Robust Design of Internal Models by Nonlinear Regression

Francesco Forte * **Alberto Isidori** ** **Lorenzo Marconi** *

* *CASY - DEI, University of Bologna, Italy*
** *"Sapienza" Università di Roma, Italy*

Abstract: This paper focuses on the design of robust internal model-based regulators for a class of nonlinear systems and exosystems such that the desired steady state control law fulfills a nonlinear regression formula affine in possible uncertainties. The design methodology builds upon the ideas presented in [9] in the context of linear adaptive output regulation. The design methodology relies upon an assumption of left invertibility of a matrix. Through numerical analysis we show how the design methodology succeeds in relevant cases, such as the cases in which the desired steady state is generated by uncertain Van der Pol, Duffing, and Lorentz oscillators.

1. INTRODUCTION

The paper deals with the problem of output regulation for a class of nonlinear systems under the assumption that the desired steady state input needed to enforce a regulation error that is identically zero fulfills a regression formula parametrized by possible uncertainties in a linear way. The design procedure builds upon the general ideas proposed in [9] in which the theory of nonlinear high-gain observers, proved to be effective in [2] within the context of nonlinear output regulation, is used in order to design a nonlinear adaptive internal model which does not rely upon an "explicit" adaptation law. The goal of [9] was to enrich the design strategies for adaptive output regulation by presenting a tool alternative to the ones already available in literature (see [11], [3] and [4] among others) more inspired by standard adaptive design strategies and relying upon an "explicit" estimation of the uncertainties. The main assumption of [9] is that the desired steady state input able to enforce an identically zero regulation error fulfills a *linear* regression law. This assumption practically limited the applicability of the tool to the case in which the desired steady state input is generated by an uncertain linear oscillator, as originally proposed in [12]. In this paper we make a step further by showing how nonlinear regression laws can be successfully dealt with, by thus enlarging the class of exosystem that can be handled. In order to prove the effectiveness of the design strategy, the particular cases in which the desired steady state input is generated by the Van der Pol, the Duffing, and the Lorentz *uncertain* nonlinear oscillators are specifically addressed. The presented methodology strongly relies on high-gain tools and on the design of high-gain observers (see [5]). It thus expected that the method presents intrinsic limitations in applicative fields where the dimension of the exosystem, and thus of the internal model, is large and the measured variables are affected by high-frequency noise.

The paper frames in the already rich literature on nonlinear output regulation pioneered with the work [8] and still a field of active research nowadays (see [6] for general presentation of the problem and of the techniques, and [7] for applicative scenarios).

2. PROBLEM DESCRIPTION AND PRELIMINARIES

We consider the class of nonlinear systems described by the following normal form

$$\begin{aligned} \dot{z} &= f(w,z,e) \\ \dot{e} &= q(w,z,e) + b(w,z,e)u \end{aligned} \tag{1}$$

with state $(w,z,e) \in \mathbb{R}^s \times \mathbb{R}^n \times \mathbb{R}$, control input $u \in \mathbb{R}$ and with $f(w,z,e)$, $q(w,z,e)$ and $b(w,z,e)$ that are smooth functions of their arguments. The function $b(w,z,e)$, referred to as high-frequency gain, is assumed to be bounded from below by a positive number $\underline{b}$, namely $b(w,z,e) \geq \underline{b}$ for all $(w,z,e) \in \mathbb{R}^s \times \mathbb{R}^n \times \mathbb{R}$. As usual in output regulation, the exogenous variable w is generated as solution of an exosystem having the form

$$\dot{w} = s(w)\,. \tag{2}$$

The initial conditions $(w(0), z(0), e(0))$ of the system range in a known compact set $W \times Z \times E$ of $\mathbb{R}^s \times \mathbb{R}^n \times \mathbb{R}$, with W assumed to be invariant for ((2)).

Note that (1) has *relative degree* 1 between the input u and the output e. All the results presented in the paper can be extended to the case of higher relative degree by mean of standard arguments that are omitted.

The problem of nonlinear output regulation amounts to design a output feedback smooth controller of the form

$$\begin{aligned} \dot{\xi} &= \varphi(\xi, e) \\ u &= \psi(\xi, e) \end{aligned} \tag{3}$$

with state $\xi \in \mathbb{R}^d$ and a compact set $\Xi \subset \mathbb{R}^d$, such that the closed loop trajectories originating from $W \times Z \times E \times \Xi$ are bounded and the error $e(t)$ asymptotically decays to 0 uniformly in the initial conditions.

By following [1], we assume the existence of a function $\pi : W \to \mathbb{R}^n$ solution of the regulator equation

$$L_{s(w)}\pi(w) = f(w, \pi(w), 0) \quad \forall w \in W. \tag{4}$$

Furthermore, we let $u^\star : W \to \mathbb{R}$ the function

$$u^\star(w) = -\frac{q(w, \pi(w), 0)}{b(w, \pi(w), 0)} \tag{5}$$

that represents the control action capable to render the set

$$\mathcal{M} = \{(w, z, e) | w \in W, z = \pi(w), e = 0\} \tag{6}$$

invariant for (1)-(2).

The design methodology presented in paper relies upon high-gain arguments to make the set $\mathcal{M}$ semiglobally attractive. For this reason the following minimum-phase assumption will be used.

Assumption 1 (minimum-phase): The set

$$\text{graph}(\pi) = \{(w, z) \in W \times \mathbb{R}^n : z = \pi(w)\} \tag{7}$$

is locally asymptotically stable for the system

$$\dot{w} = s(w) \qquad \dot{z} = f(w, z, 0) \tag{8}$$

with a domain of attraction of the form $W \times \mathcal{D}$ where $\mathcal{D}$ is an open set satisfying $\mathcal{D} \supset Z$. ◃

The structure of the regulator proposed in [10] and used in this paper is a system of the form

$$\begin{aligned} \dot{\xi} &= F(\xi) + Gu \qquad \xi \in \mathbb{R}^d \\ u &= \gamma(\xi) + v \\ v &= -\kappa(e) \end{aligned} \tag{9}$$

in which $F : \mathbb{R}^d \to \mathbb{R}^d$, $\gamma : \mathbb{R}^d \to \mathbb{R}$, $\kappa : \mathbb{R} \to \mathbb{R}$ are continuous function and $G \in \mathbb{R}^d$ is a vector to be chosen as indicated in the next proposition (see [10]).

Proposition 1. Assume that the triplet $(F(\cdot), \gamma(\cdot), G)$ can be designed so that for some smooth function $\tau : W \to \mathbb{R}^d$ the following equations are fulfilled

$$\begin{aligned} L_{s(w)}\tau(w) &= F(\tau(w)) + G\gamma(\tau(w)) \\ u^\star(w) &= \gamma(\tau(w)) \end{aligned} \tag{10}$$

and the set

$$\text{graph}(\tau) = \{(w, \xi) \in W \times \mathbb{R}^d : \xi = \tau(w)\} \tag{11}$$

is locally asymptotically stable for

$$\dot{w} = s(w) \qquad \dot{\xi} = F(\xi) + Gu^\star(w) \tag{12}$$

with domain of attraction of the form $W \times \mathcal{H}$ with $\mathcal{H}$ an open set satisfying $\mathcal{H} \supset \Xi$. Then there exists a continuous function $\kappa : \mathbb{R} \to \mathbb{R}$ such that the regulator (9) solves the problem of output regulation.

A possible way of designing the triplet $(F(\cdot), \gamma(\cdot), G)$ fulfilling the properties indicated in the previous proposition has been proposed in [2]. The method relies upon the existence of an integer d and a locally Lipschitz function $\phi : \mathbb{R}^d \to \mathbb{R}$ such the following regression relation involving the function $u^\star(w)$ is fulfilled

$$L^d_{s(w)}u^\star(w) = \phi(u^\star(w), \ldots, L^{d-1}_{s(w)}u^\star(w)) \quad \forall w \in W. \tag{13}$$

Let

$$\tau(w) = \begin{bmatrix} \tau_0(w) \\ \vdots \\ \tau_{d-1}(w) \end{bmatrix} := \begin{bmatrix} u^\star(w) \\ \vdots \\ L^{d-1}_{s(w)}u^\star(w) \end{bmatrix} \tag{14}$$

and let $\phi_s : \mathbb{R}^d \to \mathbb{R}$ be a locally Lispchitz and bounded function that agrees with $\phi(\cdot)$ on $\tau(W)$. Then, it is possible to show that if the triplet $(F(\cdot), \gamma(\cdot), G)$ is chosen so that

$$F(\xi) = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_{d-1} \\ \phi_s(\xi_0, \ldots, \xi_{d-1}) \end{bmatrix} - G\xi_0 \tag{15}$$

$\gamma(\xi) = \xi_0$, and

$$G = [g\lambda_0, g^2\lambda_1, \ldots, g^d\lambda_{d-1}]^T \tag{16}$$

where $(\lambda_0, \lambda_1, \ldots, \lambda_{d-1})$ are coefficients of an Hurwitz polynomial and g is a design parameter, then there exists a $g^\star > 0$ such that for all $g \geq g^\star$ the triplet in question fulfills the properties of Proposition 1.

In the next section we present a simple condition under which a function ϕ fulfilling (13) can be found, and thus a regulator of the form (9) can be designed.

3. DESIGN IN CASE OF NONLINEAR REGRESSION LINEARLY PARAMETRIZED IN THE UNCERTAINTIES

The starting point in the design methodology is the existence of a regression formula that governs the k-th time derivative of the desired steady state input $u^\star$. The formula is specified in the next Assumption. For ease of notation, here and in the following we let $u^\star_{[a,b]} := (u^{\star(a)}, \ldots, u^{\star(b)})^T$, with $0 \leq a < b$, the vector of time derivatives of $u^\star$.

Assumption 2. There exist $k > 0$, $p > 0$, locally Lipschitz functions $h : \mathbb{R}^k \to \mathbb{R}$ and $L : \mathbb{R}^k \to \mathbb{R}^p$ such that

$$u^{\star(k)}(w) = h(u^\star_{[0,k-1]}(w)) + L(u^\star_{[0,k-1]}(w))\,\mu \quad \forall\, w \in W. \tag{17}$$

where $\mu \in \mathbb{R}^p$ is a vector of uncertainties. ◃

In the second part of this section we show how the previous assumption is fulfilled in a number of relevant cases.

By differentiating $i \geq 0$ times relation (17) and collecting the resulting equations, we obtain the following set of equations

$$u^\star_{[k,k+i]}(w) = H_i(u^\star_{[0,k+i-1]}(w)) + A_i(u^\star_{[0,k+i-1]}(w))\,\mu \tag{18}$$

where

$$\begin{aligned} A_i(u^\star_{[0,k+i-1]}) &= \text{col}\left[L_0(u^\star_{[0,k-1]}) \cdots L_i(u^\star_{[0,k+i-1]})\right] \\ H_i(u^\star_{[0,k+i-1]}) &= \text{col}\left[h_0(u^\star_{[0,k-1]}) \cdots h_i(u^\star_{[0,k+i-1]})\right] \end{aligned} \tag{19}$$

where $L_0(\cdot) = L(\cdot)$, $h_0(\cdot) = h(\cdot)$, $L_{j+1}(\cdot) = \dot{L}_j(\cdot)$, $h_{j+1}(\cdot) = \dot{h}_j(\cdot)$, $j = 0, \ldots, i-1$, and where for compactness we have omitted the argument w of $u^\star$.

The proposed methodology relies upon the following crucial assumption.

Assumption 3: There exists a $m \geq p$ and $\epsilon > 0$ such that

$$\det(A_m^T(u^\star_{[0,k+m-1]}(w))\, A_m(u^\star_{[0,k+m-1]}(w))) \geq \epsilon$$

for all $w \in W$. ◁

The previous assumption implies that

$$\mathrm{rank}(A_m(u^\star_{[0,k+m-1]}(w))) = p \quad \forall\, w \in W$$

and, in turn, that the uncertain vector μ can be obtained from (18) as a function of $u^\star$ and its first $(k+m)$-th time derivatives. In particular, taking the $(m+1)$-th time derivative of (17) and replacing μ with the estimation obtained by left-inverting (18) for $i = m$, one obtains

$$u^{\star(m+k+1)} = h_{m+1}(u^\star_{[0,k+m]}) + L_{m+1}(u^\star_{[0,k+m]}) \cdot A_m^\dagger(u^\star_{[0,k+m-1]})[u^\star_{[k,k+m]} - h_m(u^\star_{[0,k+m-1]})]$$

where $A_m^\dagger$ represents a pseudoinverse of A_m given by

$$A_m^\dagger(\cdot) = [A_m^T(\cdot)A_m(\cdot)]^{-1}A_m(\cdot)\,.$$

This relation, in turn, is equivalent to (13) for an appropriately defined $\phi(\cdot)$ with $d = m + k + 1$.

In the remaining part of the section we show how the previous assumptions are fulfilled in a number of relevant cases in which $u^\star$ is generated by nonlinear oscillators. The three cases of Van der Pol, Duffing, and Lorentz *uncertain* oscillators are considered and are dealt with in the following subsections.

3.1 Van der Pol Oscillator

As exosystem (2) consider the Van der Pol oscillator described by

$$\begin{aligned} \dot w_1 &= w_2 \\ \dot w_2 &= -\omega^2 w_1 + \epsilon(1 - w_1^2)w_2 \end{aligned} \qquad (20)$$

in which ω and ϵ are uncertain parameters, and consider the case in which the desired steady state input $u^\star(w) = w_1$. the set W is the omega limit set where the steady state trajectories of the Van der Pol take place. It turns out that

$$\ddot u^\star(w) = -u^\star(w)\omega^2 + (1 - u^{\star 2}(w))\dot u^\star(w)\epsilon \qquad (21)$$

and thus Assumption 2 is fulfilled with $\kappa = 2$, $h(\cdot) = 0$, $L(\cdot) = \big(-u^\star(w)\,,\ (1 - u^{\star 2}(w))\dot u^\star(w)\big)$ and $\mu = \big(\omega^2\,,\ \epsilon\big)^T$. We start now to take time derivatives of (21) to identify an $m \geq 2$ for which Assumption 3 is fulfilled. By differentiating once, we obtain

$$u^\star_{[2,3]}(w) = A_1(u_{[0,2]})\mu \qquad (22)$$

where

$$A_1(u^\star_{[0,2]}) = \begin{bmatrix} -u^\star & (1 - u^{\star 2})\dot u^\star \\ -\dot u^\star & \ddot u^\star - 2u^\star \dot u^{\star 2} - u^{\star 2}\ddot u^\star \end{bmatrix}. \qquad (23)$$

It turns out that there are points of W where A_1 is singular (see Fig. 1). By thus taking a further derivative we obtain

$$u^\star_{[2,4]}(w) = A_2(u^\star_{[0,3]}(w))\ \mu \qquad (24)$$

with

$$A_2(u^\star_{[0,3]}) = \begin{bmatrix} -u^\star & (1 - u^{\star 2})\dot u^\star \\ -\dot u^\star & \ddot u^\star - 2u^\star \dot u^{\star 2} - u^{\star 2}\ddot u^\star \\ -\ddot u^\star & u^{\star(3)} - 2\dot u^{\star 3} - 6u^\star \dot u^\star \ddot u^\star - u^{\star 2}u^{\star(3)} \end{bmatrix}. \qquad (25)$$

A numerical analysis of the minors of A_2 (see Fig. 2) reveals that the matrix has rank 2 for all $w \in W$ and thus assumption 3 is fulfilled.

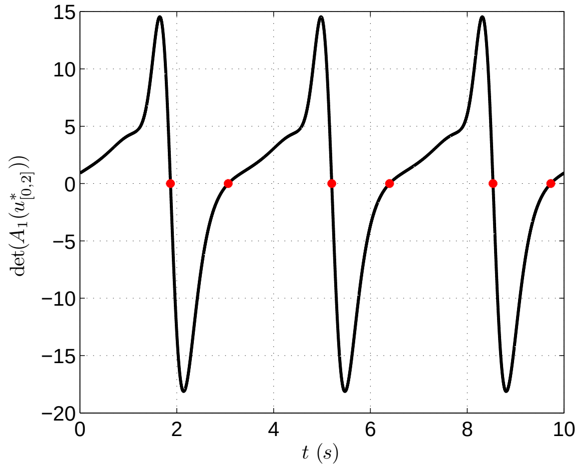

Fig. 1. Determinant of $A_1(u^\star_{[0,2]})$ on the limit cycle ($\omega^2 = 1$ and $\epsilon = 1$).

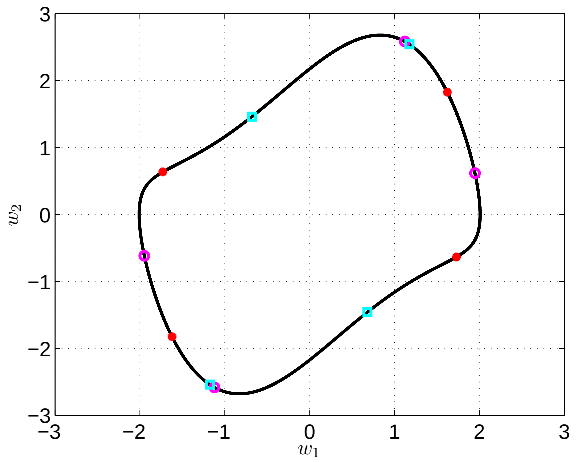

Fig. 2. Limit cycle for the VdP oscillator with $\omega^2 = 1$, $\epsilon = 1$ and singularity points for each minors of matrix $A_2(u^\star_{[0,3]})$. The red points are the singularity points for the minor $A^1 := A^{12}$ having selected the first two rows of the starting matrix; the magenta points for the minor $A^2 := A^{13}$ (first and third rows) and the cyan points for the minor $A^3 := A^{23}$ (second and third rows).

3.2 Duffing Oscillator

We consider now the case in which $u^\star(w)$ is generated by the Duffing oscillator modeled by

$$\begin{aligned} \dot w_1 &= w_2 \\ \dot w_2 &= -w_1^3\alpha - w_1\beta \end{aligned} \qquad (26)$$

where α and β are uncertain parameters and $u^\star(w) = w_1$. the set W is the limit cycle of the oscillator. It turns out that

$$\ddot u^\star(w) = -u^{\star 3}(w)\alpha - u^\star(w)\beta\,, \qquad (27)$$

namely Assumption 2 is fulfilled with $k = 2$, $h(\cdot) = 0$, $L(\cdot) = \left(-u^{\star 3}(w),\ -u^{\star}(w)\right)$ and $\mu = (\alpha\,,\ \beta)^T$. By differentiating once relation (27) we obtain $u^{\star}_{[2,3]} = A_1(u^{\star}_{[0,1]})\mu$ with

$$A_1(u^{\star}_{[0,1]}) = \begin{bmatrix} -u^{\star 3} & -u^{\star} \\ -3u^{\star 2}\dot{u}^{\star} & -\dot{u}^{\star} \end{bmatrix} \tag{28}$$

that is singular in some point of the limit cycle. Taking a further derivative we get $u^{\star}_{[2,4]} = A_2(u_{[0,2]})\mu$ with

$$A_2(u^{\star}_{[0,2]}) = \begin{bmatrix} -u^{\star 3} & -u^{\star} \\ -3u^{\star 2}\dot{u}^{\star} & -\dot{u}^{\star} \\ -3\ddot{u}^{\star}u^{\star 2} - 6u^{\star}\dot{u}^{\star 2} & -\ddot{u}^{\star} \end{bmatrix} \tag{29}$$

that is still rank-deficient. By thus taking a further derivative we get $u^{\star}_{[2,5]} = A_3(u^{\star}_{[0,3]})\mu$ with

$$A_3(u^{\star}_{[0,3]}) = \begin{bmatrix} -u^{\star 3} & -u^{\star} \\ -3u^{\star 2}\dot{u}^{\star} & -\dot{u}^{\star} \\ -3\ddot{u}^{\star}u^{\star 2} - 6u^{\star}\dot{u}^{\star 2} & -\ddot{u}^{\star} \\ -3u^{\star(3)}u^{\star 2} - 18u^{\star}\dot{u}^{\star}\ddot{u}^{\star} - 6\dot{u}^{\star 3} & -u^{\star(3)} \end{bmatrix} \tag{30}$$

which, finally, has rank 2 (see Figures 3-4).

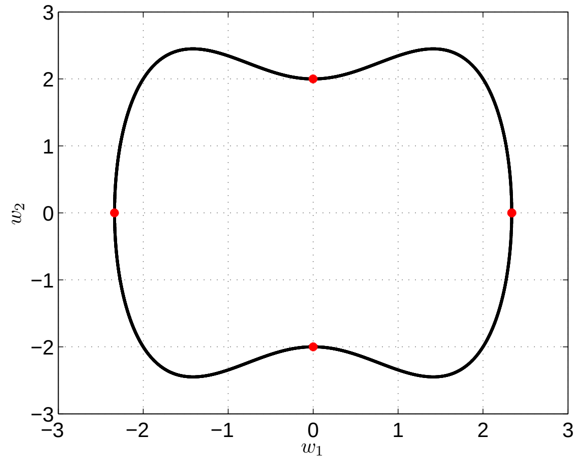

Fig. 3. Limit cycle for the Duffing oscillator with $\alpha = 1$, $\beta = -2$. In the red points at least one minor out of six (of matrix $A_3(u^{\star}_{[0,3]})$) is not singular.

3.3 Lorenz Oscillator

As a third example we consider the case in which $u^{\star}$ coincides with the w_1 component of the Lorentz oscillator described by

$$\begin{aligned} \dot{w}_1 &= \sigma(w_2 - w_1) \\ \dot{w}_2 &= w_1(\rho - w_3) - w_2 \\ \dot{w}_3 &= w_1 w_2 - \beta w_3 \end{aligned} \tag{31}$$

where (σ, ρ, β) are positive uncertain parameters. We let the set W coincide with the Lorentz attractor by assuming a persistence of excitation condition of the oscillator. Specifically we assume there exists a $\epsilon > 0$ such that

$$w_1^2 + \dot{w}_1^2 = \|u^{\star}_{[0,1]}\|^2 \geq \epsilon \qquad \forall w \in W\,.$$

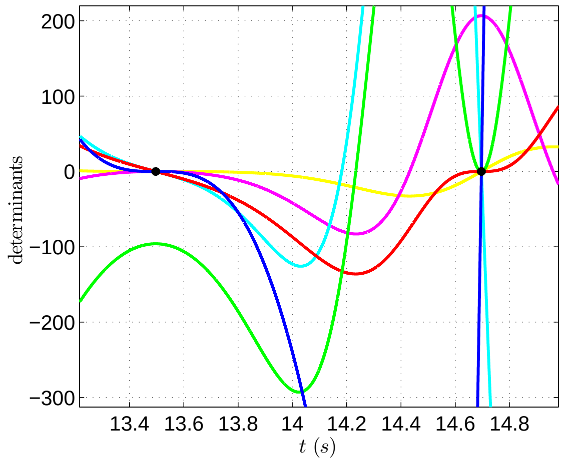

Fig. 4. The plot shows two of four singularity points of Fig. 3 in which is visible that five out of six determinants pass always through zero but, in the same points, the remaining one is always different from zero.

We start differentiating $u^{\star}$ in order to obtain the regression formula (17) and to fulfill Assumption 2. We have $w_1 = u^{\star}(w)$ and $\dot{u}^{\star}(w) = \sigma(w_2 - u^{\star}(w))$ from which $w_2 = u^{\star}(w) + \dot{u}^{\star}(w)/\sigma$. By differentiating further $\dot{u}^{\star}$ we get

$$\begin{aligned} \ddot{u}^{\star} &= \sigma[u^{\star}(\rho - w_3) - w_2 - u^{\star}] \\ &= -\dot{u}^{\star} + c_1 u^{\star} + c_2\dot{u}^{\star} + c_2 u^{\star} w_3 \end{aligned} \tag{32}$$

with $c_1 := \sigma(\rho - 1)$, $c_2 := -\sigma$. Furthermore,

$$\dot{w}_3 = u^{\star 2}(w) + \frac{u^{\star}(w)\dot{u}^{\star}(w)}{\sigma} - \beta w_3\,.$$

By differentiating once more (32) and using the previous expression of $\dot{w}_3$, we obtain

$$u^{\star(3)} = -u^{\star 2}\dot{u}^{\star} - \ddot{u}^{\star} + c_1\dot{u}^{\star} + c_2\ddot{u}^{\star} + c_2 u^{\star 3} + (c_2\dot{u}^{\star} - c_2\beta u^{\star})w_3\,. \tag{33}$$

Relations (32) and (33) can be compactly rewritten as

$$u^{\star}_{[2,3]} = \rho(u^{\star}_{[0,2]}) + C(\rho, \sigma)\,\varphi(u^{\star}_{[0,2]}) + M(\sigma, \beta)\,u^{\star}_{[0,1]}\,w_3$$

where

$$\varphi = \begin{pmatrix} u^{\star}_{[0,2]} \\ u^{\star 3} \end{pmatrix},\ \rho = \begin{pmatrix} -\dot{u}^{\star} \\ -\ddot{u}^{\star} - u^{\star 2}\dot{u}^{\star} \end{pmatrix} \tag{34}$$

and

$$C := \begin{bmatrix} c_1 & c_2 & 0 & 0 \\ 0 & c_1 & c_2 & c_2 \end{bmatrix},\ M := \begin{bmatrix} c_2 & 0 \\ -c_2\beta & c_2 \end{bmatrix}. \tag{35}$$

By taking advantage from the persistence of excitation condition, the previous relation can be used to express w_3 as a function of $u^{\star}_{[0,3]}$, namely

$$w_3 = \frac{1}{\|u^{\star}_{[0,1]}\|^2} {u^{\star}_{[0,1]}}^T M^{-1}\left(\rho(u^{\star}_{[0,2]}) + C\,\varphi(u^{\star}_{[0,2]})\right)$$

or, equivalently,

$$\begin{aligned} w_3 = &\frac{{u^{\star}_{[0,1]}}^T \otimes \rho(u^{\star}_{[0,2]})^T}{\|u^{\star}_{[0,1]}\|^2}\mathrm{vect}(M(\sigma,\beta)^{-1}) + \\ &\frac{{u^{\star}_{[0,1]}}^T \otimes \varphi(u^{\star}_{[0,2]})^T}{\|u^{\star}_{[0,1]}\|^2}\mathrm{vect}(M(\sigma,\beta)^{-1}C(\rho,\sigma)) \end{aligned}$$

where $\otimes$ denotes the Kronecker product and $\mathrm{vect}(T)$ is the column vector obtained by taking row-wise the elements of the matrix T.

Furthermore, by taking another derivative of (33) we get

$$\begin{aligned} u^{\star(4)} &= -3u^\star \dot{u}^{\star 2} - u^{\star 2}\ddot{u}^\star - u^{\star(3)} + \\ & c_1\ddot{u}^\star + c_2(u^{\star(3)} + 4u^{\star 2}\dot{u}^\star) - c_2\beta u^{\star 3} + \beta u^{\star 2}\dot{u}^\star \\ & c_2(\ddot{u}^\star - 2\beta\dot{u}^\star + \beta^2 u^\star)w_3 \end{aligned} \tag{36}$$

by which, using the expression of w_3 above and compacting the terms, we obtain

$$u^{\star(4)} = h(u^\star_{[0,3]}) + L(u^\star_{[0,3]})\mu \tag{37}$$

with $\mu \in \mathbb{R}^{10}$ defined as

$$\mu := \left(\sigma,\ \beta\sigma\rho,\ \beta^2\sigma\rho,\ \beta^3\sigma\rho,\ \beta\sigma,\ \beta^2\sigma,\ \beta^3\sigma,\ \beta,\ \beta^2,\ \beta^3\right)^T$$

and where $h(\cdot)$ and $L(\cdot)$ are appropriately defined functions. This proves that Assumption 2 is fulfilled. To check if there exists a value of m such that Assumption 3 is fulfilled, we go further by simplifying a bit the analysis by assuming that the parameter β is known. This implies, by rearranging a bit the terms in (37), that the following relation

$$u^{\star(4)} = \tilde{h}(u^\star_{[0,3]}) + \tilde{L}(u^\star_{[0,3]})\tilde{\mu} \tag{38}$$

holds, where $\tilde{h}$ and $\tilde{L}$ are known functions (dependent on β) and $\tilde{\mu} \in \mathbb{R}^2$ is defined as $\tilde{\mu} = (\sigma,\ \rho\sigma)^T$.
By differentiating once the equation (38) we get the following compact form

$$u^\star_{[4,5]} = \tilde{H}_1(u^\star_{[0,4]}) + \tilde{A}_1(u^\star_{[0,4]})\tilde{\mu}$$

with

$$\tilde{H}_1(u^\star_{[0,4]}) := \begin{bmatrix} \tilde{h}(u^\star_{[0,3]}) \\ \tilde{h}_1(u^\star_{[0,4]}) \end{bmatrix} \text{ and } \tilde{A}_1(u^\star_{[0,4]}) := \begin{bmatrix} \tilde{L}(u^\star_{[0,3]}) \\ \tilde{L}_1(u^\star_{[0,4]}) \end{bmatrix}$$

To check whether the 2×2 matrix $\tilde{A}_1(u^\star_{[0,4]})$ fulfills Assumption 3, we ran simulations with different values of the parameters and of initial conditions and we found that the matrix is singular in certain points of the Lorentz attractor. A further time derivative is thus taken by obtaining

$$u^\star_{[4,6]} = \tilde{H}_2(u^\star_{[0,5]}) + \tilde{A}_2(u^\star_{[0,5]})\tilde{\mu}$$

in which

$$\tilde{H}_2(u^\star_{[0,5]}) := \begin{bmatrix} \tilde{h}(u^\star_{[0,3]}) \\ \tilde{h}_1(u^\star_{[0,4]}) \\ \tilde{h}_2(u^\star_{[0,5]}) \end{bmatrix} \text{ and } \tilde{A}_2(u^\star_{[0,5]}) := \begin{bmatrix} \tilde{L}(u^\star_{[0,3]}) \\ \tilde{L}_1(u^\star_{[0,4]}) \\ \tilde{L}_2(u^\star_{[0,5]}) \end{bmatrix}$$

with $\tilde{A}_2(u^\star_{[0,5]})$ that is a 3×2 matrix. Numerical tests obtained with different values of the parameters and of the initial conditions showed that the three determinants of each minor of the matrix are never simultaneously zero, namely that the matrix has rank 2 on the Lorentz attractor for the numerical values used in the simulation. Assumption 3 is thus numerically verified and we obtain that relation (13) is fulfilled with a $\phi(\cdot)$ of the form

$$u^{\star(7)} = \tilde{h}_3(u^\star_{[0,6]}) + \tilde{L}_3(u^\star_{[0,6]})\tilde{A}_2^\dagger(u^\star_{[0,5]})(u^\star_{[4,6]} - \tilde{H}_2(u^\star_{[0,5]})).$$

where $\tilde{A}_2^\dagger$ is the left inverse of $\tilde{A}_2$.

4. EXAMPLE

In this section we propose an example in order to validate numerically the proposed method. We consider as controlled plant, a linear oscillator

$$\dot{x}_1 = x_2\,, \qquad \dot{x}_2 = -x_1 + u - w_1 \tag{39}$$

forced by the control variable u and by a matched exogenous disturbance w_1 generated by a Van der Pol exosystem modeled as in (20) with the two uncertain parameters ϵ and ω^2 both equal to 1. The control goal is to regulate x_1 to zero by means of a state feedback control. Output feedback solutions can be easily obtained by the state feedback solution derived below by means of standard arguments that are here omitted. In this case we can define the error $e = x_1 + x_2$ whose dynamics are governed by $\dot{e} = e + u - w_1 - 2x_1$. The system is in the form (1) with $z = x_1$ governed by $\dot{z} = -z + e$ that trivially fulfills Assumption 1. The desired state state input is clearly $u^\star = w_1$. The goal can be achieved using the regulator (9) with $d = 5$, $g = 10$ and $\lambda_0, \ldots, \lambda_4$ such that the polynomial $s^5 + \lambda_4 s^4 + \cdots + \lambda_1 s + \lambda_0 = 0$ is Hurwitz with two roots in -1 and three roots in -2. By following the theory in Section 3.1 the function $\phi(\cdot)$ in (13) is of the form

$$\phi(\xi) = \left[\xi_3,\ \xi_4 - \xi_0^2\xi_4 - 12\xi_1^2\xi_2 - 6\xi_0\xi_2^2 - 8\xi_0\xi_1\xi_3\right]\hat{\mu}$$

with $\hat{\mu}$ the vector of estimated parameters given by

$$\hat{\mu} = A_2^\dagger(u^\star_{[0,3]})[\xi_2, \xi_3, \xi_4]^T$$

where $A_2^\dagger$ is the left inverse of A_2. This function, properly saturated outside $\tau(W)$, has been used as $\phi_s(\cdot)$ in the expression (15) for F. As far as the stabilizer is concerned, the function $\kappa(\cdot)$ has been chosen as linear function $\kappa = 40$. As shown in Figure 5, the harmonic oscillator starts from an initial condition with $x_1 = 1$ and asymptotically converges to zero.

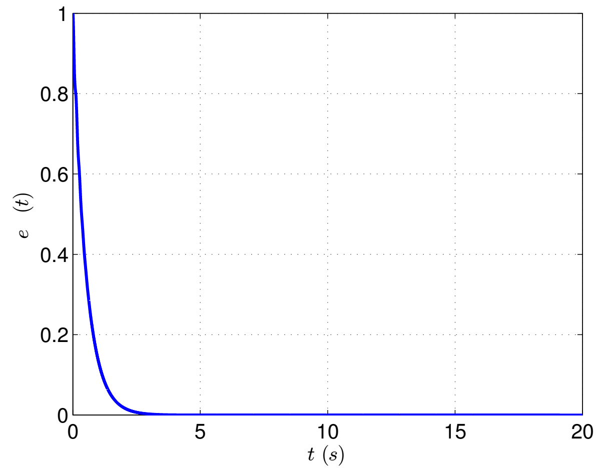

Fig. 5. Regulation error $e(t)$

In Figure 6 is shown the control input $u(t)$ asymptotically converging to the exogenous signal w_1.

5. CONCLUSIONS

The problem of designing internal model-based regulators has been considered for the class of systems (1) in the

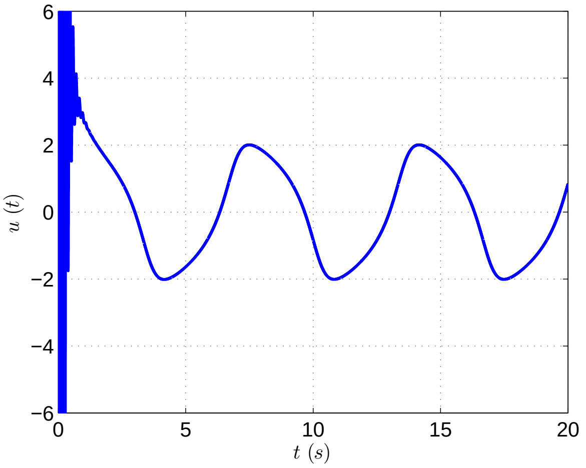

Fig. 6. The control input $u(t)$

special case in which the desired steady state input defined in (5) fulfills a regression law specified in (17). The crucial assumption behind the proposed methodology is Assumption 3. In the relevant cases in which (5) is generated as output of the Van der Pol, Duffing and Lorentz oscillators affected by uncertainties, we have shown the method applies and robust internal model-based regulator can be designed. Numerical analysis has been used to practically support the analysis. The presented methodology strongly relies on high-gain tools and on the design of high-gain observers (see [5]). It thus expected that the method presents intrinsic limitations in applicative fields where the dimension of the exosystem, and thus of the internal model, is large and the measured variables are affected by high-frequency noise.

REFERENCES

[1] C.I. Byrnes and A. Isidori, "Nonlinear Internal Models for Output Regulation", *IEEE Trans. on Automatic Control*, 49, pp. 2244-2247, 2004.

[2] Byrnes, C.I., & Isidori, A., (2004). Nonlinear internal models for output regulation. *IEEE Trans. on Automatic Control*, 49, 2244-2247.

[3] F. Delli Priscoli, L. Marconi, A. Isidori, "A New Approach to Adaptive Nonlinear Regulation", *SIAM Journal on Control and Optimization*, 45(3), pp. 829-855, 2006.

[4] Z. Ding, "Global stabilization and disturbance suppression of a class of nonlinear systems with uncertain internal model", *Automatica*, 39(3), pp. 471-479, 2003.

[5] J.P. Gauthier and I. Kupka, "Deterministic Observation Theory and Applications", Cambridge University Press, 2004.

[6] J. Huang, *Nonlinear Output Regulation Theory and Applications*. SIAM Series: Advances in Design and Control, Cambridge University Press, 2007.

[7] A. Isidori, L. Marconi, A. Serrani, "Robust Autonomous Guidance: An Internal Model-based Approach", Springer Verlag London, Limited series Advances in Industrial Control, 2003.

[8] A. Isidori and C.I. Byrnes, "Output regulation of nonlinear systems", *IEEE Trans. on Automatic Control*, 35, pp. 131-140, 1990.

[9] A. Isidori, L. Marconi, L. Praly, "Robust design of nonlinear internal models without adaptation", *Automatica*, 48, pp. 2409-2419, 2012.

[10] L. Marconi and Isidori, "A unifying approach to the design of nonlinear output regulators". In C. Bonivento, A. Isidori, L. Marconi, C. Rossi (Eds), "Advances in Control Theory and Applications", Lecture Notes in Control and Information Sciences, Springer Verlag Berlin, 2007.

[11] R. Marino, P. Tomei, "Output Regulation for linear systems via adaptive internal model", *IEEE Transaction on Automatic Control*, 48(12), n. 5, pp. 2199-2202, 2003.

[12] A. Serrani, A. Isidori and L. Marconi, "Semiglobal nonlinear output regulation with adaptive internal model", *IEEE Trans. Autom. Contr.*, 46(8), pp. 1178-1194, 2001.