Quadrotor tracking control based on a moving frame

M. Konz [*] **J. Rudolph** [*]

[*] *Chair of Systems Theory and Control Engineering,*
Saarland University, 66123 Saarbrücken, Germany,
(e-mail: {m.konz, j.rudolph}@lsr.uni-saarland.de)

Abstract: A model for a quadrotor helicopter, its flatness-based parameterization, and its control are investigated. The model is transformed by expressing the configuration in terms of a reference trajectory and the deviation from the latter. A flat output for the error system is introduced. A dynamic and a quasi-static feedback for asymptotic stabilization of reference trajectories are derived. The approach avoids introducing artificial singularities and provides the possibility for tracking "acrobatic" trajectories. A simulation result with the quadrotor flying a loop is shown.

Keywords: quadrotor, flatness, tracking control, attitude parameterization, singularities

1. INTRODUCTION

A quadrotor helicopter is a vertical take-off and landing (VTOL) aircraft with four propellers. It has gained large interest within the scientific community for its challenging problems in actuation, measurement, and signal processing. Moreover, it leads to interesting control theoretic questions.

The present work focuses on the mechanical model and control design for a quadrotor rather than on practical issues encountered in its realization. There are a variety of control approaches for quadrotors modeled as a rigid body with the propellers providing force and torque pairs. Many approaches parameterize the attitude of the rigid body by Euler angles, thus, necessarily introducing artificial singularities into the model (e.g. Lee et al. [2009], Castillo et al. [2004], Bouabdallah and Siegwart [2007], Zhang et al. [2009]). Another potential source of artificial singularities is feedback linearization (e.g. Fritsch et al. [2012]). Even though these singularities are usually far away from standard operating regimes, they restrict the domain of admissible trajectories.

The objective of the present contribution is to motivate and to derive a continuous control scheme which does not artificially restrict admissible trajectories and enables tracking complex maneuvers. A similar objective is addressed by (Lee et al. [2010]) leading to a different solution.

The present contribution is organized as follows. First the mechanical model is given and its properties are investigated. In section 3, the model is rewritten in terms of a reference configuration and the configuration error. A flat output for the error dynamics is introduced. Two possible approaches to position and orientation tracking are derived and discussed. Finally, the advantage of the control approach are illustrated by means of a simulation example.

2. MODEL

Neglecting the dynamics of the propellers and aerodynamic effects, a simple model for a quadrotor is a rigid body. Let $\hat{\boldsymbol{e}} = [\hat{\boldsymbol{e}}_x, \hat{\boldsymbol{e}}_y, \hat{\boldsymbol{e}}_z]$ be an orthonormal right handed inertial basis of $\mathbb{R}^3$. The vector to the center of mass of the quadrotor is denoted as

$$\boldsymbol{r} = \hat{r}^x \hat{\boldsymbol{e}}_x + \hat{r}^y \hat{\boldsymbol{e}}_y + \hat{r}^z \hat{\boldsymbol{e}}_z = [\hat{\boldsymbol{e}}_x, \hat{\boldsymbol{e}}_y, \hat{\boldsymbol{e}}_z] \underbrace{\begin{bmatrix} \hat{r}^x \\ \hat{r}^y \\ \hat{r}^z \end{bmatrix}}_{= \hat{r}}.$$

Here and in the sequel, vectors are indicated by bold symbols and the triples of their coefficients w.r.t. particular bases carry the accents of these bases.

Another orthonormal, right handed basis $\boldsymbol{e} = [\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z]$ is fixed to the rigid body, as illustrated in Fig. 1. The attitude of the quadrotor can then be parameterized by the rotation matrix $R \in \mathsf{SO}(3) = \{R \in \mathbb{R}^{3 \times 3} | R^T R = I, \det R = 1\}$ which relates the two bases according to

$$[\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z] = [\hat{\boldsymbol{e}}_x, \hat{\boldsymbol{e}}_y, \hat{\boldsymbol{e}}_z] \underbrace{\begin{bmatrix} R_x^x & R_y^x & R_z^x \\ R_x^y & R_y^y & R_z^y \\ R_x^z & R_y^z & R_z^z \end{bmatrix}}_{= R},$$

where $R_j^i = \langle \hat{\boldsymbol{e}}_i, \boldsymbol{e}_j \rangle$, $i, j = x, y, z$.

The coefficients $\omega = [\omega^x, \omega^y, \omega^z]^T \in \mathbb{R}^3$ of the angular velocity $\boldsymbol{\omega} = \boldsymbol{e}\, \omega$ w.r.t. the body fixed basis are related to the derivative $\dot{\boldsymbol{e}}$ of the latter by

$$\dot{\boldsymbol{e}} = \hat{\boldsymbol{e}} \dot{R} = \boldsymbol{e} \underbrace{R^T \dot{R}}_{= \mathrm{Skw}\, \omega}, \quad \mathrm{Skw}\, \omega = \begin{bmatrix} 0 & -\omega^z & \omega^y \\ \omega^z & 0 & -\omega^x \\ -\omega^y & \omega^x & 0 \end{bmatrix}. \quad (1)$$

The inverse to the operator Skw is denoted as skw.

The dynamics of a rigid body in the field of gravitational acceleration $\boldsymbol{g}$, with forces $\boldsymbol{f}$ and torques $\boldsymbol{\tau}$ is described by

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(m\boldsymbol{v}) = \boldsymbol{f} + m\boldsymbol{g}, \qquad \tfrac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\Theta}\boldsymbol{\omega}) = \boldsymbol{\tau}, \qquad (2)$$
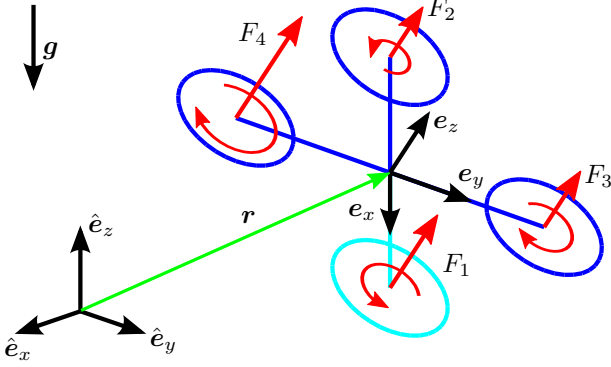
Fig. 1. Sketch of a quadrotor with its frames.

where $\boldsymbol{v} = \dot{\boldsymbol{r}}$ is the translational velocity, m is the mass, and $\boldsymbol{\Theta}$ is the inertia tensor of the rigid body. This tensor is most conveniently expressed with respect to the body fixed frame $\boldsymbol{\Theta} = \sum_{i,j=x,y,z} \Theta_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$, where $\Theta \in \mathbb{R}^{3\times3}$ is constant w.r.t. this basis.

The four propellers of the quadrotor produce four independent forces $F_i, i = 1, \ldots, 4$ which act along straight lines parallel to $\boldsymbol{e}_z$ through the points $\pm a\boldsymbol{e}_x$ and $\pm a\boldsymbol{e}_y$. In addition, each propeller produces a torque about its axis, which is proportional to its force by a factor $\pm b$. The different sign is a consequence of the opposite spinning directions of the propellers (cf. Fig. 1).

The propeller forces F_i can be mapped to the equivalent force $\boldsymbol{f} = \boldsymbol{e}_z f^z$ at the center of mass and torque $\boldsymbol{\tau} = \tau^x \boldsymbol{e}_x + \tau^y \boldsymbol{e}_y + \tau^z \boldsymbol{e}_z = \boldsymbol{e}\tau$ w.r.t. to the body frame. The relation is

$$\begin{bmatrix} f^z \\ \tau^x \\ \tau^y \\ \tau^z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ a & -a & 0 & 0 \\ 0 & 0 & a & -a \\ b & b & -b & -b \end{bmatrix}}_{=A} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}, \quad \det A = 8a^2 b \neq 0.$$

Using this invertible relation, the thrust magnitude f^z and the torques $\tau = [\tau^x, \tau^y, \tau^z]^T$ are regarded as the control input components rather than $F_i, i = 1, \ldots, 4$.

The balance of momentum and angular momentum (2) can be projected on the inertial resp. the body fixed basis in order to get a coordinate based representation. In summary, the model for the quadrotor is

$$\Sigma : \quad \begin{cases} m\ddot{\hat{r}} = m\hat{g} + R_z f^z \\ \dot{R} = R\,\mathrm{Skw}\,\omega, \ R^T R = I, \ \det R = 1 \\ \Theta\dot{\omega} + \omega \times \Theta\omega = \tau \end{cases}$$

where $R_z = [R_z^x, R_z^y, R_z^z]^T$ indicates the last column of R. The same model is considered in (Lee et al. [2010]).

This model is globally defined, i.e. it does not have singularities (as a model based on Euler angles would). Its drawback is that the parameterization is not minimal. The attitude R is represented by 9 scalar quantities, even though it evolves on the three dimensional manifold $\mathsf{SO}(3)$.

2.1 Flatness

As it is now well known, differential flatness (see e.g. Fliess et al. [1999]) is a useful property for trajectory planning

and control of nonlinear systems. Therefore, the flatness of Σ is briefly discussed first. This is done by successive elimination of inputs.

The angular velocity ω, and subsequently, the torque τ are expressed by the attitude R and its derivatives as

$$\omega = \mathrm{skw}(R^T \dot{R}), \qquad \tau = \Theta\dot{\omega} + \omega \times \Theta\omega.$$

Since $R_z \in \mathsf{S}^2 = \{v \in \mathbb{R}^3 \,|\, ||v|| = 1\}$ the force balance can be decomposed in its magnitude f^z and its direction R_z

$$f^z = \pm m\,||\ddot{\hat{r}} - \hat{g}||, \qquad R_z = \pm \frac{\ddot{\hat{r}} - \hat{g}}{||\ddot{\hat{r}} - \hat{g}||}. \qquad (3)$$

The singularity at $\ddot{\hat{r}} - \hat{g} = 0 \Leftrightarrow f^z = 0$ corresponds to 'free falling' and the sign ambiguity reflects that the same trajectory of the center of mass $\hat{r}$ can be flown 'upside-down' with opposite thrust f^z. (It can be shown that this singularity is intrinsic, i.e., a local loss of controllability of the first order approximation.) Since most quadrotor realizations can only produce positive thrusts, a restriction to the positive sign is appropriate.

Popular examples with similar constraints (and singularities) are the PVTOL (Rudolph and Fröhlich [2003]) and the $2k\pi$-juggling robot (Lenoir et al. [1998]). Flatness based control is used in both cases.

Summing up, the flatness of the quadrotor model Σ boils down to the flatness of

$$R_z = \frac{\ddot{\hat{r}} - \hat{g}}{||\ddot{\hat{r}} - \hat{g}||}, \qquad R^T R = I, \ \det R = 1. \qquad (4)$$

Obviously, the position $\hat{r}$ should be part of a flat output, corresponding to three of its four components. The trajectory of the position already fixes a part, R_z, of the attitude. Therefore, the fourth component has to somehow parameterize the remaining part R_x, R_y.

As already pointed out by Hamel et al. [2002], there is no 'correct' parameterization for this remaining degree of freedom. Every attempt will result in a corresponding (artificial) singularity.

In the motion planning, the parameterizations can be (piecewise) chosen as appropriate to the trajectory. For the feedback the moving frame approach (which will be developed in the next section) is used to 'push' the unavoidable singularity as far away from nominal conditions as possible.

3. MODEL IN TERMS OF REFERENCE AND ERROR

Assume a valid reference trajectory $t \mapsto (\hat{r}_R(t), R_R(t))$ has been planned, i.e. one that obeys the dynamic and algebraic constraints of (4) and is sufficiently smooth. The reference attitude R_R defines a reference basis $\bar{\boldsymbol{e}} = \hat{\boldsymbol{e}}R_R$. The corresponding angular velocity is $\bar{\omega}_R = \mathrm{skw}(R_R^T \dot{R}_R)$.

The deviation from the reference can be described by the position error $\hat{d}$ and the attitude error D:

$$\hat{d} := \hat{r} - \hat{r}_R, \qquad D := R_R^T R. \qquad (5)$$

Plugging this into (4) again yields structurally similar constraints for the errors:

$$D_z = \frac{R_R^T(\ddot{\hat{r}}_R + \ddot{\hat{d}} - \hat{g})}{||\ddot{\hat{r}}_R + \ddot{\hat{d}} - \hat{g}||}, \qquad D^T D = I, \ \det D = 1. \quad (6)$$
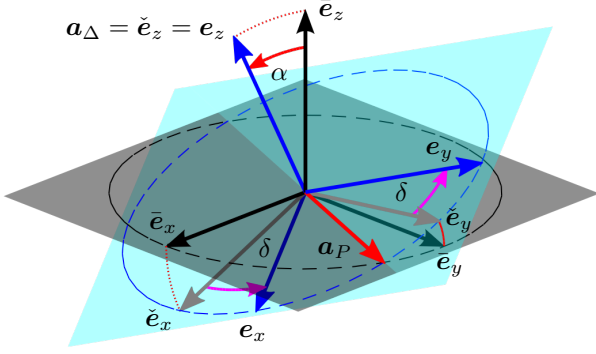
Fig. 2. Illustration for the rotation decomposition.

3.1 Decomposition of the attitude error

Motivated by the structure of (6), the attitude error $D \in \mathsf{SO}(3)$ is decomposed into a product of two rotation matrices $P, \Delta \in \mathsf{SO}(3)$, where Δ represents a rotation about the axis $a_\Delta = [0, 0, 1]^T$, which is the normal to the propeller plane. In terms of an axis-angle representation of the rotation matrices this is

$$D = P\Delta = \mathrm{e}^{\mathrm{Skw}(\bar{a}_P)\alpha}\, \mathrm{e}^{\mathrm{Skw}(a_\Delta)\delta}, \qquad \bar{a}_P \in \mathsf{S}^2 \quad (7)$$

(see e.g. (Piovan and Bullo [2012]) for a general treatment of rotation decomposition). The problem statement (7) is ambiguous since the right side has four free parameters.

One solution for $\bar{a}_P$ and α of (7) is

$$\bar{a}_P = \frac{[-D_z^y, D_z^x, 0]^T}{\sqrt{(D_z^x)^2 + (D_z^y)^2}}, \qquad \alpha = \arccos D_z^z.$$

Using Rodrigues' rotation formula, the rotation matrix is

$$P = \mathrm{e}^{\mathrm{Skw}(\bar{a}_P)\alpha} = \begin{bmatrix} 1 - \frac{(D_z^x)^2}{1+D_z^z} & -\frac{D_z^x D_z^y}{1+D_z^z} & D_z^x \\ -\frac{D_z^x D_z^y}{1+D_z^z} & 1 - \frac{(D_z^y)^2}{1+D_z^z} & D_z^y \\ -D_z^x & -D_z^y & D_z^z \end{bmatrix}. \quad (8)$$

It can be shown (see Appendix A) that this solution is the one with the minimal rotation angle $|\alpha|$. It is important to remark that P is defined solely by $D_z = [D_z^x, D_z^y, D_z^z]^T$. The singularity at $D_z^z = -1$ will be discussed below.

The angle δ can now be obtained by simply comparing the entries in

$$\begin{bmatrix} \cos\delta & -\sin\delta & 0 \\ \sin\delta & \cos\delta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \Delta = P^T D = \begin{bmatrix} \frac{D_x^x + D_y^y}{1+D_z^z} & \frac{D_x^y - D_y^x}{1+D_z^z} & 0 \\ \frac{D_y^x - D_x^y}{1+D_z^z} & \frac{D_x^x + D_y^y}{1+D_z^z} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which gives

$$\delta = \mathrm{atan2}(D_x^y - D_y^x, D_x^x + D_y^y). \quad (9)$$

Summing up, the decomposition can be regarded as a parameterization of the rotation matrix D by the quantities $D_z = [D_z^x, D_z^y, D_z^z]^T \in \mathsf{S}^2 \backslash [0,0,-1]^T$ and $\delta \in (-\pi, \pi]$.

Graphical interpretation   The attitude error D relates the reference basis $\bar{e}$ to the body fixed basis $e = \bar{e}D$. This rotation is decomposed in the way that first by $\bar{e}P =: [\check{e}_x, \check{e}_y, \check{e}_z] =: \check{e}$ the basis vector $\bar{e}_z$ is rotated into $\check{e}_z = e_z$ (Fig. 2). The corresponding axis and angle in terms of the basis vectors are

$$a_P = \frac{\bar{e}_z \times e_z}{||\bar{e}_z \times e_z||} = \bar{e}\bar{a}_P, \qquad \cos\alpha = \langle \bar{e}_z, e_z \rangle.$$

As the orthogonal basis vectors $\check{e}_x$ and $\check{e}_y$ lie in the plane of the body fixed basis vectors e_x, e_y, on $(-\pi, \pi)$ the angle δ is uniquely defined by

$$\cos\delta = \langle \check{e}_x, e_x \rangle, \qquad \sin\delta = \langle \check{e}_y, e_x \rangle.$$

An important fact is that the rotation matrix P is still well defined if $e_z = \bar{e}_z \Rightarrow a_P = 0$, as the angle α vanishes also, and so $P = I$. The singularity corresponds to the opposite case $e_z = -\bar{e}_z \Rightarrow a_P = 0$, where the axis also vanishes but the angle is $\alpha = 180°$, and so P is undefined.

Assuming a controller works properly, the case that the thrust direction e_z is antipodal to its reference $\bar{e}_z$ should be avoided.

Angular velocity   The angular velocities of the rotation matrices D, P, Δ are defined in the same manner as in (1):

$$\omega_D = \mathrm{skw}(D^T \dot{D}),$$
$$\check{\omega}_P = \mathrm{skw}(P^T \dot{P}),$$
$$\omega_\Delta = \mathrm{skw}(\Delta^T \dot{\Delta}) = a_\Delta \dot{\delta}.$$

They are related by

$$\omega_D = \mathrm{skw}(\Delta^T P^T (\dot{P}\Delta + P\dot{\Delta})) = \Delta^T \check{\omega}_P + a_\Delta \dot{\delta} \quad (10)$$

where the property

$$\mathrm{skw}(R^T(\mathrm{Skw}\, v)R) = R^T v, \ \forall\, R \in \mathsf{SO}(3), \ v \in \mathbb{R}^3$$

has been exploited.

Since P has only two free parameters, its angular velocity coefficients $\check{\omega}_P$ are dependent. This constraint follows from the symmetry $P_y^x = P_x^y \Rightarrow \dot{P}_y^x = \dot{P}_x^y$, where $\dot{P} = P\,\mathrm{Skw}\,\check{\omega}_P$. This yields

$$D_z^x \check{\omega}_P^x + D_z^y \check{\omega}_P^y - (1 + D_z^z)\check{\omega}_P^z = 0. \quad (11)$$

Combining (10) and (11) and eliminating $\check{\omega}_P^z$ implies

$$\begin{bmatrix} \omega_D^x \\ \omega_D^y \\ \omega_D^z \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\delta & \sin\delta & 0 \\ -\sin\delta & \cos\delta & 0 \\ \frac{D_z^x}{1+D_z^z} & \frac{D_z^y}{1+D_z^z} & 1 \end{bmatrix}}_{=\,W} \underbrace{\begin{bmatrix} \check{\omega}_P^x \\ \check{\omega}_P^y \\ \dot{\delta} \end{bmatrix}}_{=\,\varpi}, \quad \det W = 1. \quad (12)$$

3.2 Input transformation

The decomposition of the attitude error motivates the choice of a new input (f^z, w) with

$$w = \dot{\varpi}, \quad \text{i.e.} \quad w_1 = \dot{\check{\omega}}_P^x, \quad w_2 = \dot{\check{\omega}}_P^y, \quad w_3 = \ddot{\delta}. \quad (13)$$

This defines the physical input τ through a static feedback. The explicit equations are obtained from the relation

$$\omega = \mathrm{skw}((R_\mathrm{R}D)^T \tfrac{\mathrm{d}}{\mathrm{d}t}(R_\mathrm{R}D)) = D^T \bar{\omega}_\mathrm{R} + W\varpi \quad (14)$$

and the balance of angular momentum. They have the form

$$\tau = h_1(w, R, \omega, R_\mathrm{R}, \bar{\omega}_\mathrm{R}, \dot{\bar{\omega}}_\mathrm{R}) \quad (15)$$

and are defined where the attitude error decomposition is defined, i.e. at $D_z^z = R_{\mathrm{R},z}^T R_z \neq -1$.

Rewriting the model Σ in terms of the attitude error decomposition $(D_z^x, D_z^y, D_z^z, \delta)$ and the new input (f^z, w) results in the two *decoupled* subsystems

$$\Sigma_d: \begin{cases} m(\ddot{r}_\mathrm{R} + \ddot{d} - \hat{g}) = R_\mathrm{R} P_z f^z \\ \dot{P}_z = \check{\omega}_P^y P_x - \check{\omega}_P^x P_y, \ ||P_z|| = 1 \\ \dot{\check{\omega}}_P^x = w_1, \quad \dot{\check{\omega}}_P^y = w_2 \end{cases}$$

$$\Sigma_\delta: \qquad \ddot{\delta} = w_3$$

where $[P_x, P_y, P_z] = P$, which was defined in (8). The reference quantities $\hat{r}_R$ and R_R are regarded as time varying parameters.

The decoupling is a consequence of the fact that the rotation matrix P is completely parameterized by the coefficients of its last column $P_z = D_z = [D_z^x, D_z^y, D_z^z]^T$. Knowing this, it is straightforward to show that $\hat{d}$ (the coefficients of which are obviously differentially independent) is a flat output for Σ_d. Moreover, δ is a flat output of Σ_δ.

For the controller design it is useful to express the input (f^z, w_1, w_2) in terms of the flat output $\hat{d}$. Again, decomposing the force balance in its direction and magnitude yields

$$P_z = \frac{R_R^T(\ddot{\hat{r}}_R + \ddot{\hat{d}} - \hat{g})}{||\ddot{\hat{r}}_R + \ddot{\hat{d}} - \hat{g}||}, \qquad f^z = m||\ddot{\hat{r}}_R + \ddot{\hat{d}} - \hat{g}||. \quad (16a)$$

The remaining directions P_x and P_y of P as defined in (8) follow from the coefficients of P_z alone. Differentiating the force balance once and projecting on P_x and P_y gives

$$\breve{\omega}_P^x = -\frac{\langle R_R^T(\hat{r}_R^{(3)} + \hat{d}^{(3)}), P_y\rangle}{||\ddot{\hat{r}}_R + \ddot{\hat{d}} - \hat{g}||} - \langle \bar{\omega}_R, P_x\rangle, \quad (16b)$$

$$\breve{\omega}_P^y = \frac{\langle R_R^T(\hat{r}_R^{(3)} + \hat{d}^{(3)}), P_x\rangle}{||\ddot{\hat{r}}_R + \ddot{\hat{d}} - \hat{g}||} - \langle \bar{\omega}_R, P_y\rangle. \quad (16c)$$

Even though $\breve{\omega}_P^z$ is not present in the model, it might be recovered from (11). Differentiating once more, finally, gives the inputs $w_1 = \dot{\breve{\omega}}_P^x$, $w_2 = \dot{\breve{\omega}}_P^y$ in a form

$$w_1 = -\frac{\langle R_R^T(\hat{r}_R^{(4)} + \hat{d}^{(4)}), P_y\rangle}{||\ddot{\hat{r}}_R + \ddot{\hat{d}} - \hat{g}||} - \langle \dot{\bar{\omega}}_R, P_x\rangle$$
$$+ h_2(\ddot{\hat{r}}_R, \hat{r}_R^{(3)}, R_R, \bar{\omega}_R, \ddot{\hat{d}}, \hat{d}^{(3)}), \quad (16d)$$

$$w_2 = \frac{\langle R_R^T(\hat{r}_R^{(4)} + \hat{d}^{(4)}), P_x\rangle}{||\ddot{\hat{r}}_R + \ddot{\hat{d}} - \hat{g}||} - \langle \dot{\bar{\omega}}_R, P_y\rangle$$
$$+ h_3(\ddot{\hat{r}}_R, \hat{r}_R^{(3)}, R_R, \bar{\omega}_R, \ddot{\hat{d}}, \hat{d}^{(3)}), \quad (16e)$$

where lower order derivatives are collected in h_2 and h_3.

From (16d) and (16e) it is clear that a valid reference position trajectory $t \mapsto \hat{r}_R(t)$ must be four times continuously differentiable. The same is required in other (not flatness-based) approaches (e.g. Hamel et al. [2002] or Lee et al. [2010]) which achieve exact tracking.

4. CONTROL DESIGN

Although a flat output $\hat{d}$ of Σ_d is known, assigning an appropriate dynamics for $\hat{d}$ is not completely straightforward since w_1 and w_2 in (16d) and (16e) depend on the fourth derivative of the position error $\hat{d}$. This means a total of 12 derivatives, but the state dimension of Σ_d w.r.t. the input (f^z, w_1, w_2) is only 10.

The orientation error δ can be stabilized by the feedback

$$w_3 = -\lambda_{\delta,1}\dot{\delta} - \lambda_{\delta,0}\delta, \qquad 0 < \lambda_{\delta,0}, \lambda_{\delta,1} \in \mathbb{R}. \quad (17)$$

4.1 Position tracking by dynamic feedback

One way to tackle the stabilization task is adding extra dynamics at the thrust magnitude f^z, i.e. extending the model Σ_d by

$$\Sigma_f : \qquad f^z = \tfrac{1}{m}\xi_1, \qquad \dot{\xi}_1 = \xi_2, \qquad \dot{\xi}_2 = u$$

with the new input u and the state (ξ_1, ξ_2).

Physically these states are

$$\xi_1 = \tfrac{1}{m}f^z = ||\ddot{\hat{r}}_R + \ddot{\hat{d}} - \hat{g}||,$$
$$\xi_2 = \tfrac{1}{m}\dot{f}^z = \langle R_R^T(\hat{r}_R^{(3)} + \hat{d}^{(3)}), P_z\rangle.$$

Using this, the inputs are expressed as

$$w_1 = -\frac{\langle R_R^T(\hat{r}_R^{(4)} + \hat{d}^{(4)}), P_y\rangle - 2(\breve{\omega}_P^x + \langle\bar{\omega}_R, P_x\rangle)\xi_2}{\xi_1}$$
$$- \langle\dot{\bar{\omega}}_R, P_x\rangle + h_4(R_R, \bar{\omega}_R, P, \breve{\omega}_P), \quad (18a)$$

$$w_2 = \frac{\langle R_R^T(\hat{r}_R^{(4)} + \hat{d}^{(4)}), P_x\rangle - 2(\breve{\omega}_P^y + \langle\bar{\omega}_R, P_y\rangle)\xi_2}{\xi_1}$$
$$- \langle\dot{\bar{\omega}}_R, P_y\rangle + h_5(R_R, \bar{\omega}_R, P, \breve{\omega}_P), \quad (18b)$$

$$u = \langle R_R^T(\hat{r}_R^{(4)} + \hat{d}^{(4)}), P_z\rangle$$
$$+ ||\breve{\omega}_P^y P_x - \breve{\omega}_P^x P_y + \bar{\omega}_R \times P_z||^2\xi_1. \quad (18c)$$

Choosing an asymptotically stable dynamics for the tracking error $\hat{d}$ as

$$\hat{d}^{(4)} = -\hat{\Lambda}_3\hat{d}^{(3)} - \hat{\Lambda}_2\ddot{\hat{d}} - \hat{\Lambda}_1\dot{\hat{d}} - \hat{\Lambda}_0\hat{d}, \quad (19)$$

with $\hat{\Lambda}_i \in \mathbb{R}^{3\times3}$, $i = 0, \ldots, 3$ and using this in (18) defines a control law which asymptotically stabilizes $\hat{d} = 0$. So the actual position $\hat{r}$ tracks the reference position $\hat{r}_R$ asymptotically.

In evaluating the error dynamics (19) the quantities

$$\ddot{\hat{d}} = R_R P_z \xi_1 + \hat{g} - \ddot{\hat{r}}_R, \quad (20a)$$

$$\hat{d}^{(3)} = R_R((\breve{\omega}_P^y P_x - \breve{\omega}_P^x P_y + \bar{\omega}_R \times P_z)\xi_1 + P_z\xi_2) - \hat{r}_R^{(3)} \quad (20b)$$

are required.

4.2 Position tracking by quasi-static feedback

Reviewing the balance of momentum in a 'vectorized' form and expressing it by the reference and the errors, i.e. using $\boldsymbol{r} = \boldsymbol{r}_R + \boldsymbol{d}$, $\boldsymbol{e}_z = \bar{\boldsymbol{e}}P_z$, yields

$$m(\ddot{\boldsymbol{r}} - \boldsymbol{g}) = \boldsymbol{e}_z f^z \quad \Leftrightarrow \quad m(\ddot{\boldsymbol{r}}_R + \ddot{\boldsymbol{d}} - \boldsymbol{g}) = \bar{\boldsymbol{e}}P_z f^z.$$

Obviously, the thrust magnitude f^z affects $\ddot{\boldsymbol{d}}$ in the direction of $\boldsymbol{e}_z$. The other directions are affected by w_1 and w_2 on the fourth derivative.

Assuming a controller works properly, the body fixed frame $\boldsymbol{e}$ should remain close to the reference frame $\bar{\boldsymbol{e}}$, i.e. $R \approx R_R \Rightarrow P_z = R_R^T R_z \approx [0, 0, 1]^T$. Then it is convenient to consider (asymptotically stable) second order dynamics

$$\ddot{\bar{d}}^z + \bar{\lambda}_{z,1}\dot{\bar{d}}^z + \bar{\lambda}_{z,0}\bar{d}^z = 0, \qquad \bar{\lambda}_{z,0}, \bar{\lambda}_{z,1} \in \mathbb{R} > 0 \quad (21a)$$

for the position error $\bar{d}^z = \langle\boldsymbol{d}, \bar{\boldsymbol{e}}_z\rangle$ in the direction of $\bar{\boldsymbol{e}}_z$ and fourth order (asymptotically stable) error dynamics for the components $\bar{d}^x = \langle\boldsymbol{d}, \bar{\boldsymbol{e}}_x\rangle$ and $\bar{d}^y = \langle\boldsymbol{d}, \bar{\boldsymbol{e}}_y\rangle$ orthogonal to $\bar{\boldsymbol{e}}_z$:

$$\bar{d}_o^{(4)} + \bar{\Lambda}_{o,3}\bar{d}_o^{(3)} + \bar{\Lambda}_{o,2}\ddot{\bar{d}}_o + \bar{\Lambda}_{o,1}\dot{\bar{d}}_o + \bar{\Lambda}_{o,0}\bar{d}_o = 0,$$
$$\bar{d}_o := [\bar{d}^x, \bar{d}^y]^T, \quad \bar{\Lambda}_{o,i} \in \mathbb{R}^{2\times2}, \quad i = 0, \ldots, 3. \quad (21b)$$

Obviously, $\bar{d} = R_R^T\hat{d}$ is a flat output of Σ_d as R_R is just a time-varying rotation.

For the derivatives of coefficients w.r.t. an accelerated basis

$$\boldsymbol{d} = \bar{e}\bar{d},$$

$$\dot{\boldsymbol{d}} = \dot{\bar{e}}\bar{d} + \bar{e}\dot{\bar{d}} = \bar{e}\underbrace{(\dot{\bar{d}} + \bar{\omega}_R \times \bar{d})}_{=: \bar{d}^{[1]}},$$

$$\ddot{\boldsymbol{d}} = \bar{e}\underbrace{(\ddot{\bar{d}} + 2\bar{\omega}_R \times \dot{\bar{d}} + \dot{\bar{\omega}}_R \times \bar{d} + \bar{\omega}_R \times (\bar{\omega}_R \times \bar{d}))}_{=: \bar{d}^{[2]}},$$

etc., introduce the notation

$$\bar{d}^{[i]} := [\langle \bar{e}_x, \boldsymbol{d}^{(i)}\rangle, \langle \bar{e}_y, \boldsymbol{d}^{(i)}\rangle, \langle \bar{e}_z, \boldsymbol{d}^{(i)}\rangle]^T.$$

The input (f^z, w_1, w_2) expressed in terms of $\bar{d}$ yields

$$f^z = m||R_R^T(\ddot{\hat{r}}_R - \hat{g}) + \bar{d}^{[2]}||, \tag{22a}$$

$$
\begin{aligned}
w_1 = & -\frac{\langle R_R^T \hat{r}_R^{(4)} + \bar{d}^{[4]}), P_y\rangle}{||R_R^T(\ddot{\hat{r}}_R - \hat{g}) + \bar{d}^{[2]}||} - \langle \dot{\bar{\omega}}_R, P_x\rangle \\
& + h_2(\ddot{\hat{r}}_R, \hat{r}_R^{(3)}, R_R, \bar{\omega}_R, \dot{\bar{\omega}}_R, \ddot{\bar{\omega}}_R, \bar{d}, \ldots, \bar{d}^{(3)}), \tag{22b}
\end{aligned}
$$

$$
\begin{aligned}
w_2 = & \frac{\langle R_R^T \hat{r}_R^{(4)} + \bar{d}^{[4]}), P_x\rangle}{||R_R^T(\ddot{\hat{r}}_R - \hat{g}) + \bar{d}^{[2]}||} - \langle \dot{\bar{\omega}}_R, P_y\rangle \\
& + h_3(\ddot{\hat{r}}_R, \hat{r}_R^{(3)}, R_R, \bar{\omega}_R, \dot{\bar{\omega}}_R, \ddot{\bar{\omega}}_R, \bar{d}, \ldots, \bar{d}^{(3)}). \tag{22c}
\end{aligned}
$$

Combining this with the error dynamics (21) defines a quasi-static state feedback (Delaleau and Rudolph [1998]).

The quantities $\ddot{\bar{d}}$ are calculated by combining the force balance and the error dynamics for $\bar{d}^z$

$$
\underbrace{\begin{bmatrix} mI & -P_z \\ [0,0,1] & 0 \end{bmatrix}}_{=: M} \begin{bmatrix} \ddot{\bar{d}} \\ f^z \end{bmatrix}
$$

$$
= \begin{bmatrix} m(R_R^T(\ddot{\hat{r}}_R - \hat{g}) - 2\bar{\omega}_R \dot{\bar{d}} - \dot{\bar{\omega}}_R \times \bar{d} - \bar{\omega}_R \times (\bar{\omega}_R \times \bar{d})) \\ -\bar{\lambda}_{z,1}\dot{\bar{d}}^z - \bar{\lambda}_{z,0}\bar{d}^z \end{bmatrix}. \tag{23}
$$

Differentiating once yields the equations required to calculate $\bar{d}^{(3)}$.

The control law is singular where $\det M = D_z^z = \langle \bar{e}_z, e_z\rangle = 0$. This is related to the sign ambiguity in (3): If the angle α between the thrust direction e_z and its reference is larger than $90°$, the position error will still converge, $\boldsymbol{d} \to 0$, but with the quadrotor being flipped upside down $\alpha \to 180°$ and having opposite thrust f^z. The singularity corresponds to the separatrix between the two regions of attraction which are both equally reasonable from a geometric point of view.

The crucial aspect is that the singularity corresponds to large deviations from the reference trajectory and not to static points in the state space.

The quasi-static feedback linearization could also be applied for the position error $\hat{d}$ w.r.t. the inertial frame, but the resulting singularity would then lie at $R_z^z = \langle \hat{e}_z, e_z\rangle = 0$ (see. e.g. Fritsch et al. [2012]), i.e. where the thrust direction e_z lies in the *inertial* $(\hat{e}_x, \hat{e}_y)$-plane. Since the goal here is to track trajectories in complex maneuvers, this would be unacceptable.

4.3 Overall control structure

Figure 3 summarizes the structure of the proposed controller. The key is the attitude error decomposition (sec.
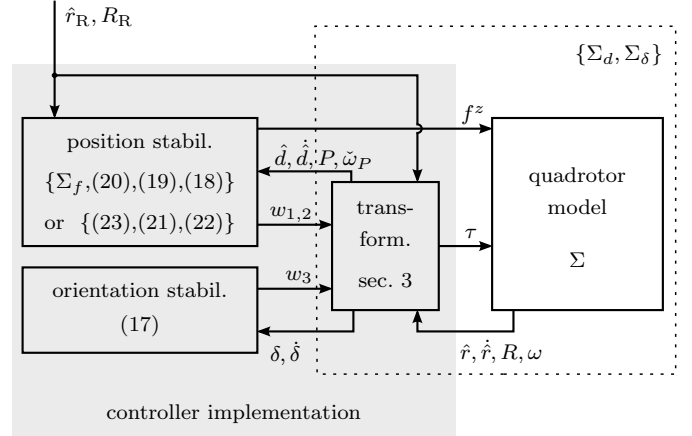

Fig. 3. Block diagram of the controlled quadrotor.

3.1) leading to the input transformation (sec. 3.2) which decouples the model into a dynamics Σ_d of the position error $\boldsymbol{d}$ and a dynamics Σ_δ of the orientation error δ. The control design uses feedback linearization based on this error.

The controller is fed with a reference trajectory of the rigid body configuration $(\hat{r}_R, R_R) \in \mathbb{R}^3 \times \mathsf{SO}(3)$, which can be planned independently. It has to obey the physical constraints (4) of the model. No artificial constraints are introduced by the controller. The only singularity at 'free falling' could, however, still be dealt with using time scaling (see Rudolph and Fröhlich [2003]).

5. SIMULATION

A numerical simulation of the model and the controller has been implemented. As no disturbances or model uncertainties were assumed, the numerical results for the tracking yield exactly the chosen tracking error. However, a small example is intended to illustrate the abilities of the proposed controller.

A cycloid curve has been chosen for the coordinates $\hat{r}_R^x$ and $\hat{r}_R^y$, and a polynomial transition for $\hat{r}_R^y$. The trajectory of the position fixes the trajectory of $R_{R,z}$. Choosing $\bar{e}_x$ to lie in the plane spanned by $\bar{e}_z$ and the velocity vector $\dot{\boldsymbol{r}}_R$ completes the parameterization of the attitude. To pick up the previous discussion about attitude parameterization at the end of section 2.1, it should be observed that this choice is only suitable if $\bar{e}_z$ is never parallel to $\dot{\boldsymbol{r}}_R$. For other trajectories different choices are convenient.

Fig. 4 illustrates the simulation result at different instances in time. The quadrotor is illustrated by the blue frame, where one arm is colored in cyan to indicate the e_x direction. The trajectory of the center of mass is the green line, its reference the black one. The reference attitude is illustrated by the black frames and the length of the red lines indicate the thrusts of the propellers.

For the sake of illustration a large initial error ($\hat{d} = [1, 0.5, 0]^T$ and $\delta = 0.99\pi$) and slow error dynamics are chosen. It can be seen that the quadrotor converges to its reference trajectory as desired. The controller has no problem with any specific attitude, like being upside down.
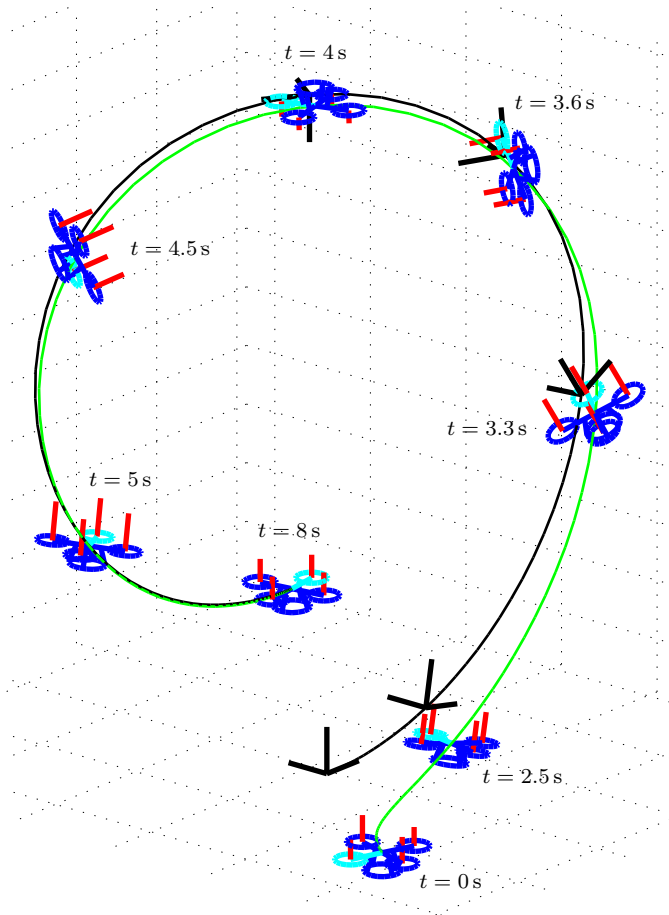
Fig. 4. Trajectory tracking example.

6. CONCLUSION

The key aspect of the control approach proposed is that error dynamics are defined for coefficients w.r.t. the moving frame of reference. Thus, (geometrically meaningful) singularities occur for large errors and not at static points in the state space. Assuming the controller implementation works properly and initial conditions are considered in the trajectory generation, all singularities should be avoided by this locally stabilizing feedback. No constraints on possible reference trajectories are introduced and so acrobatic maneuvers (e.g. a loop) can be controlled.

Another interesting feature of the proposed controller (when using the quasi-static state feedback) is that symmetries of the problem are preserved in the closed loop system. This aspect of invariant control (cf. Rouchon and Rudolph [1999], Martin et al. [2004]) should be discussed elsewhere.

REFERENCES

S. Bouabdallah and R. Siegwart. Full control of a quadrotor. In *IEEE/RSJ Int. Conf. Intelligent Robots and Systems*, pages 153–158, 2007.

P. Castillo, A. Dzul, and R. Lozano. Real-time stabilization and tracking of a four-rotor mini rotorcraft. *IEEE Trans. Contr. Syst. Technol.*, 12(4):510–516, 2004.

E. Delaleau and J. Rudolph. Control of flat systems by quasi-static feedback of generalized states. *Int. J. of Contr.*, 71(5):745–765, 1998.

M. Fliess, J. Levine, Ph. Martin, and P. Rouchon. A Lie-Bäcklund approach to equivalence and flatness of nonlinear systems. *IEEE Trans. Automat. Contr.*, 44 (5):922–937, 1999.

O. Fritsch, P. De Monte, M. Buhl, and B. Lohmann. Quasi-static feedback linearization for the translational dynamics of a quadrotor helicopter. In *Proc. American Control Conference*, pages 125–130, 2012.

T. Hamel, R. Mahony, R. Lozano, and J. Ostrowski. Dynamic Modelling and configuration stabilization for an X4-Flyer. In *Proc. 15th Triennial IFAC World Congress*, pages 217–222, 2002.

D. Lee, H. Jin Kim, and S. S. Sastry. Feedback linearization vs. adaptive sliding mode control for a quadrotor helicopter. *Int. J. Control Autom. Syst.*, 7:419–428, 2009.

T. Lee, M. Leok, and H. McClamroch. Control of complex maneuvers for a quadrotor UAV using geometric methods on SE(3). 2010.
http://arxiv.org/abs/1003.2005.

Y. Lenoir, Ph. Martin, and P. Rouchon. $2k\pi$, the juggling robot. In *Proc. 37th IEEE Conf. Decision and Control*, volume 2, pages 1995–2000, 1998.

Ph. Martin, P. Rouchon, and J. Rudolph. Invariant tracking. *ESAIM: Control, Optimisation and Calculus of Variations*, 10:1–13, 2004.

R. M. Murray, Z. Li, and S. S. Sastry. *A Mathematical Introduction to Robotic Manipulation.* CRC Press, 1994.

G. Piovan and F. Bullo. On coordinate-free rotation decomposition: Euler angles about arbitrary axes. *IEEE Trans. Rob.*, 28(3):728–733, 2012.

P. Rouchon and J. Rudolph. Invariant tracking and stabilization: problem formulation and examples. In D. Aeyels, F. Lamnabhi-Lagarrigue, and A. van der Schaft, editors, *Stability and Stabilization of Nonlinear Systems*, volume 246 of *Lecture Notes in Control and Inform. Sci.*, pages 261–273. Springer, 1999.

J. Rudolph and R. Fröhlich. Invariant tracking for planar rigid body dynamics. *Proc. Appl. Math. Mech.*, 2:9–12, 2003.

N. Zhang, G. Andrei, A. Drouin, and F. Mora-Camino. Differential flat control for rotorcraft trajectory tracking. In *Proc. Int. MultiConf. of Engineers and Computer Scientists*, 2009.

Appendix A. MINIMUM ANGLE CONDITION FOR P

From the problem statement (7) it is clear that if P is a solution, then all solutions $\tilde{P} = e^{\text{Skw}(\tilde{a}_P)\tilde{\alpha}}$ have the form $\tilde{P} = P \, e^{\text{Skw}(a_\Delta)\beta}$, $\beta \in (-\pi, \pi]$. The angle $\tilde{\alpha}$ of the rotation matrix $\tilde{P}$ is (see e.g. Murray et al. [1994])

$$\tilde{\alpha} = \arccos\left(\tfrac{1}{2}(\text{tr}\,\tilde{P} - 1)\right)$$
$$= \arccos\left(\tfrac{1}{2}(D_z^z - 1 + (D_z^z + 1)\cos\beta)\right).$$

Taking the derivative w.r.t. β,

$$\frac{\partial \tilde{\alpha}}{\partial \beta} = \frac{(1 + D_z^z)\sin\beta}{\sqrt{4 - (D_z^z - 1 + (D_z^z + 1)\cos\beta)}},$$

shows that for any given $D_z^z \neq -1$ the angle $\tilde{\alpha}$ is minimal for $\beta = 2k\pi, k \in \mathbb{Z}$. As $e^{\text{Skw}(a_\Delta)2k\pi} = I \; \forall \; k \in \mathbb{Z}$ the rotation matrix P is indeed the one with the minimum rotation angle $\alpha = \arccos D_z^z$.