Input selection in observer design for non-uniformly observable systems

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Abstract: In this paper the *problem of inputs* in observer design for *systems* which are *not uniformly observable* is considered. It is emphasized how it amounts to a *control problem*, which can be solved in a general way by some appropriate *optimization approach*. This is illustrated on the basis of a quite general *Kalman-like observer* form - possibly with *high gain*, as well as related simulation results on an application example.

Keywords: Non-uniformly observable systems, input condition, optimization, leak detection.

1. INTRODUCTION

The problem of observer design for nonlinear systems has been more and more studied over the last two decades, but still remains challenging. In particular, the required *observability property* for such an observer design *may depend on the applied input* for a nonlinear system (see eg Besançon (2007)), which makes it a specific problem. The case of observability for any input [Gauthier and Bornard (1981)] and the related famous *high gain observer* have been very largely studied since early results of Gauthier et al. (1992); Tornambe (1992), while less efforts have been dedicated to systems for which this is not true. In fact, for this latter situation, appropriate inputs have been well characterized for some classes of systems (eg *state affine systems* [Bornard et al. (1988); Besançon et al. (1996)]), but this characterization is not directly helpful for a practical observer design. Usually in practice, inputs are heuristically designed (eg as in Torres et al. (2009) for a pipeline application example), and the observer convergence is checked a posteriori.

In the present paper, the purpose is to formally address this problem, at least on the basis of a significant class of results on observer design relying on appropriate input excitation: the main point is to propose a way to *build such inputs*, meaning that unlike in standard control-oriented problems, where the observer is designed for the purpose of applying an input to a system, here instead, the input is designed for the purpose of its application to an observer. This is obviously of interest for observer applications which are not control ones - such as fault detection, or parameter estimation, but can also be useful for control purposes. The problem is quite naturally related to *identification* issues (much concerned about excitation), and our preliminary study of Rubio-Scola et al. (2013) in that direction was based on a Gramian characterization of the appropriate excitation, and limited to state affine systems. Here instead, the problem is stated directly on some observer equations affected by the input, meaning that the input selection amounts to some extent to a *control problem*, and a direct way to address it is *optimization*. This observer-oriented optimal control problem is settled for a quite general class of nonlinear systems, discussed, and illustrated by its application to a fault-detection example, for which simulation results are provided.

The paper is thus organized as follows: section 2 first formally presents the problem under consideration, and then formulates its solution as a control optimization one. Section 3 then discusses some issues related to the practical implementation of such an approach, and section 4 subsequently proposes an illustrative example, with corresponding simulation results. Section 5 finally gathers some conclusions and perspectives.

2. PROBLEM STATEMENT AND OPTIMAL SOLUTION

Let us consider systems of the following form:

$$\begin{aligned} \dot{x}(t) &= A(u(t),v(t))x(t) + f(x(t),u(t),v(t)) \\ y(t) &= Cx(t) \end{aligned} \qquad (1)$$

where $x \in I\!R^n$ denotes the state vector, $u \in I\!R^m$ some control input vector which can be used in the observer design, $y \in I\!R$ the measured output, and $v \in I\!R^q$ gathering some known signals which can be injected in the observer (possibly including t or $y(t)$ for instance).
Let us assume that $\|A(u,v)\| \leq a(u)$ for any u, v, and some smooth $a(u)$.
Let us further assume that f is globally Lipschitz in x, uniformly in u, v, with a constant γ.

The first point to be underlined is that such a system may a priori admit inputs for which observability is lost (ie is *not uniformly observable* [Besançon (2007)]).
We also remind the reader of earlier results on input conditions to guarantee a possible observer design, depending on the system structure (eg as in Besançon and Ţiclea (2007); Torres et al. (2012); Dufour et al. (2012)).

Under such conditions, an observer can typically be written in the following form:

$$\dot{\hat{x}}(t) = A(u(t), v(t))\hat{x}(t) + f(\hat{x}(t), u(t), v(t)) + +\Lambda(\lambda)P(t)C^T(y(t) - C\hat{x}(t)); \quad (2)$$

$$\dot{P}(t) = \lambda[\sigma P(t) + P(t)A^T(u(t), v(t)) + +A(u(t), v(t))P(t) - P(t)C^TCP(t)]; \quad (3)$$

$$P(0) \succ 0;$$

for some appropriate positive tuning parameters λ, σ, and related matrix Λ and $\succ$ meaning positive definite.

In the case when $f(x, u, v) = f(u, v)$ for instance, λ can be set to 1, Λ to the identity matrix (I), and one gets the *Kalman-like observer* (with forgetting factor) [Besançon et al. (1996); Ţiclea and Besançon (2009)].
If A, f satisfy the "high gain structure" [Gauthier et al. (1992)], system (2)-(3) can become an observer for (1) provided that λ is large enough - and Λ a diagonal matrix with increasing powers of λ as diagonal entries [Besançon and Ţiclea (2007)].
It can also be an observer if $\lambda = 1, \Lambda = I$ and σ is chosen large enough [Dufour et al. (2012)].
In any of those cases, the observer convergence is associated with a Lyapunov function defined from P^{-1}, and the existence of the latter is related to appropriate *input excitation*.
Looking then at (3) as some state equation driven by input u, the observer problem turns into a *control* problem, in the sense that u is to be designed so that P^{-1} remains defined.
If such an input exists (corresponding to some *observability* of the system), it can even be looked for via some *optimal control* approach, by minimizing P. This indeed will maximize P^{-1}, and can even be done by taking into account the input energy at the same time, typically via some quadratic cost function. Some additional constraints on u can also be taken into account, as well as specific constraints on P if required (for observer stability for instance).
In this way, the input selection amounts to a nonlinear optimal control problem, which allows to rely on available optimization tools in that respect.
Notice that this generalizes our former approach of [Rubio-Scola et al. (2013)] on this topic, which was limited to systems (1) with $f(x, u, v) = f(u, v)$, and based on the optimization of some ad-hoc criterion related to observability characterization in terms of Gramian instead of P.

In order to summarize, let us consider that (2)-(3) is an observer for (1) if the solution of (3) admits uniform positive lower and upper bounds and can be used to characterize the observer stability in a quadratic Lyapunov function, and let us denote by $\mathcal{P}$ the set of such appropriate positive definite matrices.

Then we can claim the following:

Proposition *2.1.* Consider a system of the form (1), and assume that for given λ, σ there exists some admissible u such that (2)-(3) is an observer for it (with a related set $\mathcal{P}$), then such an appropriate input can be found on some time interval $[0, T]$, by solving a problem of the form:

$$min_u \int_0^T \left[\rho\|u(s)\|^2 + \|P(s)\|^2\right] ds \quad \text{under (3), } P \in \mathcal{P}, \text{ and } u \in \mathcal{U} \quad (4)$$

with $\mathcal{U}$ the set of admissible inputs, and $\rho > 0$.

In the case when $f(x, u, v) = f(u, v)$, the constraint on P (set $\mathcal{P}$) reduces to an upper bound, since in that case λ can be set to 1, and it is known that a lower bound on P is guaranteed for a σ large enough whenever $a(u)$ is bounded (see e.g. Besançon et al. (1996)).
Hence, the usual stability analysis for the corresponding observer holds (cf eg Hammouri and de Leon Morales (1991)).
In the case when f depends on x in a $\gamma-$Lipschitz way, with A, f satisfying the "high gain structure", one needs to consider an additional constraint on P to ensure the observer stability, which can for instance be of the form:

$$\gamma^2P^2 - \lambda\sigma P + I < 0, \quad (5)$$

with $\lambda\sigma > 2\gamma$ for the feasibility of the problem, and $P(0)$ to be chosen satisfying (5).
In that case indeed, the lower bound on P is again guaranteed by appropriate choice of σ, while an upper bound is guaranteed by (5), and one can then consider $V(e) = e^T\Lambda^{-1}P^{-1}\Lambda^{-1}e$ as a candidate Lyapunov function for the error dynamics in $e := \hat{x} - x$.
From direct computations, it results that:

$$\dot{V} \leq -\lambda\sigma V + 2e^T\Lambda^{-1}P^{-1}\Lambda^{-1}\Delta f$$

where $\Delta f = f(\hat{x}, u, v) - f(x, u, v)$.
From this, we can get:

$$\dot{V} \leq -\lambda\sigma V + e^T\Lambda^{-1}P^{-1}P^{-1}\Lambda^{-1}e + \Delta f^T\Lambda^{-2}\Delta f$$

and using the analysis of the high gain observer [Gauthier et al. (1992)], we obtain:

$$\dot{V} \leq -\lambda\sigma V + e^T\Lambda^{-1}P^{-1}P^{-1}\Lambda^{-1}e + \gamma^2\|\Lambda^{-1}e\|^2.$$

Finally, (5) implies that:

$$-\lambda\sigma P^{-1} + P^{-1}P^{-1} + \gamma^2 I < 0$$

which gives the convergence to zero of the estimation error e by standard Lyapunov arguments.
Notice that the choice of the constraints on P (that is the stability characterization) may affect the computational burden of the optimization, but we present in section 4 an example where the approach is successful, and we leave improvements in that respect for future studies.
In particular, in the above approach, observer parameters λ and σ are pre-specified, but in the spirit of recent results on *adaptive high-gain* observers (Ahrens and Khalil (2009); Andrieu et al. (2009) or Boizot et al. (2010) for instance), one could instead simultaneously look for u *and some related* λ, σ.

3. PRACTICAL IMPLEMENTATION

In order to use proposition 2.1 in practice, the input which is looked for can be approached by piecewise constant functions, updated with a period (say T_u), which can be chosen from a trade-off between approximation accuracy and frequency constraints on input variations.
This means that an appropriate input sequence can be built by solving, for every $k \geq 0$:

$$min_{u_k} \int_{kT_u}^{(k+1)T_u} \left[\rho\|u\|^2 + \|P(s)\|^2\right] ds \quad \text{under (3), } P \in \mathcal{P}, \text{ and } u \in \mathcal{U} \quad (6)$$

Considering that possible input magnitudes may additionally be limited, the set of admissible inputs $\mathcal{U}$ may just be defined by some constraint of the form $|u_i| \leq U_{imax}$ for any input u_i of vector u.

About constraints on P (set $\mathcal{P}$), they will depend on the specific class of system which is considered as discussed in the previous section.

In the same way, the choice of λ depends on the class of system, while σ is to be chosen large enough with respect to an upper bound on $a(u)$.

Parameter ρ in the criterion is a usual weighting coefficient, which can be chosen in practice very low so as to emphasize the minimization of P.

Coming now to the possible presence of some variable $v(t)$ in the dynamics of P, the implementation of the above optimization procedure requires that at time kT_u $v(s)$ be known over $s \in [kT_u, (k+1)T_u]$, for any $k \geq 0$.
If this is not the case, the variable is to be predicted.
If $v = y$ for instance, it can be predicted by a simple zero-order or first-order hold.

Notice finally that in practice, the observer gain equation (3) can also be enhanced with some additional positive (resp. definite) matrix Q (resp. R), which may be chosen according to the noise level in the state equation (resp. output equation). This in particular makes sense in the case of Kalman equations, when system (1) reduces to a state affine one, and in that case, the proposed input selection procedure results in improving filtering properties of the observer as well.

4. ILLUSTRATIVE EXAMPLE

In order to illustrate the approach discussed above, let us consider an example of observer application to *leak detection in pipelines*.This problem indeed has attracted a lot of attention over the last 2 decades (see eg (Billman and Isermann, 1984; Brunone and Ferrante, 2001; Shields et al., 2001; Verde, 2005; Torres et al., 2009, ...) and references therein). We propose here to look at the simple example of a pump-pipe-tank system recently studied in Besançon et al. (2012): the system is made of a pipeline connected to a pump at one end, and to a water tank at the other end, as depicted by Fig. 1. Also, we assume that the pipe can be subject to leaks.

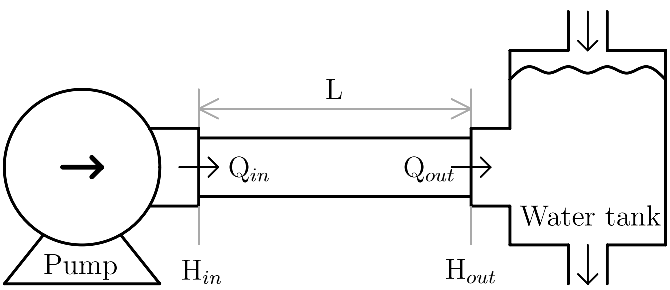

Fig. 1. Pump-pipe-tank system.

The dynamics of such a system can simply be described as:

$$\begin{aligned} \frac{d}{dt}H_{in} &= -\frac{c^2}{agL}Q_{out} - \frac{Fc^2}{agL}\sqrt{H_{out}} + \frac{c^2}{agL}Q_{in} \\ \frac{d}{dt}Q_{out} &= -\frac{ag}{L}(H_{out} - H_{in}) - \frac{f}{2Da}Q_{out}|Q_{out}| \end{aligned} \tag{7}$$

where H_{in}, H_{out} denote the input, output pressures (m), Q_{in} and Q_{out} are the input and the output flow rates in the pipeline (m^3/s), and F denotes the magnitude coefficient of the pipeline leak, if any.
Notice that the input flow Q_{in} in fact results from the pump effect, which can be modeled as a nonlinear function of the pump piezometric head H_p and the input pressure head H_{in} [Besançon et al. (2012)]:

$$Q_{in} = -\frac{1}{2}\left(\theta - \sqrt{\theta^2 - 4\frac{\theta}{B}(-A + H_{in} - H_p)}\right) \tag{8}$$

for constant parameters A, B, θ characterizing the pump.

In this example, the pipe parameters are taken from an experimental prototype for instance described in [Padilla and Begovich (2012)], and all numerical values are summarized in table 1.

Table 1. Constant parameters of the system

Letters	Values	Units	Description
a	$3.4 \cdot 10^{-3}$	m^2	Cross sectional area
g	9.81	m/s^2	Gravitational acceleration
L	85	m	Length of the pipe
c	372.567	m/s	Wave speed in the fluid
f	0.0189		Nominal friction coefficient
D	0.0661	m	Diameter of the pipe
θ	$1.91 \cdot 10^{-4}$		Pump parameter
A	27.3		Pump parameter
B	274		Pump parameter

Let us then assume that we only measure $y = Q_{out}$, and that the output pressure H_{out} can be modified as an input $u = H_{out}$.
Let us also assume that the pump pressure H_p is fixed ($H_p = 6.5m$ in the simulations), and known, as well as all other constant parameters, except the friction coefficient: the latter indeed in general depends on the flow rate in the pipe [Chaudry (1979)], which means in particular that when a leak occurs, it changes.

An observer can then be used to monitor the system and detect leaks taking into account this friction change, by directly estimating the leak coefficient F together with the friction one f.
To that end, one just needs to rewrite equations (7) as a state-space representation, by considering an extended state vector as follows:

$$x = (Q_{out}\ H_{in}\ f\ F)$$

with approximations $\dot{f} = 0,\ \dot{F} = 0.$

Let us then illustrate observer results in two situations: (i) the case when the input flow of the pipe (Q_{in}) is known; (ii) the case when Q_{in} is unknown (but its model (8) is).

In case (i), the state space model can be written in the form:

$$\begin{aligned} \dot{x} &= A_1(u, y)x + B_1(u, y, v) \\ y &= Cx \end{aligned} \tag{9}$$

with $v = Q_{in}$ and $A_1 = \begin{pmatrix} 0 & \frac{ag}{L} & -\frac{y|y|}{2Da} & 0 \\ 0 & 0 & 0 & -\frac{c^2\sqrt{u}}{agL} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$,

$$B_1 = \begin{pmatrix} \frac{-agu}{L} \\ \frac{c^2(v-y)}{agL} \\ 0 \\ 0 \end{pmatrix}, \qquad C = (1\ 0\ 0\ 0).$$

Hence an observer via classical Kalman-like equations can be obtained as in (2)-(3), ie with $\lambda = 1,\ \Lambda = I$.

In case (ii), the model becomes:

$$\begin{aligned} \dot{x} &= A_2(u, y)x + B_2(u, y, x) \\ y &= Cx \end{aligned} \tag{10}$$

with $A_2 = A_1$, $B_2 = \begin{pmatrix} \frac{-agu}{L} \\ \frac{c^2(Q_{in}(u, x_2) - y)}{agL} \\ 0 \\ 0 \end{pmatrix}$ and C as before, with Q_{in} given by (8).

In that case, the observer needs to be of some "high gain" type, either with λ large enough, as in Besançon and Ţiclea (2007), or with $\lambda = 1$ and σ large enough, as in Dufour et al. (2012).

In both cases (i) and (ii), it is clear that observability depends on the input, and even if one could find some candidate input via trial and error approach, our point here is to show what can be obtained via the proposed optimization approach.

Various simulation results are thus provided hereafter, either related to case (i) or to case (ii). In both cases, simulations are started in nominal operation conditions (with a flow rate around $4.3 \cdot 10^{-3} m^3/s$), and a leak effect is added in the system at time $t = 500$, modifying both F and f coefficients, while the observer is to recover those values. In all observers here, $\sigma = 0.01$, while λ is set to 1 in case (i), and to 1.05 in case (ii).

First of all, figures 2 and 3 show the obtained estimation errors in case (i) for both states, and both parameters respectively: it can be checked that convergence is indeed achieved. The corresponding input obtained from the optimization procedure can be seen in figure 4 (constrained between 3.1 and 3.7).

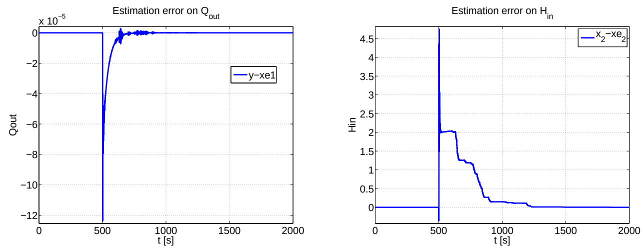

Fig. 2. State estimation errors in case (i).

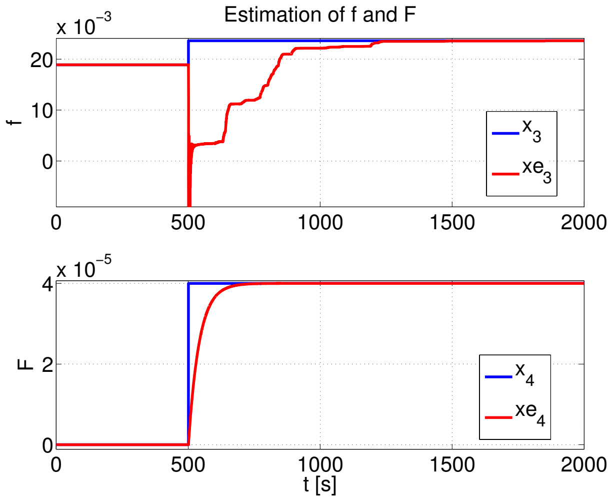

Fig. 3. Parameter estimations in case (i).

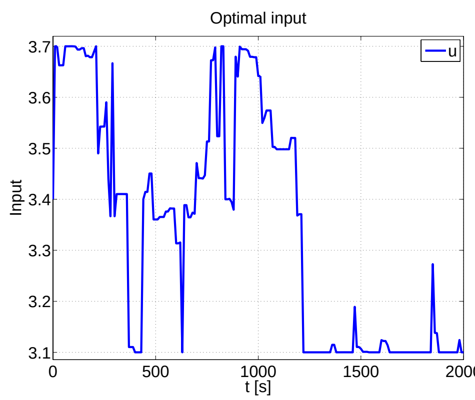

Fig. 4. Optimal input in case (i).

Figures 5 and 6 then show similar results for case (ii): the state and parameter estimation can again be checked to be successful. The corresponding input is presented in figure 7.

In order to better evaluate the observer performances, let us also present estimation results when the measurement is corrupted by some noise: here a band-limited white noise has been simulated corresponding to magnitudes of a few percents of the output nominal value.

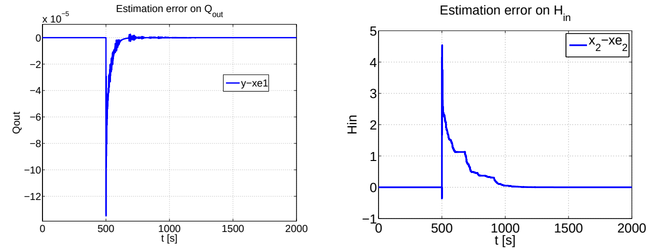

Fig. 5. State estimation errors in case (ii).

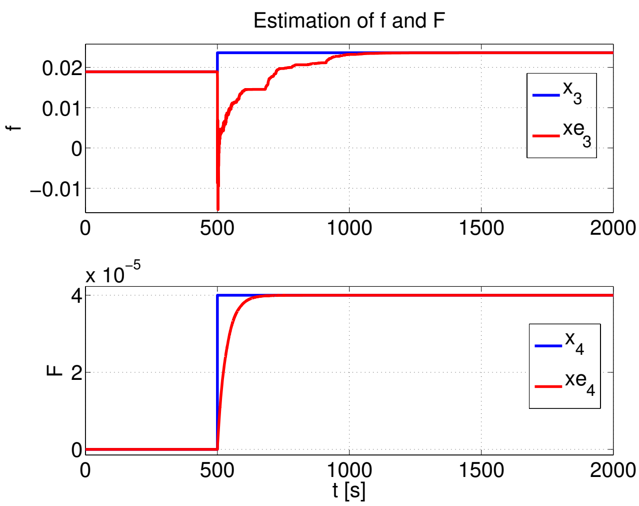

Fig. 6. Parameter estimations in case (ii).

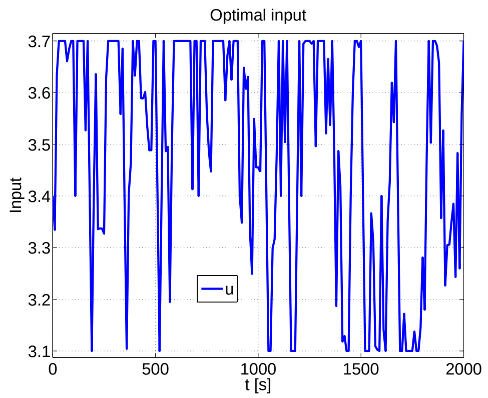

Fig. 7. Optimal input in case (ii).

Notice that observer equations include Q, R matrices in a similar way as in the standard Kalman formulation:
$\dot{P} = \lambda[\sigma P + PA^T(u, y) + A(u, y)P(t) - PC^T R^{-1} CP + Q]$
with $R = 1 \cdot 10^{-9}$, $Q = 1 \cdot 10^{-13} I_d$.

Figures 8 and 9 again show estimation errors for both states and parameters in case (i): it can be seen that the actual values are indeed recovered in spite of the noise. Figure 10 shows the corresponding input.

As for case (ii), similar results are presented in figures 11 and 12, where it can be seen how the "high gain" still performs pretty well. Further improvements can be expected via more *adaptive* versions - as formerly mentioned, and this is left for future studies.
The related input is shown in figure 13.

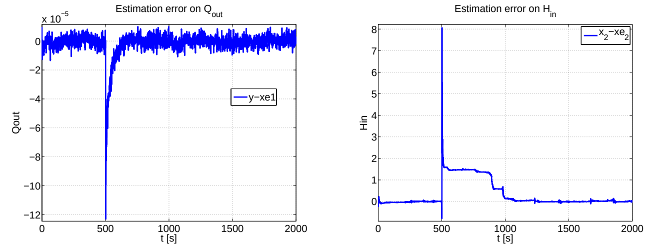

Fig. 8. State estimation errors in case (i) with noise.

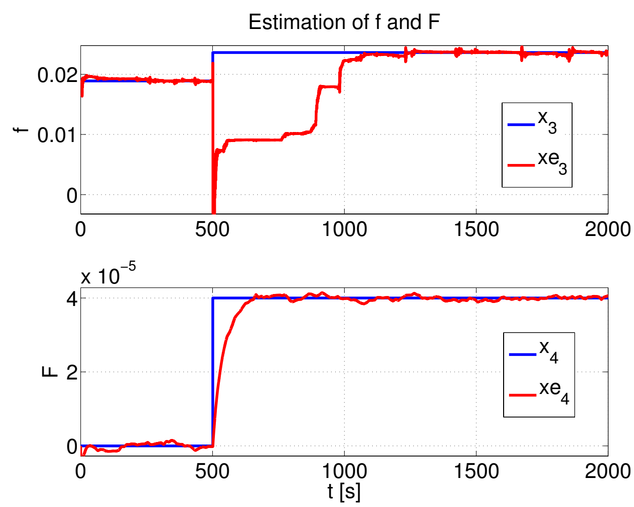

Fig. 9. Parameter estimations in case (i) with noise.

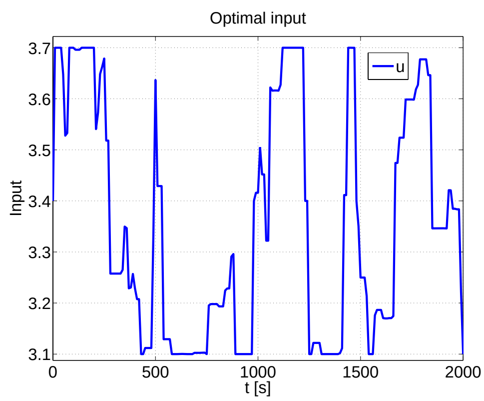

Fig. 10. Optimal input in case (i) with noise.

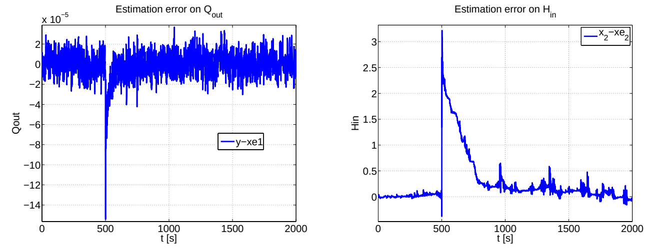

Fig. 11. State estimation errors in case (ii) with noise.

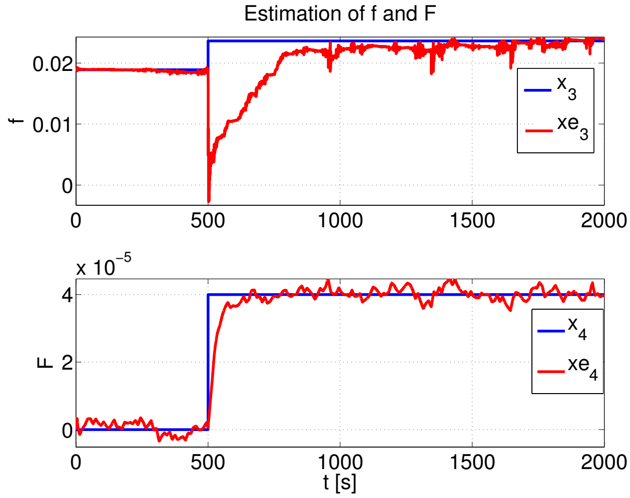

Fig. 12. Parameter estimation errors in case (ii) with noise.

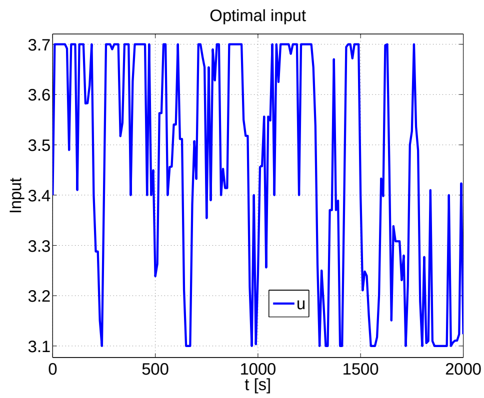

Fig. 13. Optimal input in case (ii) with noise.

5. CONCLUSIONS AND PERSPECTIVES

In this paper, we have emphasized how appropriate inputs for observers can be chosen in a systematic way for systems which are not uniformly observable, and the method has been illustrated with a simple example. Refining optimization criterion and procedure will be part of future studies, in particular with the purpose of noise attenuation.

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