Positive control for global stabilization of predator-prey systems

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Abstract: In this paper we propose a positive linear control law for the stabilization of positive equilibria in predator-prey systems. This problem is motivated by the introduction, which is by essence positive, of predators in biological control applications, to prevent excessive levels of prey (pests). We build a linear controller that we saturate at zero and prove, under some conditions, the global asymptotic stability of the equilibrium. Without one of these conditions, the point is shown to be globally attractive but may be unstable.

Keywords: Biological control, positive control, predator-prey model, stabilization, positive linear control, global stability

1. INTRODUCTION

In this paper, we present control laws for a large family of predator-prey models. These models with two variables (the prey and predator densities, x and y) describe the dynamical behavior of a biological system, and may exhibit several stable equilibria, or oscillations. Typical behaviors from this class of models are a unique stable equilibrium or a stable limit cycle around an unstable equilibrium [Kot, 2001].

But, often, man may be able to act on these systems through the introduction of predators in the system. This kind of control offers interesting problems from two point of views.

From a biological point of view, in the context of biological control [Hawkins and Cornell, 1999], the aim is to maintain the prey (the pests) below some level; we want to avoid large transients where this prey would become too large. For that, introducing predators seems a good idea, but it is well known in ecology that this action (implemented as a constant control) may destabilize a stable system and result in sustained oscillations with large values of the prey: this is known as one aspect of the paradox of enrichment for the Rosenzweig-MacArthur (RMA) model [Rosenzweig, 1971]. Our aim would be to introduce predators in a more subtle way, to reduce the number of prey without destabilizing the system.

From a mathematical point of view, our control is only positive, which makes the mathematical problem more difficult, even in the linear case [Brammer, 1972], because we cannot control everywhere around the equilibrium. There exists no general method for the design of positive controller so that their construction and stability analysis often are ad-hoc.

In this work we build feedback controls that stabilize, in a global way, a nonlinear predator prey system around an equilibrium, with a positive control. The value of the equilibrium can be chosen such that the value of the prey is arbitrarily low, and the control limits the size and number of the peaks of the transient above this low threshold. The tools are classical tools from dynamical system theory.

Similar results are rather rare in the literature. For positive linear control, see Brammer [1972], Saperstone and Yorke [1971]. For similar problems, see Grognard and Gouzé [2005] for Lotka-Volterra systems and Muratori and Rinaldi [1988] for discrete-time Leslie matrices. The paper Meza et al. [2005] considers very similar problems, but the sliding control it designs cannot stabilize the system toward a low level of prey. In fact, it creates a new stable equilibrium at a higher level, and the control is a harvest and not an introduction of predators.

This paper is structured as a build-up to Theorem 2 so that Assumptions that are given along the way stay valid in the following developments. We start by defining the model and our control action in Section 2 and 3 before developing our controller and analyzing its local (Section 4) and global (Section 5) stability. Simulations and discussions of the results are then given in Section 6.

2. MODELS

We will concentrate on predator-prey models whose positive equilibrium of interest is unstable, the most classical example in which it arises being termed the Rosenzweig-MacArthur model [Rosenzweig, 1971]:

$$\begin{cases} \dot{x} = rx\left(1 - \dfrac{x}{K}\right) - \dfrac{bx}{x+h}y \\ \dot{y} = \dfrac{cx}{x+h}y - my \end{cases} \qquad (1)$$

with all parameters being positive, with logistic growth of the prey and Holling type II predation rate, eg. to represent satiation. The non-trivial x nullcline is defined by $y = \frac{r}{b}\left(1 - \frac{x}{K}\right)(x + h)$ which is a parabola having a maximum in $x = \frac{K-h}{2}$. The non-trivial y nullcline is $x = \frac{mh}{c-m} = x^\dagger$. When it exists, the positive equilibrium $(x^\dagger, y^\dagger)$ lies at the intersection of these two non-trivial nullclines. In fact, it can easily be computed that, if the slope of the x nullcline is negative at the equilibrium $\left(\frac{K-h}{2} < x^\dagger < K\right)$, it is globally stable, while if it is positive $\left(0 < x^\dagger < \frac{K-h}{2}$, which imposes $K > h\right)$ the equilibrium is unstable and the system presents a globally attractive limit cycle [Rosenzweig, 1971].
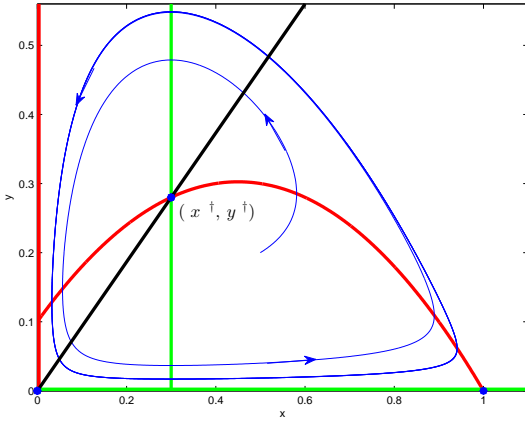

Fig. 1. Illustration of the limit cycle for the RMA model (1) (with $r = 1$, $K = 1$, $b = 1$, $c = 2$, $m = 1.5$ and $h = 0.1$). The x nullclines are in red and the y nullclines in green; the blue dots are the equilibria and the blue line a solution of the system that goes to the limit cycle. Note that (x^*, y^*) is on the left of the maximum of the non-trivial x nullcline. The black line illustrates Assumption 3 with $(x^*, y^*) = (x^\dagger, y^\dagger)$

Since we placed ourselves in the context of biological control, with x the pest and y the control agent, we will most likely desire small values of $x^\dagger$, so that the corresponding positive equilibrium will likely be unstable and we will need to stabilize it. In order to do so in a more general manner, we propose, in the spirit of Rosenzweig [1971], a generalization of the classical RMA model in which we kept the elements that ensure it has the same properties as the classical model:

$$\begin{cases} \dot{x} = f(x) - g(x)y \\ \dot{y} = h(x)y - my \end{cases} \quad (2)$$

with the following hypotheses

Assumption 1. f, g and h are $\mathcal{C}^1(\mathbb{R}^+)$ functions satisfying

(i) $f(0) = 0$, $f'(0) > 0$, and $\exists! \ x(= \bar{x}) > 0$ such that $f(\bar{x}) = 0$; it is such that $f(x) < 0$ for $x > \bar{x}$;
(ii) $g(0) = 0$, $g'(0) > 0$, and $g(x) > 0$ for $x > 0$;
(iii) $h(0) = 0$, $h(x) > 0$ for $x > 0$, $\exists! \ x^\dagger < \bar{x}$ such that $h(x^\dagger) = m$ and $h'(x) > 0$ for $x \leq x^\dagger$.

These assumptions are quite natural: the hypothesis on the prey growth rate states that there is a threshold above which some sort of prey over-crowding prevents growth;

the ones on the predator functional and numerical responses $g(.)$ and $h(.)$ mainly state that predation is unilateral: the prey never preys on the predator. Oftentimes, $g(.)$ and $h(.)$ are taken to be proportional to each other, but we do not suppose it here. In what follows, the function $\frac{f(x)}{g(x)}$ will be of paramount importance; it is obviously $\mathcal{C}^1(\mathbb{R}_*^+)$; moreover, $\lim_{x \searrow 0} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)}$, so that $\frac{f(x)}{g(x)}$ is continuous over $[0, \bar{x}]$, the interval over which it is non-negative and hence possesses an upper-bound Y.

In that framework, the x nullclines satisfy $x = 0$ or $y = \frac{f(x)}{g(x)}$ and the y nullclines satisfy $y = 0$ or $h(x) = m$ from which, and hypothesis (iii), we deduce that (2) has three equilibria in $\mathbb{R}_+^3$: $(0,0)$, $(\bar{x}, 0)$ and $(x^\dagger, y^\dagger)$ with $x^\dagger > 0$ such that $h(x^\dagger) = m$ and $y^\dagger = \frac{f(x^\dagger)}{g(x^\dagger)} > 0$. Moreover, the first two are unstable and the stability of the last one can be deduced from the Jacobian matrix ; this equilibrium is stable if $\left(\frac{f}{g}\right)'(x^\dagger) < 0$. This means that the stability of this equilibrium is determined by the slope of the x nullcline at equilibrium, that is where the vertical line $h(x^\dagger) = m$ intersects it, as in the classical RMA model.

3. DEFINING A CONTROL OF THE PREDATOR POPULATION

We consider now the important and realistic problem of Biological Control: prey x is a pest that damages the (unmodelled) crop, and the aim is to introduce predator y such that the new equilibrium for x is as low as possible, and surely under some threshold of irreversible damage to the cultures. Moreover, a second aim is to also control the transients of the system, to prevent any trajectory to cross the threshold at any time.

According to the study in the above section, a case that may arise is that the equilibrium $x^\dagger$ is too large (above the threshold), and a first action could be to move the equilibrium to the left, reducing the size of the prey population at equilibrium. To do so, we introduce a control of the predator growth rate which corresponds to a continuous introduction of predators in the system. This takes the form:

$$\begin{cases} \dot{x} = f(x) - g(x)y \\ \dot{y} = h(x)y - my + u(x,y)y \end{cases} \quad (3)$$

where we impose that $u(x,y)$ be non-negative because we can introduce predators into the system but have no mean of decreasing their population. In case we want to reduce the prey population at equilibrium to a value $x^* < x^\dagger$, we notice that this equilibrium can be achieved through a constant control $u(x,y) = \bar{u}$ such that

$$h(x^*) - m + \bar{u} = 0$$

and, since $h(x^*) < h(x^\dagger) = m$, this equation ensures that $\bar{u} > 0$. The corresponding y equilibrium is then $y^* = \frac{f(x^*)}{g(x^*)}$. With that constant control, the form of the model is unchanged with respect to (2) except that the predator mortality rate is replaced with $m - \bar{u} > 0$. The aim of having a low equilibrium x^* is achieved, but what happens is that this equilibrium is very probably unstable: in the classical RMA model, it will be the case if the threshold, and therefore the equilibrium, is lower than the maximum

of the x nullcline, as seen above. Having a low equilibrium (for x) results in an unstable equilibrium, and therefore large oscillations around the equilibrium, that will possibly exceed the threshold. This first situation is also known as the paradox of enrichment [Rosenzweig, 1971], or the paradox of biological control.

Whether $\bar{u} = 0$ or not, we possibly are left with a stabilization problem to solve with a positivity constraint on the control value. We will want to stabilize the equilibrium (x^*, y^*) (with $x^* \leq x^\dagger$) for system (3) with $u(x, y) \geq 0$ (to simplify, we suppose that our control is not bounded). The case where $\bar{u} = 0$ is of course the most difficult, because the control constraint is active at the equilibrium; if $\bar{u} > 0$, the equilibrium control being $\bar{u}$, the constraint will only be active away from the equilibrium which simplifies greatly the local analysis.

4. LOCAL LINEAR CONTROL

In this section we build a nonnegative control to stabilize the equilibrium, by a local linearized approach. The control is a saturated linear one.

4.1 Linear control

We write a linear control law in the form
$$u(x, y) = \bar{u} - \gamma \left[y - y^* - \beta(x - x^*) \right]$$
which is not positive in itself, with $\bar{u}$ chosen as in the previous section and $\gamma, \beta > 0$.

This control ensures the existence of an equilibrium in (x^*, y^*) for system (3) whose stability can be identified in the Jacobian matrix of the controlled form $\mathcal{J}_u(x^*, y^*)$: (x^*, y^*) is asymptotically stable if $det(\mathcal{J}_u) > 0$ and $trace(\mathcal{J}_u) < 0$ with
$$det(\mathcal{J}_u) = \gamma f(x^*) \left[\beta - \left(\frac{f}{g} \right)'(x^*) \right] + h'(x^*) f(x^*),$$
which is made strictly positive for all $\gamma > 0$ by taking
$$\beta \geq \left(\frac{f}{g} \right)'(x^*), \tag{4}$$
and
$$trace(\mathcal{J}_u) = f'(x^*) - g'(x^*) \frac{f(x^*)}{g(x^*)} - \gamma \frac{f(x^*)}{g(x^*)}$$
which can be made negative by taking
$$\gamma > \frac{g^2(x^*)}{f(x^*)} \left(\frac{f}{g} \right)'(x^*) \tag{5}$$
Again, this is linked to the slope of $\frac{f}{g}$ at equilibrium. If this slope is negative, a gain $\gamma = 0$ would be sufficient for the stability of the equilibrium, which is not surprising since the equilibrium was already stable.

4.2 Positive control

In order to have a control $u(x, y)$ positive, we will simply saturate the previous control at 0:
$$u(x, y) = \begin{cases} 0 & \text{if } \gamma \left[y - y^* - \beta(x - x^*) \right] > \bar{u} \\ \bar{u} - \gamma \left[y - y^* - \beta(x - x^*) \right] & \text{else} \end{cases} \tag{6}$$

If $\bar{u} > 0$, the right-hand-side of the closed-loop system (3)-(6) is $\mathcal{C}^1$ at (x^*, y^*) because $u(x^*, y^*) = \bar{u}$ is away from

the threshold, so that local asymptotic stability directly follows from the analysis of the Jacobian matrix and conditions (4)-(5).

If $\bar{u} = 0$, the analysis is trickier because the Jacobian matrix of the closed-loop system is not properly defined at (x^*, y^*) since it is on the threshold and the right-hand-side is not smooth. With $\bar{u} = 0$, we have $x^* = x^\dagger$ at the equilibrium and the linearized control system around this equilibrium is:
$$\dot{X} = \mathcal{J}_0(x^*, y^*)X + bu$$
$$\mathcal{J}_0(x^*, y^*) = \begin{pmatrix} f'(x^*) - g'(x^*) \dfrac{f(x^*)}{g(x^*)} & -g(x^*) \\ h'(x^*) \dfrac{f(x^*)}{g(x^*)} & 0 \end{pmatrix}$$
and
$$b = \begin{pmatrix} 0 \\ y^* \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \end{pmatrix}.$$
Our control (6) can then be rewritten as:
$$u = \max(0, \gamma(\beta(x - x^*) - (y - y^*))) \tag{7}$$
where β and γ are two gains to be chosen. This feedback is nonnegative, continuous, but not differentiable around (x^*, y^*). We will show that, under some conditions, it is possible to choose gains that stabilize the linearized system around the equilibrium. Indeed, this equilibrium is unstable, as can be seen on the Jacobian $\mathcal{J}_0(x^*, y^*)$ where the determinant is positive, but the trace is also positive because $(f'(x^*) - g'(x^*) \frac{f(x^*)}{g(x^*)}) = ((\frac{f}{g})'(x^*)g(x^*))$ is positive, the equilibrium point being on the left of the maximum of the first nullcline.

First we recall the original result from Saperstone and Yorke [1971] for local controllability with positive controls. The system is a linear system with one input in the form
$$\dot{X} = \mathcal{J}X + bu \tag{8}$$
Theorem 1. System (8) with controls $u(.)$ belonging to the nonnegative bounded measurable functions of $\mathbb{R}^+ \rightarrow [0, 1]$ is locally controllable at the origin if and only if

(1) *all eigenvalues of $\mathcal{J}$ have nonzero imaginary parts, and*
(2) *the controllability matrix for (8) has full rank.*

Intuitively, this result says that in the uncontrolled region $u = 0$, the solutions rotate around the equilibrium, and thus exit this region to enter the controlled region. In our case, we only need stability, therefore a sufficient condition resulting of the theorem is that the eigenvalues of $\mathcal{J}$ have real negative or complex conjugate eigenvalues (see Willems et al. [2002]). It is clear that the controllability matrix is of full rank. Therefore condition (1) of Theorem 1 is satisfied if the discriminant of the characteristic equation of (8) is negative, giving:

Assumption 2. Inequality
$$((\frac{f}{g})'(x^*)g(x^*))^2 < 4h'(x^*)f(x^*) \tag{9}$$
is satisfied.

Then the linear system has complex eigenvalues, and the above theorem applies. Now we have to build the controller. We choose the form (7), in an approach very

similar to that of Willems et al. [2002]. The system we consider is a kind of hybrid system, with two modes $u = 0$ and $u = \gamma(\beta(x - x^*) - (y - y^*))$. The system switches between these two modes along the straight line $\beta(x - x^*) - (y - y^*) = 0$, that separates the plane between two half-planes. First we remark that the control is continuous along the switching boundary, so there will be no sliding modes. If a trajectory starts in the region where $u = 0$, there is no control, and it will rotate around (x^*, y^*) until it enters the other region. In this region, because of the controllability of the system, we are able to place the eigenvalues of the closed-loop system with the gains β and γ. We choose two real distinct negative eigenvalues. Then in the controlled region there will be two half straight lines, corresponding to the two stable vector spaces (we choose them such that none of the eigenspaces is on the switching boundary). Therefore, after entering the controlled region, a trajectory stays there and converges to the equilibrium, because the invariant stable eigenspaces delimits invariant and stable regions centered on the equilibrium. Hence, the controlled equilibrium with such a positive feedback is locally stable thanks to condition (9). We will show in the next section that the conditions we impose on β and γ ensure that the eigenvalues of the linearized controlled system are negative real.

If condition (9) is not satisfied, the unstable equilibrium could have two real positive eigenvalues, and stability cannot be achieved.

5. GLOBAL STABILITY

The global stability proof will start by first building a positively invariant region bordering both the (x^*, y^*) equilibrium and the y axis. Based on that, local stability is shown to be a consequence of this invariance, while global attractiviy ensues because this positively invariant region prevents oscillations around the equilibrium.

While keeping the Assumptions and conditions that have been imposed for local stability, an other assumption is added in order to show global stability: we impose a condition that ensures that $\frac{f(x)}{g(x)}$ is above the straight line joining the equilibrium and the origin for all $0 < x < x^*$

Assumption *3.* We suppose that $\frac{f(x)}{g(x)} > \frac{f(x^*)}{g(x^*)}\frac{x}{x^*}$ (resp. $<$) for all $0 < x < x^*$ (resp. $x > x^*$) and that $\left(\frac{f}{g}\right)'(x^*) < \frac{f(x^*)}{g(x^*)}\frac{1}{x^*}$

This assumption is illustrated on Figure 1, where the black line, joining the origin and (x^*, y^*) ($y = y^*\frac{x}{x^*} = \frac{f(x^*)}{g(x^*)}\frac{x}{x^*}$) is below the red x nullcline $y = \frac{f(x)}{g(x)}$ for $x < x^*$ and above it for $x > x^*$.

The second part of Assumption 3 guarantees that the slope of $\frac{f(x)}{g(x)}$ at the equilibrium is smaller than that of the line joining the equilibrum and the origin. This ensures that there exist two different lines joining the equilibrium and the y axis with $y \geq 0$ with both staying below $\frac{f(x)}{g(x)}$ for $0 < x < x^*$. Indeed, since $\left(\frac{f(x)}{g(x)}\right)(0) = \frac{f'(0)}{g'(0)} > 0$, Assumption 3 ensures that there is room for such a construction. We can then choose $\mu > \beta > 0$ such that

$$\forall x \in [0, x^*), 0 \leq y^* + \mu(x - x^*) < y^* + \beta(x - x^*) < \frac{f(x)}{g(x)} \tag{10}$$

which implies $\frac{y^*}{x^*} \geq \mu > \beta \geq \left(\frac{f}{g}\right)'(x^*)$, so that (4) is a consequence of (10). For instance, condition (10) implies that the black line is below the magenta line which is below the red parabola for $x < x^*$ in Figure 2-top, and the two black lines are below the red parabola in Figure 2-bottom. We also impose

$$\forall x > x^*, y^* + \beta(x - x^*) > \frac{f(x)}{g(x)} \tag{11}$$

(for instance, the magenta line is above the red parabola for $x > x^*$ in Figure 2-top, not illustrated on the other one).

Positive invariance of Ω   We will now show that, for γ large enough (and satisfying (5)), the saturated control generates a positively invariant compact region Ω bounded by the two lines $y = y^* + \beta(x - x^*)$, $y = y^* + \mu(x - x^*)$ (with $x \leq x^*$) and the y axis. The invariance of the y axis ensures that no solution can escape there. We then need to show that, when $y - y^* - \beta(x - x^*) = 0$, $\dot{y} - \beta\dot{x} \leq 0$ and, when $y - y^* - \mu(x - x^*) = 0$, $\dot{y} - \mu\dot{x} \geq 0$, that is

- When $y - y^* - \beta(x - x^*) = 0$,
  $$\dot{y} - \beta\dot{x} = h(x)y - my + \bar{u}y - \beta(f(x) - g(x)y)$$
  which is non positive since $h(x) \leq m - \bar{u}$ (because $h(x^*) = m - \bar{u}$ by definition and $h(x)$ is increasing for $x \leq x^*$) and $y = y^* + \beta(x - x^*) \leq \frac{f(x)}{g(x)}$.

- When $y - y^* - \mu(x - x^*) = 0$,
  $$\dot{y} - \mu\dot{x} = h(x)y - my + \bar{u}y - \gamma[y - y^* - \beta(x - x^*)]y \\ -\mu(f(x) - g(x)y)$$
  which is non-negative for all $x \in [0, x^*]$ when

$$\gamma \geq \sup_{x \in (0, x^*)} \left[\frac{h(x) - m + \bar{u}}{(\mu - \beta)(x - x^*)} - \frac{\mu(f(x) - g(x)(y^* + \mu(x - x^*)))}{(\mu - \beta)(x - x^*)(y^* + \mu(x - x^*))} \right] \\ \triangleq \sup_{x \in (0, x^*)} \phi(x) \tag{12}$$

The argument of the supremum being continuous over $(0, x^*)$, the only problem could arise at the limit for x going to x^* or 0 (if $y^* - \mu x^* = 0$ in the latter case). If either limit value is $+\infty$, the supremum is not finite. Otherwise, there is a maximum of $\phi(x)$ over $[0, x^*]$. Taking the limit of this argument as x goes to x^*, we see that both the denominators and numerators of both terms go to 0. We then have, using L'Hospital rule

$$\lim_{x \overset{<}{\to} x^*} \phi(x)$$
$$= \lim_{x \overset{<}{\to} x^*} \frac{h'(x)}{\mu - \beta} - \frac{\mu(f'(x) - g'(x)(y^* + \mu(x - x^*)) - \mu g(x))}{(\mu - \beta)(y^* + 2\mu(x - x^*))}$$
$$= \frac{h'(x^*)}{\mu - \beta} - \frac{\mu}{\mu - \beta}\frac{f'(x^*) - g'(x^*)y^* - \mu g(x^*)}{y^*}$$
$$= \frac{h'(x^*)f(x^*) - \mu(f'(x^*)g(x^*) - g'(x^*)f(x^*)) + \mu^2 g^2(x^*)}{(\mu - \beta)f(x^*)}$$
$$= \frac{h'(x^*)f(x^*) + \mu g^2(x^*)\left[\mu - \left(\frac{f}{g}\right)'(x^*)\right]}{(\mu - \beta)f(x^*)} \tag{13}$$

which clearly is finite because $\mu \neq \beta$. A similar argument shows that the limit is finite as $x \to 0$ with $y^* - \mu x^* = 0$.

If condition (12) is satisfied, (13) and $h'(x^*)f(x^*) > 0$ imply that

$$\gamma > \frac{\mu g^2(x^*)\left[\mu - \left(\frac{f}{g}\right)'(x^*)\right]}{(\mu - \beta)f(x^*)}$$

so that, since $\mu > \beta > \left(\frac{f}{g}\right)'(x^*)$, $\frac{\left[\mu - \left(\frac{f}{g}\right)'(x^*)\right]}{(\mu-\beta)} > 1$ and

$$\gamma > \frac{\mu g^2(x^*)}{f(x^*)} > \frac{g^2(x^*)}{f(x^*)}\left(\frac{f}{g}\right)'(x^*)$$

We then have that (5) is a consequence of (12).

Back to local stability   In the case where $\bar{u} = 0$, we can then re-consider the local stability analysis in the light of the invariance of Ω. With β, μ and γ chosen satisfying (10)-(11) and (12), we have shown that (4)-(5) are verified, which implies that the real part of the eigenvalues of the system with linear control are negative. By enforcing the invariance of Ω, we prevent solutions initiated very close to (x^*, y^*) to oscillate around this equilibrium, and so ensure real eigenvalues. Also, we have in fact shown that $\dot{y} - \beta\dot{x} < 0$ on the switching surface $y = y^* + \beta(x - x^*)$ when $x < x^*$ so that no eigenvector is colinear with that line. We have then shown local asymptotic stability of (x^*, y^*) based on the results of Section 4.

Equilibria   We can then compute the equilibria of the closed-loop system (3)-(6) by noting that, above the $y = y^* + \beta(x - x^*) + \frac{\bar{u}}{\gamma}$ line the system is unchanged compared to the uncontrolled system since $u = 0$, so that the equilibria are not modified there: only (x^*, y^*) exists (if $\bar{u} = 0$). Below $y = y^* + \beta(x - x^*) + \frac{\bar{u}}{\gamma}$, the x nullclines are unchanged with respect to the uncontrolled system, which yields two pieces:

(i) $y = \frac{f(x)}{g(x)}$ when $\frac{f(x)}{g(x)} \leq y^* + \beta(x - x^*) + \frac{\bar{u}}{\gamma}$ (which includes at least all $x \geq x^*$ because of (11))

(ii) $x = 0$ for $y \leq y^* - \beta x^* + \frac{\bar{u}}{\gamma}$

The y nullclines then become

(a) $y = 0$ for all $x \geq 0$

(b) $y = y^* + \beta(x - x^*) + \frac{h(x)-m+\bar{u}}{\gamma}$, which is only below $y = y^* + \beta(x - x^*) + \frac{\bar{u}}{\gamma}$ when $\frac{h(x)-m}{\gamma} \leq 0$, that is when $x \leq x^\dagger$.

Equilibria are then generated by the intersection of these nullclines: (i)-(a) yields the classic predator less equilibrium $(\bar{x}, 0)$; (i)-(b) gives (x^*, y^*); (ii)-(a) generates the origin $(0,0)$; (ii)-(b) yields $(0, y^* - \beta x^* - \frac{m-\bar{u}}{\gamma})$. It can be shown that this last equilibrium is in the region Ω; indeed, $y - y^* - \beta(x - x^*) < 0$ at this point because $m > \bar{u}$ and $y - y^* - \mu(x - x^*) > 0$ thanks to inequality (12) in $x = 0$. System (3)-(6) therefore presents 4 equilibria among which only (x^*, y^*) is not on the boundaries of the positive orthant. See Figure 2 for illustration of the nullclines and equilibria.

Boundedness of solutions   Boundedness of the solutions comes from the analysis of a composite bound based on x, y, and $y + Mx$ (with $M > 0$). We indeed have
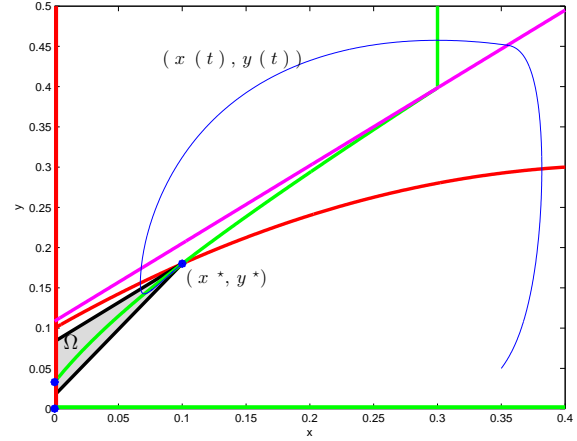

Fig. 2. Illustration of the nullclines for x (in red) and y (in green) for the closed loop system, with the equilibria at their intersections (blue dots, the $(\bar{x}, 0)$ equilibrium is out of the picture), with $\bar{u} = 0.5$. The magenta line is $y = y^* + \beta(x - x^*) + \frac{\bar{u}}{\gamma}$, the boundary between the region where $u = 0$ (above) and u is linear (below). The shaded region is the positively invariant region Ω, bounded by $y = y^* + \beta(x - x^*)$ above and $y = y^* + \mu(x - x^*)$ below. If $\bar{u} = 0$, (x^*, y^*) is moved to the basis of the green vertical line. A solution of the system, initiated in $(0.35, 0.05)$ and converging to the equilibrium, is illustrated in blue. The simulation was built for the RMA model with the parameters of Figure 1, so that $(x^\dagger, y^\dagger) = (0.3, 0.28)$; the controller with $\bar{u} = 0.5, \beta = 0.97, \mu = 1.63$ and $\gamma = 19.81$ moves the equilibrium to $(x^*, y^*) = (0.1, 0.18)$.

$$\dot{y} + M\dot{x} = y(h(x) - m + u(x,y)) + M(f(x) - g(x)y)$$
$$= (f(x) - g(x)y)\left[\frac{h(x) - m + u(x,y)}{g(x)\left(\frac{f(x)}{g(x)y} - 1\right)} + M\right]$$

We will now consider this inequality only in the region where $x \in [x^*, 2\bar{x}]$ and $y \geq \max\left(2Y, y^* + \beta(2\bar{x} - x^*)\right)$ (with $Y = \max_{x \geq 0}\frac{f(x)}{g(x)}$). The latter ensures that $u(x,y) = \bar{u}$, that $\frac{f(x)}{g(x)y} \leq \frac{1}{2}$ and $f(x) - g(x)y < 0$. Therefore, in that region, $\dot{y} + M\dot{x} < 0$ if

$$M - 2\frac{h(x) - m + \bar{u}}{g(x)} > 0$$

It then suffices to take $M > \max_{x \in [x^*, 2\bar{x}]} 2\frac{h(x)-m+\bar{u}}{g(x)}$ to ensure that $\dot{y} + M\dot{x} < 0$. Combining this with $\dot{x} < 0$ for $x > x^*$ and $\dot{y} < 0$ for $x < x^*$ and $y > y^*$ ensures boundedness of the solutions.

ω-limit sets   We then conclude from the Poincaré-Bendixson theorem [Guckenheimer and Holmes, 1983] that solutions either converge to an equilibrium, a periodic orbit or heteroclinic/homoclinic orbits. Since, inside any periodic orbit, there must be at least one equilibrium [Khalil, 1996], any periodic orbit should surround (x^*, y^*). This orbit should then go through Ω, whose positive invariance prevents such occurrence. No periodic orbit can therefore exist.

Having shown that the y nullcline (b) is initiated at $(0, y^* - \beta x^* - \frac{m-\bar{u}}{\gamma})$, and noting that $\beta x + h(x)$ is positive for all

$x > 0$, the (b) nullcline corresponding to $\dot{y} = 0$ is always above $y = y^* - \beta x^* - \frac{m - \bar{u}}{\gamma}$ and we have $\dot{y} > 0$ for all (x, y) with $y < y^* - \beta x^* - \frac{m - \bar{u}}{\gamma}$) so that no solution initiated in the positive orthant can converge to $(0, 0)$ or $(\bar{x}, 0)$ or any homoclinic or heteroclinic orbit including these points. Similarly, since $\dot{x} > 0$ inside Ω, no solution initiated in Ω (or in the positive orthant for that matter) can converge to $(0, y^* - \beta x^* - \frac{m - \bar{u}}{\gamma})$, which cannot be part of an homoclinic orbit. Finally, solutions can only converge to (x^*, y^*) or a homoclinic cycle built upon it. Since (x^*, y^*) is locally asymptotically stable, only convergence can take place and (x^*, y^*) is globally asymptotically stable inside the positive orthant.

Theorem 2. Let system (3) satisfy Assumptions 1, 2 and 3. The control law (6) with $\mu > \beta > 0$, and γ satisfying conditions (10)-(11)-(12) ensures global asymptotic stability of the equilibrium (x^*, y^*) in the positive orthant.

6. UNSTABLE BUT GLOBALLY ATTRACTIVE EQUILIBRIUM

We have seen in the course of the proof of our main theorem that stabilty heavily depended on the existence of complex (unstable) eigenvalues for the uncontrolled equilibrium. In this section, we illustrate on an example the kind of behaviours that occur if Assumption 2 is not satisfied. For that, we take $r = 3$, $K = 9$, $b = 6$, $c = 0.15$, $m = 0.05$ and $h = 1.8$, which yields eigenvalues 0.5 and 0.1 for the uncontrolled equilibrium. With the choice $\bar{u} = 0$, $\beta = 0.5$, $\mu = 1.2$ and $\gamma = 3$, we obtain the phase plane of Figure 3, where the shaded area is still Ω and the magenta line, the border between the controlled (below) and uncontrolled (above) system. Solutions in Ω obviously converge to the equilibrium. What is interesting to see is what occurs near the equilibrium: solutions initiated between the two eigenvectors of the uncontrolled linearized system (in red on Figure 3) escape far from the equilibrium before coming back to the equilibrium through Ω; on the other hand, in backward time, they converge to the equilibrium directly. In fact, they form an infinite set of potentially large homoclinic orbits, which confirms that the equilibrium is unstable and globally attractive. From the point of view of biological control, this is not a real problem since these homoclinic orbits take place with pest densities that are smaller than that of the target equilibrium.

7. CONCLUSION

We have proposed a family of control laws for the global control of a positive equilibrium for quite general predator-prey models. They have the advantage of being able to deal with small values of the desired prey equilibrium levels which, in models such as the Rosenzweig-MacArthur model, is often unstable. In biological control methods, being able to achieve that is fundamental since the purpose is to use the predators to bring the prey/pest to a very low level; The strength of our method is that it generates simple control laws which are continuous near the switching lines and it has tuning parameters that can lead them to reach desired performances. Our method presents an original approach to a rarely tackled problem,
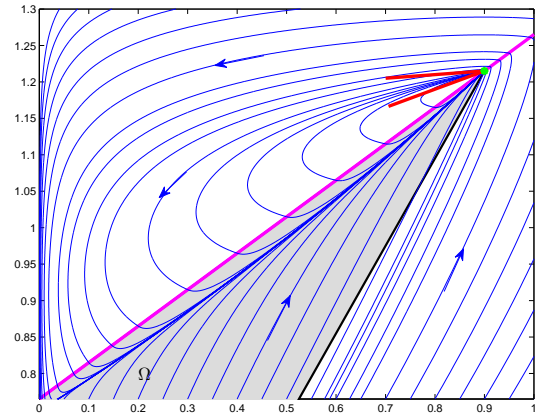

Fig. 3. Phase plane of the system as discussed in Section 6. The target equilibrium is the green dot and the red vectors represent the eigenvectors of the uncontrolled system. Large homoclinic orbits are observed.

the positivity of the control in predator prey systems, and it achieves a level of genericity that ensures it can be applied in a wide variety of predator-prey settings thanks to the qualitative form of the functions that have been used in our developments. Further developments will aim at minimizing the total control effort and the size of the transient peak in the prey population.

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