g}(\omega)u, \\ \psi = \bar{g}^T(\omega)\dfrac{\partial^T \bar{\mathcal{H}}(\omega)}{\partial \omega}, \end{cases} \tag{20}$$

where $\omega(t) \in \mathbb{R}^\nu$ and

$$\begin{aligned} \bar{J}(\omega) &= \frac{\partial \rho(x)}{\partial x}\bigg|_{x=\pi(\omega)} J(\pi(\omega)) \frac{\partial \rho^T(x)}{\partial x}\bigg|_{x=\pi(\omega)}, \\ \bar{R}(\omega) &= \frac{\partial \rho(x)}{\partial x}\bigg|_{x=\pi(\omega)} R(\pi(\omega)) \frac{\partial \rho^T(x)}{\partial x}\bigg|_{x=\pi(\omega)}, \\ \bar{g}(\omega) &= \frac{\partial \rho(x)}{\partial x}\bigg|_{x=\pi(\omega)} g(\pi(\omega)),\ \bar{\mathcal{H}}(\omega) = \mathcal{H}(\pi(\omega)), \end{aligned} \tag{21}$$

with $\rho: \mathbb{R}^n \to \mathbb{R}^\nu$. If the mapping ρ satisfies the following properties
(a) $\rho(x)|_{x=\pi(\omega)} = \rho(\pi(\omega)) = \omega$;
(b) $\frac{\partial \bar{\mathcal{H}}(\omega)}{\partial \omega}\frac{\partial \rho(x)}{\partial x}\Big|_{x=\pi(\omega)} g(\pi(\omega)) = \frac{\partial \mathcal{H}(x)}{\partial x}\Big|_{x=\pi(\omega)} g(\pi(\omega))$.
then (20) (parameterized in ρ) matches the moment $g^T(\pi(\omega))\frac{\partial^T \mathcal{H}(x)}{\partial x}\Big|_{x=\pi(\omega)}$ of the system (2) at $\{s(\omega), l(\omega)\}$,
(2) Consider the system (3). The gradient reduced model of order ν, that matches the moment $h(\pi(\omega))$ of the system (3) at $\{s(\omega), l(\omega)\}$, is given by

$$\Sigma_{\pi(\omega)grad}: \begin{cases} \bar{G}(\omega)\dot{\omega} = -\dfrac{\partial^T \bar{\mathcal{P}}(\omega)}{\partial \omega} + \dfrac{\partial^T \bar{h}(\omega)}{\partial \omega}u, \\ \psi = \bar{h}(\omega), \end{cases} \tag{22}$$

where

$$\bar{G}(\omega) = \frac{\partial^T \pi(\omega)}{\partial \omega} G(\pi(\omega)) \frac{\partial \pi(\omega)}{\partial \omega},$$
$$\bar{\mathcal{P}}(\omega) = \mathcal{P}(\pi(\omega)),\ \bar{h}(\omega) = h(\pi(\omega)). \tag{23}$$

□

Note that $\Sigma_{\rho(x)Ham}$ defines a family of reduced order port Hamiltonian models parameterized in $\rho(x)$ which match the moments of the nonlinear port-Hamiltonian system (2).

Remark 2. The linear port Hamiltonian results from Proposition 1 are a particular case of Theorem 3. Consider a linear port Hamiltonian system (2) with $J = -J^T$, $J \in \mathbb{R}^{n\times n}$, $R = R^T > 0$, $R \in \mathbb{R}^{n\times n}$ and $g(x) = B \in \mathbb{R}^n$ constant matrices and $\mathcal{H}(x) = \frac{1}{2}x^T Q x$, where $Q \in \mathbb{R}^{n\times n}$ is a constant symmetric matrix. Let $\pi(\omega) = \Pi\omega$, with $\Pi \in \mathbb{R}^{n\times\nu}$ the unique solution of the Sylvester equation (11). Hence $\bar{\mathcal{H}}(\omega) = \frac{1}{2}\omega^T \Pi^* Q \Pi \omega$ which yields $\frac{\partial \bar{\mathcal{H}}(\omega)}{\partial \omega} = \omega^* \Pi^* Q \Pi$. A linear port Hamiltonian system of order ν which matches the moments of system (2) is characterized by the matrices $\bar{J}, \bar{R}, \bar{Q}$ and $\bar{B}$ such that $\bar{B}^* \bar{Q}\omega = B^* Q \Pi \omega$. We construct the reduced order port Hamiltonian system using Theorem 3. Let $\rho(x) = \Theta x$, with $\Theta \in \mathbb{R}^{\nu\times n}$ such that

- $\Theta\Pi\omega = \omega$ and
- $\omega^* \bar{Q}\Theta B = \omega^* \Pi^* Q B$, for all ω.

The second condition is equivalent to $\Pi^* Q \Pi \Theta B = \Pi^* Q B \Leftrightarrow [\Theta - (\Pi^* Q \Pi)^{-1}\Pi^* Q]B = 0$. Hence, selecting $\Theta = (\Pi^* Q \Pi)^{-1}\Pi^* Q$, then $\rho(\Pi\omega) = (\Pi^* Q \Pi)^{-1}\Pi^* Q \Pi \omega = \omega$. Hence a port Hamiltonian reduced order model that matches the moments of (2) is given by equations of the form (22), with $\bar{J} = \bar{Q}^{-1}\Pi^* Q J Q \Pi \bar{Q}^{-1}, \bar{R} = \bar{Q}^{-1}\Pi^* Q R Q \Pi \bar{Q}^{-1}$ and $\bar{B} = P^{-1}\Pi^* Q B$, with $\bar{Q} = \Pi^* Q \Pi$. Note that this model is obtained from (equivalent to) the reduced order model (13) with the parameters $\tilde{J}, \tilde{R}, \tilde{Q}$ and $\tilde{B}$ as in (14) for the change of coordinates $\omega = \bar{Q}^{-1}\xi$. □

Consider a port Hamiltonian system (2) and a port Hamiltonian model of order ν, given by (22), with the Hamiltonian $\bar{\mathcal{H}}(\omega) = \mathcal{H}(\pi(\omega))$. Let $\pi(\omega)$ be the unique solution of (18). Then the moment matching condition is equivalent to

$$\left.\frac{\partial \mathcal{H}(x)}{\partial x}\right|_{x=\pi(\omega)} \frac{\partial \pi(\omega)}{\partial \omega}\bar{g}(\omega) = \left.\frac{\partial \mathcal{H}(x)}{\partial x}\right|_{x=\pi(\omega)} g(\pi(\omega)). \tag{24}$$

Based on (24) we compute a family of systems (2) that match the moments of (22) at $\{s(\omega), l(\omega)\}$, i.e., a converse version of Theorem 3.

Proposition 2. A port Hamiltonian system of order n that matches the moments of the port Hamiltonian model (13) of order $\nu < n$, is given by equation of the form (2) with

$$J(x)|_{x=\pi(\omega)} = \frac{\partial \pi(\omega)}{\partial \omega}\bar{J}(\omega)\frac{\partial^T \pi(\omega)}{\partial \omega},$$
$$R(x)|_{x=\pi(\omega)} = \frac{\partial \pi(\omega)}{\partial \omega}\bar{R}(\omega)\frac{\partial^T \pi(\omega)}{\partial \omega}, \tag{25}$$
$$g(x)|_{x=\pi(\omega)} = \frac{\partial \pi(\omega)}{\partial \omega}\bar{g}(\omega),\ \mathcal{H}(x)|_{x=\pi(\omega)} = \bar{\mathcal{H}}(\omega)$$

□

The following result shows that the models $\Sigma_{\pi(\omega)}$ are a subset of the family of models $\Sigma_{\delta(\xi)}$ described by equations of the form (10) and is a consequence of Theorem 3.

Corollary 1. Consider the class of ν order systems $\Sigma_{\delta(\xi)}$ as in (10). Then the following statements hold.

(1) Let $\Sigma_{\delta(\omega)}$ be a reduced order model of the system (2). Then $\Sigma_{\delta(\omega)}$ is a port Hamiltonian system $\Sigma_{\pi(\omega)Ham}$ as in (20) if and only if there exists $\rho : \mathbb{R}^n \to \mathbb{R}^\nu$, $\rho(x)|_{x=\pi(\omega)} = \omega$ such that $\delta(\omega) = \left.\frac{\partial \rho(x)}{\partial x}\right|_{x=\pi(\omega)} g(\pi(\omega))$.

(2) Let $\Sigma_{\delta(\omega)}$ be a reduced order model of the system (3). Then $\Sigma_{\delta(\omega)}$ is a gradient system $\Sigma_{\pi(\omega)grad}$ as in (22), if and only if $\delta(\omega) = \bar{G}^{-1}(\omega)\left.\frac{\partial^T h(x)}{\partial x}\right|_{x=\pi(\omega)}$, where $\bar{G}(\omega)$ is given by the relations (23). □

Proof. The claims follow using equations (**??**) and (**??**), respectively and applying Theorem 1. ■

4. EXAMPLE

Consider a double-mass, double-spring and damper system (see Fig. 1), we aim at computing the first order port Hamiltonian model which matches the moment of the system at $\{0, \omega\}$.

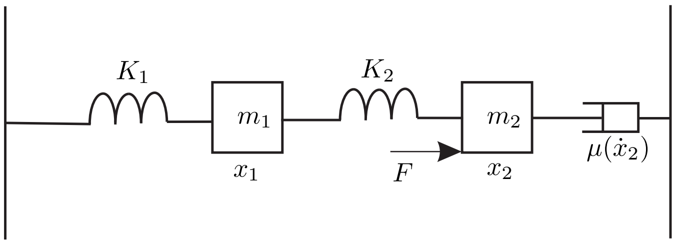

Fig. 1. Double-mass, double-spring and damper system

Let x_1 and x_2 be the positions of the mass m_1 and of the mass m_2, respectively, and let $x_3 = \dot{x}_1$ and $x_4 = \dot{x}_2$ be the corresponding velocities. We select $m_1 = m_2 = 1$ and assume that the spring forces linearly depend on the relative displacements, with $K_1 = K_2 = 1$. Let $F = x_4 u$ and the damper force be described by $\mu(x_4) = x_4^2 + 1$. Assume that the input u also influences the evolution of the velocity x_4. Hence the double-mass, double-spring and damper system is described by equations of the form (2), with

$$J(x) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},\ R(x) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu(x_4) \end{bmatrix},$$
$$g(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ x_4 \end{bmatrix},\ H(x) = \frac{1}{2}(x_3^2 + x_4^2) + \frac{1}{2}(x_2 - x_1)^2 + \frac{1}{2}x_1^2. \tag{26}$$

Let $s(\omega) = 0$, $l(\omega) = \omega$, i.e., consider the nolinear version of a Padé approximation problem, and let

$$\pi(\omega) = [\pi_1(\omega)\ \pi_2(\omega)\ \pi_3(\omega)\ \pi_4(\omega)]^T,\ \pi(\omega) : \mathbb{R} \to \mathbb{R}^4$$

satisfy (18), i.e.,

$$\pi_3(\omega) = 0,$$
$$\pi_4(\omega) + \omega = 0,$$
$$\pi_2(\omega) - 2\pi_1(\omega) = 0, \quad (27)$$
$$\pi_2(\omega) - 2\pi_1(\omega) - \pi_4(\omega)[\pi^2(\omega) + 1] + \omega\pi_4(\omega) = 0,$$

which yields the unique solution $\pi_1(\omega) = \pi_2(\omega)/2 = \omega^3 - \omega^2 + \omega$, $\pi_3(\omega) = 0$, $\pi_4(\omega) = -\omega$. Hence, the moment of (26) at $\{0, \omega\}$ is $\omega^3 + \omega$. We construct a first order port Hamiltonian system that matches the moment $\omega^3 + \omega$ at $\{0, \omega\}$. First note that $\mathcal{H}(\omega) = \frac{1}{2}\omega^2 + (\omega^3 - \omega^2 + \omega)^2$. Applying Theorem 3 we compute a solution (not unique) $\frac{\partial \rho(x)}{\partial x}\Big|_{x=\pi(\omega)} = \left[\frac{6\omega^5 - 10\omega^4 + 12\omega^3 - 7\omega^2 + 3\omega - 2}{\omega(3\omega^2 - 2\omega + 1)\beta(\omega)} \; 0 \; 0 \; -\frac{\omega^2+1}{\omega\beta(\omega)}\right]$, with $\beta(\omega) = 6\omega^4 - 10\omega^3 + 12\omega^2 - 6\omega + 3 = \frac{d\bar{\mathcal{H}}(\omega)}{d\omega}$. Thus a first order port Hamiltonian model that matches the moment $\omega^3 + \omega$ at $\{0, \omega\}$ is given by equations of the form (20), with $\bar{J}(\xi)0$, $\bar{R}(\xi) = \frac{(\omega^2+1)^3}{\omega^2\beta^2(\omega)}$ and $\bar{g}(\xi) = \frac{\omega^2+1}{\beta(\omega)}$, i.e.,

$$\begin{aligned} \dot{\omega} &= \frac{(\omega^2+1)^3}{\omega\beta(\omega)}(u - 1), \\ \psi &= \omega^3 + \omega. \end{aligned} \quad (28)$$

Note that (28) belongs to the family of first order models (10) that match the moment $\omega^3 + \omega$ at $\{0, \omega\}$, with $\delta(\xi) = \frac{(\xi^2+1)^3}{\xi\beta(\xi)}$ for $\xi = p(\omega) = \omega$, the unique solution of (8).

5. CONCLUSIONS

In this paper we have used the time-domain approach to nonlinear moment matching to compute the (families of) reduced order model(s) that preserves (preserve) the port Hamiltonian or gradient structure and matches (matches) the moments of the given nonlinear port Hamiltonian system. In other words, from the family of models that achieve moment matching, we have identified the reduced order model that inherits the port Hamiltonian/gradient form, by picking a particular (subset of) member(s), i.e., we obtain a (family of) reduced order model(s) that matches (match) the moments and inherit(s) the structure of the given system.

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