On Multi Time-Scale Form of Nonlinear Systems

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Abstract: In general, controller is designed with respect to one time scale. Time-state control form divides a system into virtual time control part and state control part, and the system is linearized based on two time scales; real and virtual time scales. This paper claims that handling of time scales for a control system should be more flexible, and introduces a multi time-scale transformation as a generalization of time scale transformation. As an example, a mechanical system is linearized under two virtual time scales.

Keywords: Linearization, Nonlinear system, Time scale transformation, Geometric approach.

1. INTRODUCTION

The use of state-dependent time scaling transformations was first considered by Sampei and Furuta (1986). This technique was applied to some methods, such as linearizing nonlinear systems (Repondek (1998); Guay (1999); Saito et al. (2010); Sekiguchi and Sampei (2013)), optimal control problem for switched system (Loxton et al. (2009); Yu et al. (2012)), and control strategies using time-state control form (Sampei (1994); Satoko (2011)), etc. When we apply a time scale transformation to a practical control system, we can also adopt a state function as a virtual time scale. In this case, it should be taken into account that a virtual time scale must be well-defined. The control method via time-state control form focuses in this point. This method separates a system into a time scale control part and state control part with respect to a new time scale. The main idea of time-state control form is to guarantee the monotonically increase of virtual time scale using time scale control part. In the other literature, researchers focus on adopting one virtual time scale.

The purpose of this paper is to present a new technique using a time scale transformation. As shown in a control strategy using a time-state control form, there is no restriction to use only one time scale. Moreover, we consider that there is no need for separating a system into time and state control parts, that is, we can separate a system into some parts, and time scale transformations can be applied to each subsystem separately. In the analytical point of view, we focused on the specific form that is decoupled in the sense of state and input, which is called multi time-scale form. In this paper, we define the decoupling matrix on multiple time scales, and confirm that the nonsingularity of this decoupling matrix is sufficient condition for partial feedback linearization with multiple time scales. As an application, we linearize a pendulum system with two inputs via multi time-scale form.

2. PRELIMINARIES

In this paper, we consider a control-affine multi-input nonlinear control system of the form,

$$\begin{aligned}\dot{x} := \frac{\mathrm{d}x}{\mathrm{d}t} &= f(x) + \sum_{i=1}^{m} g_i(x)u_i \\ &= f(x) + G(x)u, \end{aligned} \tag{1}$$

where $x \in M \subset \mathbb{R}^n$, $u \in \mathbb{R}^m$, and f and $g_1, \ldots, g_m$ are C^∞ vector fields on M, $u = [u_1, \ldots, u_m]^T$, and we assumed that $G(x) = [g_1(x), \ldots, g_m(x)]$ has rank m for all x in M. Hereinafter, we sometimes omit the coordinate representation (x) for simplicity. Using vector fields f and $g_1, \ldots, g_m$, we define following distribution :

$$\begin{aligned}\mathscr{G}_{i+1} &= \mathrm{span}\{\mathrm{ad}_f^i \mathscr{G}_1\} + \bar{\mathscr{G}}_i \quad (i = 1, 2, \ldots), \\ \mathscr{G}_1 &= \mathrm{span}\{g_1, \ldots, g_m\}\end{aligned} \tag{2}$$

where $\mathrm{ad}_f^i \mathscr{G}_1$ is set of vector fields $\mathrm{ad}_f^i g$ for $g \in \mathscr{G}_1$, and $\mathrm{ad}_f^i g = [f, \mathrm{ad}_f^{i-1} g_j]$ and $[f, g]$ is a Lie bracket of f and g. In this paper, we refer to these distributions $\mathscr{G}_i$ as system distributions, and $\bar{\mathscr{G}}_i = \mathrm{inv}\mathscr{G}_i$ denotes involutive distribution, and we assume that all distributions considered here have constant dimension.

The vector field f_1 is said to be congruent to the vector field f_2 modulo distribution $\mathscr{G}$ if there exists $g \in \mathscr{G}$ such that $f_1 = f_2 + g$, and this congruence is denoted by $f_1 \equiv f_2 \mod \mathscr{G}$. If $f_1 \in \mathscr{G}$, then we also say $f_1 \equiv 0 \mod \mathscr{G}$.

In this paper, we adopt a definition of relative degree of a scalar function as follows:

Definition 1. (Relative degree). A smooth scalar function $h(x)$ is said to have relative degree r with respect to (1) if and only if $h(x)$ satisfies following conditions:

$$\begin{aligned}\mathcal{L}_{\mathscr{G}_1} h = \cdots = \mathcal{L}_{\mathscr{G}_{r-1}} h &= 0, \\ \mathcal{L}_{\mathscr{G}_r} h &\neq 0,\end{aligned}$$

where $\mathcal{L}_{\mathscr{G}} h = 0$ represents $\mathcal{L}_g h = 0$ for any $g \in \mathscr{G}$.

Note that this definition of relative degree is the same with a traditional relative degree for outputs.

2.1 Partial Feedback Linearization

Partial feedback linaerization is a method to transform a system into a partially linearized system via coordinate and input transformations, as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\xi\\ \eta\end{bmatrix} = \begin{bmatrix}A\xi\\ f_\eta(\xi,\eta)\end{bmatrix} + \begin{bmatrix}B\\ G_\eta(\xi,\eta)\end{bmatrix} v,$$

where $\xi = [\xi_1^T, \ldots, \xi_m^T]^T$, $\xi_i = [\xi_{i1}, \ldots, \xi_{ir_i}]^T$, and r_i is a relative degree of ξ_{i1}. Hence the dimension of linear subsystem is the sum of relative degree r_i. The necessary and sufficient condition for linearizing a system partially with $h = \{h_1, \ldots, h_m\}$ is that the functions h have vector relative degree defined below.

Definition 2. (Vector relative degree). Functions $\{h_1, \ldots, h_m\}$ are said to have vector relative degree $[r_1, \ldots, r_m]$ with respect to a system (1) if r_i is a relative degree of h_i with respect to (1) and the following matrix is nonsingular:

$$\begin{bmatrix} \mathcal{L}_{\mathrm{ad}_f^{r_1-1} g_1} h_1 & \cdots & \mathcal{L}_{\mathrm{ad}_f^{r_1-1} g_m} h_1 \\ \vdots & \ddots & \vdots \\ \mathcal{L}_{\mathrm{ad}_f^{r_m-1} g_1} h_m & \cdots & \mathcal{L}_{\mathrm{ad}_f^{r_m-1} g_m} h_m \end{bmatrix}. \tag{3}$$

This matrix is called decoupling matrix.

2.2 Time Scale Transformation

Consider a virtual time scale τ satisfying

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = s(x), \qquad \tau(t_0) = \tau_0, \tag{4}$$

where $s(x)$ is called a time scaling function and satisfies the following condition

$$0 < s(x) < \infty. \tag{5}$$

With the new time scale, the system (1) is represented as

$$\begin{aligned} \breve{x} :=& \frac{\mathrm{d}x}{\mathrm{d}\tau} = s(x)f(x) + \sum_{i=1}^{m} g_i(x)\tilde{u}_i \\ =& s(x)f(x) + G(x)\tilde{u}, \end{aligned} \tag{6}$$

where $\tilde{u} = [\tilde{u}_1, \ldots, \tilde{u}_m]$, and $\tilde{u}_i = s(x)u_i$. Let system distribution with respect to new time scale be given as

$$\mathscr{G}_{i+1}^s = \mathrm{span}\left\{\mathrm{ad}_{sf}^i \mathscr{G}_1\right\} + \bar{\mathscr{G}}_i^s, \quad (i = 1, 2, \ldots), \tag{7}$$

where $\mathscr{G}_1^s = \mathscr{G}_1$. To make following discussion clear, we adopt the notation $\mathrm{rd}_\tau(h)$ to describe a relative degree of h with respect to time scale τ.

In the rest of the paper, it is assumed that the virtual time scale is well defined, that is, the derivative of a virtual time scale with respect to a real time scale is positive and bounded. Moreover, we assume the time scaling function is a class of C^∞. The latter assumption is required to retain the smoothness of vector fields, and it is not essential for the time scale transformation.

At the end of this section, we define the relative degree with respect to a time scale τ.

Definition 3. (Relative degree w.r.t τ). A smooth function $h(x)$ is said to have relative degree r with respect to (1) and a virtual time scale τ if and only if $h(x)$ satisfies following conditions on M:

$$\begin{aligned} \mathcal{L}_{\mathscr{G}_1^s} h = \cdots = \mathcal{L}_{\mathscr{G}_{r-1}^s} h = 0, \\ \mathcal{L}_{\mathscr{G}_r^s} h \neq 0. \end{aligned}$$

3. KEY IDEA

Consider coordinate and input transformations:

$$\begin{aligned} \xi &= \Phi(x), \\ u &= \alpha(x) + \beta(x)v, \end{aligned}$$

where ξ is new state, v is new input, and $\beta(x)$ is nonsingular matrix. Consider multiple time scales satisfying

$$\frac{\mathrm{d}t}{\mathrm{d}\tau_i} = s_i(x), \qquad (i = 1, \ldots, k).$$

Applying these transformations, system (1) becomes

$$\begin{aligned} \frac{\mathrm{d}\xi_1}{\mathrm{d}\tau_1} &= \tilde{f}_1(\xi) + \tilde{G}_1(\xi)v, \\ &\vdots \\ \frac{\mathrm{d}\xi_k}{\mathrm{d}\tau_k} &= \tilde{f}_k(\xi) + \tilde{G}_k(\xi)v, \end{aligned}$$

where $\xi_i = [\xi_{i1}, \ldots, \xi_{ir_i}]^T$, $\xi = [\xi_1^T, \ldots, \xi_k^T]^T$, $\sum_{i=1}^k r_i = n$, and $\tilde{f}_i, \tilde{G}_i$ are projection of vector fields with respect to a time scale τ_i to ξ_i, that is,

$$\begin{aligned} \tilde{f}_i(\xi) &= \mathrm{Pr}_i\left(s_i f \circ \Phi^{-1}(\xi)\right) \in \mathbb{R}^{r_i}, \\ \tilde{G}_i(\xi) &= [\tilde{g}_{i1}(\xi), \ldots, \tilde{g}_{im}(\xi)] \in \mathbb{R}^{r_i \times m}, \\ \tilde{g}_{ij}(\xi) &= \mathrm{Pr}_i\left(s_i g_j \circ \Phi^{-1}(\xi)\right) \in \mathbb{R}^{r_i}, \end{aligned}$$

where $\mathrm{Pr}_i(\cdot)$ denotes the projection map to ξ_i. Each subsystem evolves on a different time scale. However these subsystems have interactions via ξ and feedback input v, and the interactions make it difficult to analyze a system.

Hence, we focus on the partially decoupled form:

$$\begin{aligned} \frac{\mathrm{d}\xi_1}{\mathrm{d}\tau_1} &= \tilde{f}_1(\xi_1) + \tilde{g}_1(\xi_1)v_1, \\ &\vdots \\ \frac{\mathrm{d}\xi_m}{\mathrm{d}\tau_m} &= \tilde{f}_m(\xi_m) + \tilde{g}_m(\xi_m)v_m, \\ \frac{\mathrm{d}\xi_{m+1}}{\mathrm{d}\tau_{m+1}} &= \tilde{f}_{m+1}(\xi) + \tilde{g}_{m+1}(\xi)v, \end{aligned} \tag{8}$$

and this form is called multi time-scale form.

One of the easiest ways to transform a system into multi time-scale form is partial feedback linearization. Next we define the vector relative degree with multiple time scales to discuss a partial feedback linearization for (8).

Definition 4. Functions $h = \{h_1, \ldots, h_m\}$ are said to have **time scaled vector relative degree** $[r_1, \ldots, r_m]$ with respect to a system (1) and virtual time scales $\tau = \{\tau_1, \ldots, \tau_m\}$ if r_i is a relative degree of h_i with respect to a system (1) and virtual time scale τ_i, and the following matrix is nonsingular:

$$D_\tau := \begin{bmatrix} \mathcal{L}_{\mathrm{ad}_{s_1 f}^{r_1-1} s_1 g_1} h_1 & \cdots & \mathcal{L}_{\mathrm{ad}_{s_1 f}^{r_1-1} s_1 g_m} h_1 \\ \vdots & \ddots & \vdots \\ \mathcal{L}_{\mathrm{ad}_{s_m f}^{r_m-1} s_m g_1} h_m & \cdots & \mathcal{L}_{\mathrm{ad}_{s_m f}^{r_m-1} s_m g_m} h_m \end{bmatrix}. \tag{9}$$

D_τ is referred as **time scaled decoupling matrix**.

Lemma 1. D_τ is equal to the following matrix

$$\begin{bmatrix} \mathcal{L}_{s_1 g_1}\mathcal{L}_{s_1 f}^{r_1-1} h_1 & \cdots & \mathcal{L}_{s_1 g_m}\mathcal{L}_{s_1 f}^{r_1-1} h_1 \\ \vdots & \ddots & \vdots \\ \mathcal{L}_{s_m g_1}\mathcal{L}_{s_m f}^{r_m-1} h_m & \cdots & \mathcal{L}_{s_m g_m}\mathcal{L}_{s_m f}^{r_m-1} h_m \end{bmatrix}. \tag{10}$$

Proof. Let r be the relative degree of h with time scaling function s. To prove the lemma, it is enough to show the equation:

$$\mathcal{L}_{sg_i}\mathcal{L}_{sf}^{r-1} h = \mathcal{L}_{\mathrm{ad}_{sf}^{r-1} sg_i} h, \tag{11}$$

for $i \in \{1,\dots,m\}$, and this equation is trivial because it is closed in one virtual time scale. $\triangleleft$

If functions $h = \{h_1,\dots,h_m\}$ have time scaled vector relative degree $[r_1,\dots,r_m]$ with respect to $\tau = \{\tau_1,\dots,\tau_m\}$, then there exist a coordinate transformation

$$\xi = \left[\, \xi_1^T, \xi_2^T, \dots, \xi_m^T, \xi_{m+1}^T \,\right]^T,$$

$$\xi_i = \begin{bmatrix} h_i \\ \frac{\mathrm{d}h_i}{\mathrm{d}\tau_i} \\ \vdots \\ \frac{\mathrm{d}^{r_i-1}h_i}{\mathrm{d}\tau_i^{r_i-1}} \end{bmatrix} \quad \text{for } i = 1,\dots,m,$$

where ξ_{m+1} is chosen to make Jacobian be nonsingular, and an input transformation

$$u = \alpha(x) + (D_\tau)^{-1} v,$$

$$\alpha(x) = \begin{bmatrix} s_1 \mathcal{L}_f \frac{\mathrm{d}^{r_1-1}h_1}{\mathrm{d}\tau_1^{r_1-1}} \\ s_2 \mathcal{L}_f \frac{\mathrm{d}^{r_2-1}h_2}{\mathrm{d}\tau_2^{r_2-1}} \\ \vdots \\ s_m \mathcal{L}_f \frac{\mathrm{d}^{r_m-1}h_m}{\mathrm{d}\tau_m^{r_m-1}} \end{bmatrix},$$

that transform a system (1) into following patially linear multi time-scale form:

$$\begin{aligned} \frac{\mathrm{d}\xi_1}{\mathrm{d}\tau_1} &= A_{r_1}\xi_1 + b_{r_1}v_1, \\ &\vdots \\ \frac{\mathrm{d}\xi_m}{\mathrm{d}\tau_m} &= A_{r_m}\xi_m + b_{r_m}v_m, \\ \frac{\mathrm{d}\xi_{m+1}}{\mathrm{d}\tau_{m+1}} &= \tilde{f}_{m+1}(\xi) + \tilde{G}_{m+1}(\xi)v, \end{aligned}$$

where we applied Lemma 1, and (A_i, b_i) is controllable canonical form for i-th dimensional system.

Each linear subsystem is decoupled, and the multi time-scale form is realized. Zero dynamics is represented as follows:

$$\frac{\mathrm{d}\xi_{m+1}}{\mathrm{d}\tau_{m+1}} = \tilde{f}_{m+1}(0,\dots,0,\xi_{m+1}).$$

By summarizing up to here, we get the following theorem.

Theorem 2. If functions $\{h_1,\dots,h_m\}$ have time scaled vector relative degree $[r_1,\dots,r_m]$ with respect to virtual time scales $\{\tau_1,\dots,\tau_m\}$, then there exists coordinate and input transfromations that transform a system (1) into following multi time-scale form:

$$\begin{aligned} \frac{\mathrm{d}\xi_1}{\mathrm{d}\tau_1} &= A_{r_1}\xi_1 + b_{r_1}v_1, \\ &\vdots \\ \frac{\mathrm{d}\xi_m}{\mathrm{d}\tau_m} &= A_{r_m}\xi_m + b_{r_m}v_m, \\ \frac{\mathrm{d}\xi_{m+1}}{\mathrm{d}\tau_{m+1}} &= \tilde{f}_{m+1}(\xi) + \tilde{G}_{m+1}(\xi)v, \end{aligned}$$

where $\xi_i = \left[h_i, \dots, \frac{\mathrm{d}^{r_i-1}h_i}{\mathrm{d}\tau_i^{r_i-1}} \right]^T$ for $i = 1,\dots,m$.

Remark 1. Dimension of the largest linearizable subsystem is one of the great interests. Let d_t, d_{τ_0} and d_τ denote the dimension of the largest linearizable subsystem via only feedback transformation, single time-scale transformation, and multi time-scale transformation respectively, then these dimensions have following relationship:

$$d_t \le d_{\tau_0} \le d_\tau.$$

This relation is clear because single time scale transformation is a special case of multi time-scale transformation. However, the calculation of d_τ remains an open problem.

There is a system that can be linearized by multi time-scale transformation even if the system cannot be linearizable via single time-scale transformation, and we show an example in the next section.

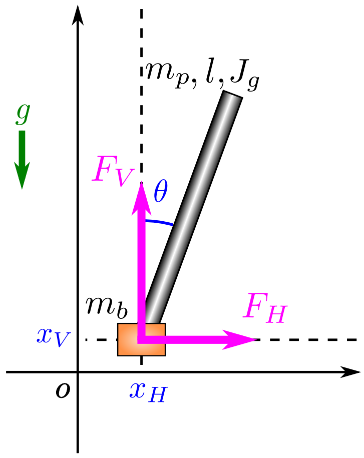

Fig. 1. Simplified model of the inverted pendulum with horizontal and vertical movement

4. MULTI TIME-SCALE LINEARIZATION : PENDULUM SYSTEM WITH VERTICAL AND HORIZONTAL FREEDOM

Consider a pendulum system mounted on a base that moves in the vertical plane with two inputs. We consider only the pendulum system with two inputs shown in Fig. 1, and it is known that this system is linearizable using dynamic feedback transformation, see Sekiguchi and Sampei (2010). In this paper, we linearize this system without dynamics extension. First derive the model of the inverted pendulum with horizontal and vertical movement. Notations of generalized coordinate and inputs in Fig. 1 are defined in Table 1. The definitions of physical parameters are as follows. Let the mass of the pendulum and base be m_p and m_b respectively, and let J_g denote the moment of inertia with respect to the center of gravity (COG) of the pendulum. Furthermore, let l be the distance from the pivot to the center of gravity, and g denotes the gravity acceleration. It is assumed that friction is very small and can be neglected. By using these notations, equations of motion are

$$M(q)\ddot{q} + C(q,\dot{q}) + G(q) = \left[\, F_H \;\; F_V \;\; 0 \,\right]^T \tag{12}$$

where

Table 1. Notation of variables

x_H	horizontal position of the base
x_V	vertical position of the base
θ	angle between the vertical line and the pendulum
F_H	horizontal force input
F_V	vertical force input

$$q = [x_H \ x_V \ \theta]^T,$$

$$M(q) = \begin{bmatrix} m_b + m_p & 0 & m_p l \cos\theta \\ 0 & m_b + m_p & -m_p l \sin\theta \\ m_p l \cos\theta & -m_p l \sin\theta & m_p l^2 + J_g \end{bmatrix},$$

$$C(q, \dot{q}) = \begin{bmatrix} -m_p l \dot{\theta}^2 \sin\theta \\ -m_p l \dot{\theta}^2 \cos\theta \\ 0 \end{bmatrix}, \quad G(q) = \begin{bmatrix} 0 \\ (m_b + m_p) g \\ -m_p g l \sin\theta \end{bmatrix}.$$

Then the state space realization of the system is

$$\dot{x}_q = f_q(x_q) + [g_H(x_q), g_V(x_q)]F, \tag{13}$$

$$f_q = \begin{bmatrix} \dot{q} \\ \hline M^{-1}(C+G) \end{bmatrix}, \quad [g_H, g_V] = \begin{bmatrix} 0 \ 0 \\ \hline M^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{bmatrix},$$

$$F = [\, F_H \ F_V \,]^T,$$

where

$$x_q = [\, x_H \ x_V \ \theta \ \dot{x}_H \ \dot{x}_V \ \dot{\theta} \,]^T.$$

In order to calculate the relative degree structure introduced by Sekiguchi et al. (2010), the some distributions are defined using vector fields f, g_H and g_V as follows.

$$\mathscr{G}_1 = \text{span}\{g_H, g_V\},$$

$$\mathscr{G}_2 = \text{span}\left\{\text{ad}_{f_q} g_H, \text{ad}_{f_q} g_V, \bar{\mathscr{G}}_1\right\},$$

where $\bar{\mathscr{G}}_1$ is involutive closure of the distribution $\mathscr{G}_1$. In this system, $\bar{\mathscr{G}}_1$ is equal to $\mathscr{G}_1$, and $\bar{\mathscr{G}}_2$ spans full space. The dimension of these distributions are

$$\dim \mathscr{G}_1 = \dim \bar{\mathscr{G}}_1 = 2,$$

$$\dim \mathscr{G}_2 = 4, \qquad \dim \bar{\mathscr{G}}_2 = 6.$$

Then the relative degree structure of this system is $(2,2,2,2,1,1) - [2,2]$. The relative degree structure is invariant under the feedback transformation, and $(2,2,2,2,1,1) - [2,2]$ indicates that this system is not feedback linearizable and that there exist feedback transformation and coordinate (ξ_1, ξ_2, η) such that the transformed system has two linear subsystem as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \xi_1 \\ \xi_2 \\ \hline \eta \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \xi_1 \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \xi_2 \\ \hline \alpha(\xi_1, \xi_2, \eta) \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1 \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_2 \\ \hline 0 \end{bmatrix}, \tag{14}$$

where u_1, u_2 are new inputs. Moreover, the relative degree structure $(2,2,2,2,1,1) - [2,2]$ means that some functions that are independent from ξ_1, ξ_2 and have relative degree 2 can be selected as the state of nonlinear subsystem. Indeed there is a coordinate transformation that realizes (14) as follows:

$$h = (\dot{x}_H \cos\theta - \dot{x}_V \sin\theta)\rho + \dot{\theta}, \qquad \rho = \frac{lm}{J + l^2 m},$$

$$x = [\, x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6 \,]^T,$$

$$:= [\, x_H \ x_V \ \dot{x}_H \ \dot{x}_V \ h \ \theta \,]^T,$$

where h is selected from the functions with relative degree 2. Using this coordinate, we get the following normal form:

$$\dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2, \tag{15}$$

where

$$f = [\, x_3 \ x_4 \ 0 \ 0 \ f_5(x) \ f_6(x) \,]^T, \tag{16}$$

$$g_1 = [\, 0 \ 0 \ 1 \ 0 \ 0 \ 0 \,]^T, \tag{17}$$

$$g_2 = [\, 0 \ 0 \ 0 \ 1 \ 0 \ 0 \,]^T, \tag{18}$$

$$f_5(x) = \begin{pmatrix} \left(\rho^2(x_3^2 - x_4^2)\cos x_6 - \rho x_3 x_5 + g\rho\right)\sin x_6 \\ -2\rho^2 x_3 x_4 \sin^2 x_6 - \rho x_4 x_5 \cos x_6 + \rho^2 x_3 x_4 \end{pmatrix},$$

$$f_6(x) = \rho x_4 \sin x_6 - \rho x_3 \cos x_6 + x_5,$$

and input transformation is derived as follows:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -\delta^{-1}\gamma + \delta^{-1}\begin{bmatrix} F_H \\ F_V \end{bmatrix},$$

$$\gamma = \begin{bmatrix} l m_p \sin x_6 (\dot{x}_6^2 - \rho \cos x_6) \\ g m_b + m_p(g - l\dot{x}_6^2 \cos x_6) - m_p l g \rho \sin^2 x_6 \end{bmatrix},$$

$$\delta = \begin{bmatrix} m_b + m_p\rho - m_p l\rho\cos^2 x_6 & m_p l\rho \sin x_6 \cos x_6 \\ m_p l\rho \sin x_6 \cos x_6 & m_b + m_p\rho - m_p l\rho \sin^2 x_6 \end{bmatrix}.$$

To linearize the system via a multi time-scale form, let us consider the following time scaling function:

$$s = \frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{h_2}{\frac{\mathrm{d}h_1}{\mathrm{d}t}}, \tag{19}$$

where h_1 and h_2 are scalar function, and their relative degrees are r with respect to real time scale t. The relative degree of h_1 with respect to a new time scale τ is calculated as follows:

$$\frac{\mathrm{d}h_1}{\mathrm{d}\tau} = \frac{\mathrm{d}h_1}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = h_2.$$

Hence, the relative degree has a relationship

$$\text{rd}_\tau(h_1) = \text{rd}_\tau(h_2) + 1.$$

Moreover, next lemma says that $\text{rd}_\tau(h_2) \geq r$ because $\text{rd}_t(s) = r - 1$.

Lemma 3. If $\text{rd}_t(s)$ is r, then $\mathscr{G}_i^s = \mathscr{G}_i$ for $i = 1, \ldots, r$.

Proof of this lemma is in Appendix.

Therefore, the relative degree of h_1 with respect to a new time scale is larger than the original one. This type of time scaling function has been applied to two-link robot known as Acrobot by Saito et al. (2010), and the stabilization was achieved via linearization with time scale transformation.

This time-scale design method is applied to the target pendulum system. The pendulum system has 4 independent functions whose relative degrees are 2. Choose x_1 and x_5, and define the time scaling function

$$s_1 = \frac{\mathrm{d}t}{\mathrm{d}\tau_1} = \frac{x_1}{\frac{\mathrm{d}x_5}{\mathrm{d}t}}.$$

After applying this time scale transformation, x_5 has relative degree 3. The relative degree structure of transformed system is calculated as $(3,2,2,2,1,1) - [3,2]$, and hence the time scaled system is still not feedback linearizable. The relative degree of x becomes $\{2,1,2,1,3,2\}$ on the new time scale τ_1. Therefore, we can choose functions x_5, x_1, and $s_1\dot{x}_3$ as a coordinate of 3-dimensional linear subsystem.

In the same manner, we get another set of functions $\{x_6, x_2, s_2 x_4\}$ as a coordinate of linear subsystem with respect to a virtual time scale:

$$s_2 = \frac{\mathrm{d}t}{\mathrm{d}\tau_2} = \frac{x_2}{\frac{\mathrm{d}x_6}{\mathrm{d}t}}.$$

$$D_\tau = \begin{pmatrix} -\frac{x_1^2(\sin(x_5)x_4+x_6)}{\rho^2(-\cos(x_5)x_3+\sin(x_5)x_4+x_6)^3} & \frac{\sin(x_5)x_1^2x_3}{\rho^2(-\cos(x_5)x_3+\sin(x_5)x_4+x_6)^3} \\ \frac{x_2^2x_4(\sin(2x_5)x_3+\cos(2x_5)x_4-\sin(x_5)x_6)}{\rho^2\left(\cos(2x_5)x_3x_4+\frac{\sin(x_5)(g-\rho x_3x_6)}{\rho}-\cos(x_5)\left(\sin(x_5)\left(x_4^2-x_3^2\right)+x_4x_6\right)\right)^3} & \frac{\sin(x_5)x_2^2\left(\frac{g}{\rho}+\cos(x_5)\left(x_3^2+x_4^2\right)-x_3x_6\right)}{\rho^2\left(-\cos(2x_5)x_3x_4+\sin(x_5)\left(x_3x_6-\frac{g}{\rho}\right)+\cos(x_5)\left(\sin(x_5)\left(x_4^2-x_3^2\right)+x_4x_6\right)\right)^3} \end{pmatrix} \tag{20}$$

The relative degree of x with respect to the new time scale τ_2 is $\{2, 1, 2, 1, 2, 3\}$, and it is also not feedback linearizable.

Next, we separate the system into two subsystems defined by

$$\xi_1 = [x_5, x_1, s_1 x_3]^T,$$
$$\xi_2 = [x_6, x_2, s_2 x_4]^T,$$

where x_5 and x_6 have relative degree 3 with respect to τ_1 and τ_2 respectively.

In order to linearize the system in the sense of multiple time scales, the functions $\{x_5, x_6\}$ must have time scaled vector relative degree [3,3] with respect to virtual time scales $\tau = [\tau_1, \tau_2]$, that is, the time scaled decoupling matrix D_τ defined in (9) must be nonsingular. D_τ is described in (20). Therefore, the system is transformed into multi time-scale form only if the time scaled decoupling matrix (20) is nonsingular. Moreover, the system is linearizable in that case.

5. CONCLUDING REMARKS

In this paper, we proposed the usage of multiple time scales as a tool for analyzing and control a multi-input system. Multi time-scale form was defined as a system representation under the multiple time scales that was decoupled about coordinates and input. Moreover, we also discussed the partial feedback linearization under the multiple time scales. We defined decoupling matrix for multiple time scales and checked that the nonsingularity of decoupling matrix is required to transform a system into multi time-scale partial feedback linear form.

Through a specific example, we presented that the system is linearizable under the multiple time scales even though the system is not orbitally feedback linearizable. However, designed time scales and decoupling matrix have a lot of singular points, for example, D_τ is singular when $x_3 = x_4 = x_6 = 0$. Hence, the presented linear form cannot be applied to stabilize the system.

For multi-input systems, the orbital feedback linearizability with one virtual time scale was solved in Guay (2001). However, the conditions for orbital feedback linearization with multiple time scales remain to be solved. Moreover, in order to apply multi time-scale transformation to practical systems, more investigations are required to establish checkable linearizability conditions and design method of virtual time scales as well as linearization problem via a single time scale transformation.

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APPENDIX

First, to prove Lemma 3, we prepare a following lemma.

Lemma 4. Lie brackets of time scaled system (6) with time scaling function s can be written as

$$\mathrm{ad}^i_{sf} g_k = \sum_{j=0}^{i} \xi^{(i,j)} \mathrm{ad}^j_f g_k + \xi^i_{(k,f)} f, \tag{21}$$

where functions $\xi^{(i,j)}$ and $\xi^i_{(k,f)}$ are determined by x, s and Lie derivatives of s as follows:

$$\begin{cases} \xi^{(i+1,0)} = s\mathcal{L}_f \xi^{(i,0)} \\ \xi^{(i+1,j)} = s\xi^{(i,j-1)} + s\mathcal{L}_f \xi^{(i,j)} \\ \xi^{(i+1,i+1)} = s\xi^{(i,i)} = s^{i+1} \\ \xi^{i+1}_{(k,f)} = s\mathcal{L}_f \xi^i_{(k,f)} - \xi^i_{(k,f)}\mathcal{L}_f s - \sum_{j=0}^{i} \xi^{(i,j)} \mathcal{L}_{\mathrm{ad}_f^j g_k} s \end{cases} \tag{22}$$

where $1 \leq j \leq i$, and $\xi^0_{(k,f)} = 0$, and $\xi^{(0,0)} = 1$.

Proof. The proof is the same with Lemma 7 in Sampei and Furuta (1986). ◁

Now, we prove Lemma 3

Proof of Lemma 3 $\mathrm{rd}_t(s) = r$ means that

$$\mathcal{L}_{\mathscr{G}} s = \mathcal{L}_{\mathrm{ad}_f \mathscr{G}} s = \cdots = \mathcal{L}_{\mathrm{ad}_f^{r-2} \mathscr{G}} s = 0 \quad \text{for all } x \in U. \tag{23}$$

By substituting (23) into (22), we get

$$\xi^i_{(k,f)} = 0 \qquad \text{for } i = 0, \ldots, r-1 \text{ and} \tag{24}$$

$$\xi^r_{(k,f)} = -\sum_{j=0}^{r-1} \xi^{(r-1,j)} \mathcal{L}_{\mathrm{ad}_f^j g_k} s = -s^{r-1}\left(\mathcal{L}_{\mathrm{ad}_f^{r-1} g_k} s\right). \tag{25}$$

Moreover,

$$\mathrm{ad}^i_{sf} g_k = \sum_{j=0}^{i} \xi^{(i,j)} \mathrm{ad}^j_f g_k, \qquad \text{for } i = 0, \ldots, r-1, \tag{26}$$

where $\xi^{(i,i)} = s^i$ is not zero. Therefore, $\mathscr{G}^s_i = \mathscr{G}_i$ for $i = 1, \ldots, r$. ◁

