Definition of eigenvalues for a nonlinear system

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Abstract: In this paper the concept of eigenvalues and eigenvectors of nonlinear systems, both continuous- and discrete-time, is suggested. It represents a generalization of the concept known from linear control theory. Some basic properties, like invariance of eigenvalues under a (nonlinear) change of coordinates, possibility to transform the system to the diagonal form and, respectively, to the feedforward form are then shown.

Keywords: nonlinear systems, eigenvalues, eigenvectors, diagonal form, feedforward form, discrete-time systems

1. INTRODUCTION

The concept of eigenvalues and eigenvectors play an important role in the control theory of linear time-invariant systems, both continuous- and discrete-time. Besides the fact that eigenvalues determine the stability of a system, the concept is also useful for various system transformations, as for instance to the diagonal form. This paper attempts to introduce notions of eigenvalues and eigenvectors for nonlinear control systems, both continuous- and discrete-time. The eigenvalues are defined in the way they are invariant under a (nonlinear) change of coordinates, and that they represent a generalization of the notion known from the linear systems. Considering the continuous-time case, the concept introduced in this paper can be viewed as an extension of that of Wu (1980), which addresses the linear time-varying case, once the tangent linear systems is associated with the nonlinear one. This concept is also equivalent to that of Menini and Tornambe (2011) where the respective object representing an eigenvalue is called a characteristic function, though the definition is given in terms of the Lie brackets.
The concept of eigenvalues and eigenvector is generalized also to the nonlinear discrete-time systems. An application of the concept is studied too. First, it is shown that the nonlinear system can be transformed to the (nonlinear) diagonal form using the respective eigenvectors, as in the linear case. The diagonal form is useful for many reasons. One can, for instance, find the solution to the system equations simply by integrating the respective equations in the diagonal form, as they do not depend on other state variables. Second, the concept is used to transform the nonlinear system to the (nonlinear) feedforward form, which is a less restrictive form than diagonal. However, it preserves the property that one can find the solution to the system equations by integrating the respective equations. The feedforward form is useful for instance for designing stabilizers for the system, see for instance Tall and Respondek (2000); Respondek and Tall (2004). In the continuous-time case, the geometric interpretation has been given in Astolfi and Mazenc (2000) in terms of invariant distributions. However, no algorithm for computing such distributions was given. In that respect, the corresponding result for discrete-time systems, which was studied in Aranda-Bricaire and Moog (2004), is stronger, as it is accomplished by an algorithm. Note also that in Respondek and Tall (2004) the problem has been studied in terms of vector fields. Nevertheless, the adaptation of the formalism developed in Aranda-Bricaire and Moog (2004) to continuous-time systems is not trivial, as it would need to find a solution to a set of higher-order partial differential equations. The concept of eigenvalues introduced in our paper needs, however, to find a solution to a set of first order differential (or difference) equations only. Though, the results that can be obtained depend heavily on the fact that, in general, the respective transformations might not be defined at the origin.
Finally, some additional comments, perspectives, and concluding remarks related to the proposed concept are given.

2. PRELIMINARIES

In this paper we will use the algebraic setting of Conte et al. (2007) adapted to the case of uncontrolled systems (i.e. systems without input) which are considered here for the sake of simplicity.

Consider the nonlinear system defined by differential equations of the form

$$\dot{x} = f(x) \tag{1}$$

where $x \in \mathbb{R}^n$ and elements of f are assumed to be from the field of meromorphic functions of variables $\{x_1, \dots, x_n\}$ denoted by $\mathcal{K}$.

Let $\mathcal{E}$ denote the formal vector space of differential one-forms defined as

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$$\mathcal{E} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\xi; \xi \in \mathcal{K}\}$$

Elements of $\mathcal{E}$ are called (differential) one-forms.
Recall that a one-form $\omega \in \mathcal{E}$ is called exact (or integrable), if there exists $F \in \mathcal{K}$ such that $\mathrm{d}F = \omega$, and a subspace $\mathcal{V} \subset \mathcal{E}$ is called exact (or integrable) if it has a basis that consists of exact one-forms only.

Define the derivative operator $\frac{\mathrm{d}}{\mathrm{d}t}$ that acts on $\mathcal{K}$ in the usual way. The operator $\frac{\mathrm{d}}{\mathrm{d}t}$ induces the derivative operator, which is by abuse of notation denoted by the same symbol $\frac{\mathrm{d}}{\mathrm{d}t}$, that acts on $\mathcal{E}$ as follows.
Let $\omega = \sum_i \alpha_i \mathrm{d}\xi_i$ be in $\mathcal{E}$, then

$$\dot{\omega} = \sum_i (\dot{\alpha}_i \mathrm{d}\xi_i + \alpha_i \mathrm{d}\dot{\xi}_i)$$

Finally, the tangent linear system associated to the nonlinear system (1) is given by

$$\mathrm{d}\dot{x} = A\mathrm{d}x \tag{2}$$

where $A = (\partial f/\partial x) \in \mathcal{K}^{n\times n}$.

3. CONCEPT OF EIGENVALUES AND EIGENVECTORS FOR A NONLINEAR SYSTEM

A concept of eigenvalues and eigenvectors play a key role in linear control theory. For a linear system $\dot{x} = Ax$, where $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$, a scalar $\lambda \in \mathbb{R}$ and a column vector $e \in \mathbb{R}^n$ are said to be an eigenvalue and an eigenvector respectively, if

$$\lambda e = Ae \tag{3}$$

One of the very fundamental properties that implies from such a definition is that the eigenvalues are invariant with respect to a linear change of coordinates $\xi = Tx$ with $\mathrm{rank}_{\mathbb{R}} T = n$ (i.e. a similarity transformation TAT^{-1}). This seems to be the property we need to carry over to the nonlinear case.

3.1 Eigenvalues and eigenvectors of a nonlinear system

Definition 1. A function $\lambda \in \mathcal{K}$ and a nonzero vector $e \in \mathcal{K}^n$ are said to be an eigenvalue and, respectively, an eigenvector (associated with the eigenvalue λ) of the nonlinear system (1), if

$$\lambda e + \dot{e} = Ae \tag{4}$$

where A is defined by (2).

Note that this definition represents a generalization of that for linear systems, as for $e \in \mathbb{R}^n$ one has $\dot{e} = 0$, and (4) reduces to (3).

Theorem 2. Eigenvalues of the nonlinear system (1) are invariant with respect to a change of coordinates $\xi = \phi(x)$ where $\phi \in \mathcal{K}$.

Proof. Let λ be an eigenvalue of the system (1) associated with an eigenvector e. For any change of coordinates $\xi = \phi(x)$ one has $\mathrm{d}\xi = T\mathrm{d}x$ where $T = (\partial\phi/\partial x)$ and $\mathrm{rank}_{\mathcal{K}} T = n$.
In the new coordinates we get $\dot{\xi} = \tilde{f}(\xi)$ for some $\tilde{f} \in \mathcal{K}$, and

$$\mathrm{d}\dot{\xi} = \tilde{A}\mathrm{d}\xi$$

where $\tilde{A} = TAT^{-1} + \dot{T}T^{-1}$.
The corresponding eigenvector then is

$$\tilde{e} = Te$$

λ	e_1	e_2
0	x_1	0
	$2x_1$	0
	x_1x_2	x_2^2
1	x_2	0
	$2x_2$	0
$1 + x_2$	1	0
	2	0
$2x_2$	x_1/x_2	1
	$2x_1/x_2$	2

Table 1. Possible eigenvalues and various eigenvectors for the system in Example 3.

It suffices to show that λ is an eigenvalue of the nonlinear system $\dot{\xi} = \tilde{f}(\xi)$ too (associated with the eigenvector $\tilde{e}$).

$$\begin{aligned}\lambda\tilde{e} + \dot{\tilde{e}} &= \tilde{A}\tilde{e}\\ \lambda Te + (Te)^{\cdot} &= (TAT^{-1} + \dot{T}T^{-1})Te\\ \lambda Te + T\dot{e} + \dot{T}e &= TAe + \dot{T}e\\ \lambda Te + T\dot{e} &= TAe\\ \lambda e + \dot{e} &= Ae\end{aligned}$$

which completes the proof. □

Eigenvalues of a nonlinear system, though being invariant, are not unique. That is, there might be (possibly infinitely) many eigenvalues for a system and various eigenvectors associated with the respective eigenvalue.

Example 3. Consider the system

$$\begin{aligned}\dot{x}_1 &= x_1 + x_1x_2\\ \dot{x}_2 &= x_2^2\end{aligned}$$

We are looking for $\lambda \in \mathcal{K}$ and a nonzero vector $(e_1, e_2)^T \in \mathcal{K}^2$ such that

$$\lambda\begin{pmatrix}e_1\\e_2\end{pmatrix} + \begin{pmatrix}\dot{e}_1\\\dot{e}_2\end{pmatrix} = \begin{pmatrix}1+x_2 & x_1\\0 & 2x_2\end{pmatrix}\begin{pmatrix}e_1\\e_2\end{pmatrix}$$

Table 1 shows various solutions for λ, e_1 and e_2 that can be found.

Example 4. Consider the system

$$\dot{x} = x$$

One can show that any $k \in \mathbb{Z}$ is an eigenvalue. Note that for $\lambda = k$ we have

$$ke + \dot{e} = e$$

which can be solved by, for instance, $e = 1/x^{k-1}$.
This example and the result is not that trivial. For it means the linear system $\dot{x} = x$ can be transformed to the linear system $\dot{\xi} = k\xi$, $k \in \mathbb{Z}$, by a nonlinear change of coordinates while it cannot be by any linear. See Example 8, and also Example 15.

3.2 Transformation to the diagonal form

The potential of having various eigenvalues for a system seems interesting and needs to be exploited further. One of the first natural result is that the system (1) can be diagonalized in a similar manner as in the linear case.

Definition 5. A system of the form (1) is said to be in a diagonal form, if

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$$\frac{\partial f}{\partial x} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

where λ_i, $i = 1, \ldots, n$, are in $\mathcal{K}$.

Theorem 6. Given a system of the form (1), there exists a change of coordinates $\xi = \phi(x)$ that transforms the system into a diagonal form if and only if there exist n eigenvalues $\lambda_1, \ldots, \lambda_n$ associated with n eigenvectors $e_1, \ldots, e_n$ such that

$$T^{-1} = (e_1|e_2|\cdots|e_n)$$

is nonsingular and $T\mathrm{d}x = (\omega_1, \ldots, \omega_n)^T$ where the one-forms $\omega_1, \ldots, \omega_n$ are exact.

Proof. Consider the system (1) and the matrix A given by (2). Then

$$\begin{aligned} AT^{-1} &= (Ae_1|Ae_2|\cdots|Ae_n) \\ &= (\lambda_1 e_1 + \dot{e}_1|\lambda_2 e_2 + \dot{e}_2|\cdots|\lambda_n e_n + \dot{e}_n) \\ &= T^{-1}\Lambda + (T^{-1})^{\cdot} \\ &= T^{-1}\Lambda - T^{-1}\dot{T}T^{-1} \end{aligned}$$

where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

Since T^{-1} is nonsingular, one has

$$\Lambda = TAT^{-1} + \dot{T}T^{-1}$$

Finally, there exists a change of coordinates $\xi = \phi(x)$, such that $\mathrm{d}\xi = T\mathrm{d}x$ where $T = (\partial\phi/\partial x)$, if and only if the rows of $T\mathrm{d}x$ are exact one-forms. □

Example 7. Consider the system from Example 3. Since the matrix T^{-1} needs to be nonsingular it is not true we can transform the system to the diagonal form for any pair λ_1, λ_2. The corresponding eigenvectors need to be independent. We can transform the system to the diagonal form for instance for $\lambda_1 = 2x_2$, $\lambda_2 = 1$, in which case we can pick the eigenvectors as

$$e_1 = \begin{pmatrix} \frac{x_1}{x_2} \\ 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}$$

Then

$$T^{-1} = \begin{pmatrix} \frac{x_1}{x_2} & x_2 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ \frac{1}{x_2} & -\frac{x_1}{x_2^2} \end{pmatrix}$$

and

$$\mathrm{d}\xi = T\mathrm{d}x = \begin{pmatrix} \mathrm{d}x_2 \\ \frac{1}{x_2}\mathrm{d}x_1 - \frac{x_1}{x_2^2}\mathrm{d}x_2 \end{pmatrix}$$

where the rows are exact one forms. Thus, the change of coordinates $(\xi_1, \xi_2) = (x_2, x_1/x_2)$ transforms the system to the diagonal form

$$\begin{aligned} \dot{\xi}_1 &= \xi_1^2 \\ \dot{\xi}_2 &= \xi_2 \end{aligned}$$

Another interesting choice of eigenvalues is $\lambda_1 = 0$, $\lambda_2 = 0$ for which we can pick two independent eigenvectors

$$e_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} x_1 x_2 \\ x_2^2 \end{pmatrix}$$

Then

$$T^{-1} = \begin{pmatrix} x_1 & x_1 x_2 \\ 0 & x_2^2 \end{pmatrix}, \quad T = \begin{pmatrix} 1/x_1 & -1/x_2 \\ 0 & 1/x_2^2 \end{pmatrix}$$

and

$$\mathrm{d}\xi = T\mathrm{d}x = \begin{pmatrix} \frac{1}{x_1}\mathrm{d}x_1 - \frac{1}{x_2}\mathrm{d}x_2 \\ \frac{1}{x_2^2}\mathrm{d}x_2 \end{pmatrix}$$

consists of exact one forms. Thus, the change of coordinates $(\xi_1, \xi_2) = (\ln x_1 - \ln x_2, -1/x_2)$ transforms the system to

$$\begin{aligned} \dot{\xi}_1 &= 1 \\ \dot{\xi}_2 &= 1 \end{aligned}$$

which is the diagonal form according to Definition 5.

Example 8. Consider the system from Example 4 where $\lambda = k$, $k \in \mathbb{Z}$, and $e = 1/x^{k-1}$ are an eigenvalue and, respectively, a corresponding eigenvector. Since $T^{-1} = (e)$, and since $\mathrm{d}\xi = T\mathrm{d}x = x^{k-1}\mathrm{d}x$ is an exact one-form, the change of coordinates $\xi = \frac{1}{k}x^k$, for $k \neq 0$, transforms the system to

$$\dot{\xi} = k\xi$$

For $k = 0$ the corresponding change of coordinates is $\xi = \ln x$.

3.3 Transformation to the feedforward form

Another application of the concept of eigenvalues of a nonlinear system lies in the system transformation to the feedforward form.

Definition 9. A system of the form (1) is said to be in the feedforward form, if

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \ldots, x_n) \\ \dot{x}_2 &= f_2(x_2, \ldots, x_n) \\ &\vdots \\ \dot{x}_n &= f_n(x_n) \end{aligned} \tag{5}$$

Note that the diagonal form is, in fact, the special case of the feedforward form. It is, therefore, natural to expect the conditions for the system transformation to such a form being less restrictive than those of Theorem 6.

Theorem 10. Given a system of the form (1), there exists a change of coordinates $\xi = \phi(x)$ that transforms the system into the feedforward form if and only if there exist n eigenvalues $\lambda_1, \ldots, \lambda_n$ associated with n eigenvectors $e_1, \ldots, e_n$ such that

$$T^{-1} = (e_1|e_2|\cdots|e_n)$$

is nonsingular and $T\mathrm{d}x = (\omega_1, \ldots, \omega_n)^T$ where the one-forms $\omega_1, \ldots, \omega_n$ are such that $\Omega_1 = \mathrm{span}_{\mathcal{K}}\{\omega_1, \omega_2, \ldots, \omega_n\}$, $\Omega_2 = \mathrm{span}_{\mathcal{K}}\{\omega_2, \ldots, \omega_n\}$, …, $\Omega_n = \mathrm{span}_{\mathcal{K}}\{\omega_n\}$ are integrable.

Sketch of the proof. *Sufficiency.* As a result of Theorem 6 we have $\dot{\omega}_i = \lambda_i \omega_i$ for $i = 1, \ldots, n$. Therefore, for any vector $\nu \in \Omega_i$ also $\dot{\nu} \in \Omega_i$. Since all Ω_i, $i = 1, \ldots, n$, are integrable there exist functions $\phi_i \in \mathcal{K}$, $i = 1, \ldots, n$, such that $\Omega_1 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\phi_1, \mathrm{d}\phi_2, \ldots, \mathrm{d}\phi_n\}$, $\Omega_2 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\phi_2, \ldots, \mathrm{d}\phi_n\}$, …, $\Omega_n = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\phi_n\}$. Hence, the change of coordinates $\xi = \phi(x)$ transforms the system into the feedforward form.

Necessity. Suppose there exists a change of coordinates

$\xi = \phi(x)$ that transforms the system into the feedforward form. Let $\Omega_1 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\xi_1, \mathrm{d}\xi_2, \ldots, \mathrm{d}\xi_n\}$, $\Omega_2 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\xi_2, \ldots, \mathrm{d}\xi_n\}$, ..., $\Omega_n = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\xi_n\}$. Since the system is in the feedforward form one has that for any $\nu \in \Omega_i$ also $\dot{\nu} \in \Omega_i$. Since $\Omega_n \subset \ldots \subset \Omega_2 \subset \Omega_1$ there exist $\omega_1, \ldots, \omega_n$ such that $\Omega_1 = \mathrm{span}_{\mathcal{K}}\{\omega_1, \omega_2, \ldots, \omega_n\}$, $\Omega_2 = \mathrm{span}_{\mathcal{K}}\{\omega_2, \ldots, \omega_n\}$, ..., $\Omega_n = \mathrm{span}_{\mathcal{K}}\{\omega_n\}$ and $\dot{\omega}_i = \lambda_i \omega_i$, $i = 1, \ldots, n$, for some $\lambda_i \in \mathcal{K}$. Therefore λ_i, $i = 1, \ldots, n$, are eigenvalues of the system. □

4. DISCRETE-TIME COUNTERPART

The concept of eigenvalues and eigenvectors can easily be extended to the case of nonlinear discrete-time systems.

To deal with the nonlinear discrete-time systems we will use the algebraic setting of Grizzle (1993); Aranda-Bricaire et al. (1996) adapted again to the case of uncontrolled systems (for the sake of simplicity).

Consider the nonlinear discrete-time system defined by difference equations of the form

$$x^+ = f(x) \tag{6}$$

where $x \in \mathbb{R}^n$ and elements of f are assumed to be from the field of meromorphic functions of variables $\{x_1, \ldots, x_n\}$ denoted by $\mathcal{K}$. The abridged notation x^+ stands for $x(t+1)$.

Let again $\mathcal{E}$ denote the formal vector space of differential one-forms defined as

$$\mathcal{E} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\xi; \xi \in \mathcal{K}\}$$

Again, a one-form $\omega \in \mathcal{E}$ is called exact (or integrable), if there exists $F \in \mathcal{K}$ such that $\mathrm{d}F = \omega$, and a subspace $\mathcal{V} \subset \mathcal{E}$ is called exact (or integrable) if it has a basis that consists of exact one-forms only.

In discrete-time case, one defines the forward-shift operator δ that acts on $\mathcal{K}$ in the usual way. That is, $\delta(\xi) = \xi^+$ for any $\xi \in \mathcal{K}$. The operator δ induces the forward-shift operator, which is by abuse of notation denoted by the same symbol δ, that acts on $\mathcal{E}$ as follows.
Let $\omega = \sum_i \alpha_i \mathrm{d}\xi_i$ be in $\mathcal{E}$, then

$$\omega^+ = \sum_i \alpha_i^+ \mathrm{d}\xi_i^+$$

It is important that δ is an automorphism over $\mathcal{K}$, in which case the pair $(\mathcal{K}, \delta)$ really forms a difference field. This is satisfied if and only if (Halás et al., 2009) for the system (6) the following assumption holds

$$\mathrm{rank}_{\mathcal{K}} \frac{\partial f}{\partial x} = n$$

Then, the backward-shift operator δ^{-1} exists.

Finally, the tangent linear system associated to the nonlinear system (6) is given by

$$\mathrm{d}x^+ = A\mathrm{d}x \tag{7}$$

where $A = (\partial f/\partial x) \in \mathcal{K}^{n\times n}$.

4.1 Eigenvalues and eigenvectors of a nonlinear system

The definition of eigenvalues and eigenvectors can now be adapted to the discrete-time counterpart.

Definition 11. A function $\lambda \in \mathcal{K}$ and a nonzero vector $e \in \mathcal{K}^n$ are said to be an eigenvalue and, respectively, an eigenvector (associated with the eigenvalue λ) of the nonlinear system (6), if

$$\lambda e^+ = Ae \tag{8}$$

where A is defined by (7).

Note that this definition again represents a generalization of that for linear systems, as for $e \in \mathbb{R}^n$ one has $e^+ = e$, and (8) reduces to (3).

Theorem 12. Eigenvalues of the nonlinear system (6) are invariant with respect to a change of coordinates $\xi = \phi(x)$ where $\phi \in \mathcal{K}$.

Proof. Let λ be an eigenvalue of the system (6) associated with an eigenvector e. For any change of coordinates $\xi = \phi(x)$ one has $\mathrm{d}\xi = T\mathrm{d}x$ where $T = (\partial\phi/\partial x)$ and $\mathrm{rank}_{\mathcal{K}} T = n$.
In the new coordinates we get $\xi^+ = \tilde{f}(\xi)$ for some $\tilde{f} \in \mathcal{K}$, and

$$\mathrm{d}\xi^+ = \tilde{A}\mathrm{d}\xi$$

where $\tilde{A} = T^+ A T^{-1}$.
The corresponding eigenvector then is

$$\tilde{e} = Te$$

It again suffices to show that λ is an eigenvalue of the nonlinear system $\xi^+ = \tilde{f}(\xi)$ too.

$$\begin{aligned}\lambda\tilde{e}^+ &= \tilde{A}\tilde{e}\\ \lambda T^+ e^+ &= (T^+ A T^{-1})Te\\ \lambda T^+ e^+ &= T^+ A e\\ \lambda e^+ &= Ae\end{aligned}$$

which completes the proof. □

4.2 Transformation to the diagonal form

Analogous result can now be shown for the transformation of the system (6) to the diagonal form.

Theorem 13. Given a system of the form (6), there exists a change of coordinates $\xi = \phi(x)$ that transforms the system into a diagonal form if and only if there exist n eigenvalues $\lambda_1, \ldots, \lambda_n$ associated with n eigenvectors $e_1, \ldots, e_n$ such that

$$T^{-1} = (e_1|e_2|\cdots|e_n)$$

is nonsingular and $T\mathrm{d}x = (\omega_1, \ldots, \omega_n)^T$ where the one-forms $\omega_1, \ldots, \omega_n$ are exact.

Proof. Consider the system (6) and the matrix A given by (7). Then

$$\begin{aligned}AT^{-1} &= (Ae_1|Ae_2|\cdots|Ae_n)\\ &= \left(\lambda_1 e_1^+|\lambda_2 e_2^+|\cdots|\lambda_n e_n^+\right)\\ &= \left(T^+\right)^{-1}\Lambda\end{aligned}$$

where

$$\Lambda = \begin{pmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & & \vdots\\ \vdots & & \ddots & 0\\ 0 & \cdots & 0 & \lambda_n\end{pmatrix}$$

Since T^{-1} is nonsingular, one has

$$\Lambda = T^+ A T^{-1}$$

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Finally, there exists a change of coordinates $\xi = \phi(x)$, such that $\mathrm{d}\xi = T\mathrm{d}x$ where $T = (\partial\phi/\partial x)$, if and only if the rows of $T\mathrm{d}x$ are exact one-forms. □

4.3 Transformation to the feedforward form

Theorem 14. Given a system of the form (6), there exists a change of coordinates $\xi = \phi(x)$ that transforms the system into the feedforward form if and only if there exist n eigenvalues $\lambda_1, \ldots, \lambda_n$ associated with n eigenvectors $e_1, \ldots, e_n$ such that

$$T^{-1} = (e_1|e_2|\cdots|e_n)$$

is nonsingular and $T\mathrm{d}x = (\omega_1, \ldots, \omega_n)^T$ where the one-forms $\omega_1, \ldots, \omega_n$ are such that $\Omega_1 = \mathrm{span}_{\mathcal{K}}\{\omega_1, \omega_2, \ldots, \omega_n\}$, $\Omega_2 = \mathrm{span}_{\mathcal{K}}\{\omega_2, \ldots, \omega_n\}$, …, $\Omega_n = \mathrm{span}_{\mathcal{K}}\{\omega_n\}$ are integrable.

Proof. The proof is an adaptation of the proof of Theorem 10. □

4.4 Examples

Example 15. The following system was considered in Aranda-Bricaire and Moog (2004)

$$\begin{aligned} x_1^+ &= x_2 \\ x_2^+ &= -x_1 \end{aligned}$$

We are looking for $\lambda \in \mathcal{K}$ and a nonzero vector $(e_1, e_2)^T \in \mathcal{K}^2$ such that

$$\lambda \begin{pmatrix} e_1^+ \\ e_2^+ \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

Table 2 shows various solutions for λ, e_1 and e_2 that can be found.

Note that though the system is linear, this example is not trivial. If we restrict our attention to linear systems and the linear theory, the system does not admit any real eigenvalue (we have $\lambda_{1,2} = \pm i$). However, it does ($\lambda_{1,2} = \pm 1$) according to Definition 11. In other words, the system cannot be transformed into the diagonal form by any linear change of coordinates, but it can be by nonlinear. If we pick for instance

$$e_1 = \begin{pmatrix} 1/x_1 \\ 1/x_2 \end{pmatrix}, \quad e_2 = \begin{pmatrix} -1/x_1 \\ 1/x_2 \end{pmatrix}$$

then

$$T^{-1} = \begin{pmatrix} 1/x_1 & -1/x_1 \\ 1/x_2 & 1/x_2 \end{pmatrix}, \quad T = \begin{pmatrix} 2x_1 & 2x_2 \\ -2x_1 & 2x_2 \end{pmatrix}$$

and

$$\mathrm{d}\xi = T\mathrm{d}x = \begin{pmatrix} 2x_1\mathrm{d}x_1 + 2x_2\mathrm{d}x_2 \\ -2x_1\mathrm{d}x_1 + 2x_2\mathrm{d}x_2 \end{pmatrix}$$

consists of exact one forms. Thus, the change of coordinates $(\xi_1, \xi_2) = (x_1^2 + x_2^2, -x_1^2 + x_2^2)$ transforms the system to the diagonal form

$$\begin{aligned} \xi_1^+ &= \xi_1 \\ \xi_2^+ &= -\xi_2 \end{aligned}$$

To transform the system into the feedforward form we use $\omega_1 = 2x_1\mathrm{d}x_1 + 2x_2\mathrm{d}x_2$ and $\omega_2 = -2x_1\mathrm{d}x_1 + 2x_2\mathrm{d}x_2$ from $T\mathrm{d}x$. Then we have $\Omega_1 = \mathrm{span}_{\mathcal{K}}\{2x_1\mathrm{d}x_1 + 2x_2\mathrm{d}x_2, -2x_1\mathrm{d}x_1 + 2x_2\mathrm{d}x_2\} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_1, \mathrm{d}x_2\}$ and $\Omega_2 = \mathrm{span}_{\mathcal{K}}\{-2x_1\mathrm{d}x_1 + 2x_2\mathrm{d}x_2\}$. We can choose for instance $\phi_1 = x_1^2$ and $\phi_2 = -x_1^2 + x_2^2$ for which $\mathrm{d}\phi_1 \in \Omega_1$ and $\mathrm{d}\phi_2 \in \Omega_2$ respectively. Then the change of coordinates $(\xi_1, \xi_2) = (x_1^2, -x_1^2 + x_2^2)$ transforms the system into the feedforward form

$$\begin{aligned} \xi_1^+ &= \xi_1 + \xi_2 \\ \xi_2^+ &= -\xi_2 \end{aligned}$$

λ	e_1	e_2
1	x_1	x_2
	$-x_2$	x_1
	$1/x_1$	$1/x_2$
	$-1/x_2$	$1/x_1$
-1	x_2	x_1
	$-x_1$	x_2
	$1/x_2$	$1/x_1$
	$-1/x_1$	$1/x_2$

Table 2. Possible eigenvalues and various eigenvectors for the system in Example 15.

Example 16. Consider the system

$$\begin{aligned} x_1^+ &= x_1 x_2 \\ x_2^+ &= x_2 \end{aligned}$$

We are looking for $\lambda \in \mathcal{K}$ and a nonzero vector $(e_1, e_2)^T \in \mathcal{K}^2$ such that

$$\lambda \begin{pmatrix} e_1^+ \\ e_2^+ \end{pmatrix} = \begin{pmatrix} x_2 & x_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

Easy computation shows that $\lambda_1 = x_2$, $\lambda_2 = 1$ are eigenvalues of the system with the corresponding eigenvectors for instance $(1, 0)$, $(0, 1)$ respectively.

Example 17. Consider the system

$$x^+ = 2x$$

The change of coordinates $\xi = x^k$, $0 \neq k \in \mathbb{Z}$ transforms the system to the form

$$\xi^+ = 2^k \xi$$

which implies that (any) $\lambda = 2^k$ is an eigenvalue for the system.

5. DISCUSSION, PERSPECTIVES AND CONCLUDING REMARKS

In this paper the notions of an eigenvalue and an eigenvector for a nonlinear system have been introduced. Both continuous- and discrete-time systems were studied. Characteristic property of this concept is that (allowing nonlinear change of coordinates) the system can admit (possibly infinitely) many eigenvalues. However, this does not mean that anything can be an eigenvalue for the system. Note for instance that in Example 15 we were not able to find any other eigenvalue in $\mathcal{K}$ besides 1 and -1. One of the main issues here is, therefore, to get an interpretation of the eigenvalues obtained. At the moment, such an interpretation is given mainly by the algebraic properties we expect of eigenvalues. Though not unique, the eigenvalues of the system are invariant under a change of coordinates. Therefore, the system can be transformed to the diagonal form whenever the corresponding matrix constructed form the respective eigenvectors is nonsingular and consists of exact one-forms. Another interpretation comes from an analogous concept that has been introduced in Menini and Tornambe (2011) in terms of Lie brackets, called characteristic function. In terms of our notation the definition of a characteristic function λ for the system (1) reads

$$[e, f] = \lambda e$$

After computing $[e, f] = \frac{\partial f}{\partial x} e - \frac{\partial e}{\partial x} f = Ae - \dot{e}$ one gets $Ae = \lambda e + \dot{e}$. Therefore, the concept of eigenvalues introduced in our paper is an alternative way of obtaining a characteristic function for the system in the sense of Menini and Tornambe (2011).

Another possibility how to use the concept of eigenvalues of a nonlinear system is to transform the system to the feedforward form. The reader is referred to Astolfi and Mazenc (2000); Tall and Respondek (2000); Respondek and Tall (2004); Aranda-Bricaire and Moog (2004) for additional facts about systems in the feedforward form. The concept introduced in this paper seems to be easier than that of Astolfi and Mazenc (2000) and Aranda-Bricaire and Moog (2004), for it is applicable for both continuous- and discrete-time systems and one has to find the solution to first order differential equations only (or difference equations in case of discrete-time systems). In addition, the equations do not involve here time-derivatives (or time-shifts respectively) of the eigenvalue λ.
The problem left for the future research here consists of determining the way how to find the parameter λ (the eigenvalue). Note that this problem has not been addressed in Aranda-Bricaire and Moog (2004). An interesting question is whether the eigenvalues can be computed as the roots of the so-called Ore determinant (Ore, 1931, 1933) of the matrix $(sI - A)$ where s stands for a time-derivative operator (in continuous-time case); that is, $s\,\mathrm{d}\xi = \mathrm{d}\dot{\xi}$ for any $\mathrm{d}\xi \in \mathcal{K}$. If so, they could be identified with the poles of the transfer function of a nonlinear system (Halás, 2008). That is, the eigenvalues could be computed as the roots of the respective polynomial from the (non-commutative) skew polynomial ring $\mathcal{K}[s]$. See also Zheng et al. (2001) for such a polynomial description of a nonlinear system. Once the eigenvalues are found, the respective eigenvectors can be found by solving the set of differential equations of the form (4).

Another interesting question, left for the future research, is the relation of the concept of eigenvalues introduced in this paper to the concept of stability of a nonlinear system. In that respect the fact that the proposed transformations (for instance to the diagonal form) might, in general, not be defined at the origin has to be taken into account.

Finally, the results of this paper can also be adapted to the systems with input. At the moment, the extension seems more or less straightforward and we expect changes of the technical nature only.

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