Nonlinear observer normal form with output injection and extended dynamic

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Abstract: This paper presents a new extended output depending nonlinear observer normal form. A sufficient geometrical conditions that guarantee a change of coordinates allowing the transformation of a given nonlinear dynamical system into the proposed observer form are given. Throughout this work, it will be showed that, unlike to the existing observer normal forms, this new form enables to design an observer for the Susceptible, Exposed, Infected, and Recovered (SEIR) model of population under an infectious disease.

1. INTRODUCTION

Occurrence of new disease as influenza pandemic in 2009, with all its economic and health implications, recall us that humanity is very weak. Moreover the unpredictable features of diseases and virus mutation forces us to acquire tools and means allowing to follow, predict and control these phenomena propagation. The evolution of an epidemic through a population takes several steps, some are visible or measurable and other intermediaries, which are crucial to analyze and estimate the evolution and the threat of an epidemic on the population, are invisible or unmeasurable. Therefore, to establish an effective plan to fight against an epidemic, it is necessary to know the hidden and the unmeasurable steps.

One of the most used systems to studies the propagation of an infectious disease, is the Susceptible Exposed Infected and Recovered (SEIR) model. This model is characterized by nonlinearities which are not easy to handle. Many works are dedicated to improve this model Beretta and Capasso [1986], to study its stability Bonzi et al. [2011], Fall et al. [2007] and to develop vaccination process De la Sen and Alonso-Quesada [2011]. But, few initiatives are oriented to synthesize a nonlinear observer Bernard et al. [1998].

Our study is motivated by a desire to analyze and reconstruct the future trend of contagious disease through a population. To achieve our aim, we propose an approach based on class of normal form and high gain observer.

The idea of observer normal form, introduced by Krener and Isidori [1983] for single output dynamical systems, is to turn the given system into a form allowing an observer design. However, the form proposed in Krener and Isidori [1983] requires very strict conditions for the existence of diffeomorpism. To overcome this limitation, several approach was developed as geometric approach in Phelps [1991], Keller [1987], Boutat et al. [2006], Boutat and Busawon [2011], Kazantzis and Kravaris [1998], Lynch and Bortoff [2001], Marino and Tomei [1996], Rudolph and Zeitz [1994], Noh et al. [2004], Jo and Seo [2002] ; and direct transformation in Lopez et al. [1999], Hou and Pugh [1999]. Also, several variants of observer normal forms was proposed, as the output depending normal forms which is introduced by Respondek et al. [2004] for single output dynamical systems and improved in Zheng et al. [2007], Wang and Lynch [2009]; and the extended nonlinear observer normal form which is studied in Jouan [2003], Back et al. [2006], Noh et al. [2004], Boutat [2007], Yang et al. [2011], Boutat and Busawon [2011].

For the SEIR model, the only accessible measurement is the infected population, it is provided by health department. We will show in the paper that the existing results in the literature are not sufficient to study the SEIR model. For this, the paper proposes a new extended output depending observer normal form, which mixes both the extended normal form (ENF) and the output depending nonlinear observer normal form Respondek et al. [2004], Krener and Respondek [1985], Zheng et al. [2007], Wang and Lynch [2009] and generalizes various previous results on this fields.

This paper is organized as follows. Section 2 presents the epidemic model, its observer normal form and the high gain observer. Section 3, recalls a background about nonlinear observer normal forms. It also states the motivation of this work by pointing out that the existing results are not sufficient. In section 4, a new extended output depending observer normal form is proposed, and a set of sufficient conditions are given to guarantee the transformation of nonlinear dynamical systems into the proposed form. Section 5 applies the proposed result to study SEIR model.

2. MATHEMATIC EPIDEMIC MODEL

Mathematical models are a useful tool to understand the dynamics of infectious diseases. The model considered here brings into play four variables; $S(t)$ the susceptibility

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of the host population to the contagious disease, $E(t)$ exposed population but not yet expressing symptoms, $I(t)$ infectious population and $R(t)$ is the recovered population. The SEIR model is given as follow

$$\begin{cases} \dfrac{dS}{dt} = bN - \mu S - \beta\dfrac{SI}{N} - pbE - qbI \\ \dfrac{dE}{dt} = \beta\dfrac{SI}{N} + pbE + qbI - (\mu+\varepsilon)E \\ \dfrac{dI}{dt} = \varepsilon E - (r+\delta+\mu)I \\ \dfrac{dR}{dt} = rI - \mu R \\ \dfrac{dN}{dt} = (b-\mu)N - \delta I \\ y = I \end{cases} \tag{1}$$

where b is the rate of the natural birth, μ rate of fecundity, the transmission rate is the parameter β, δ death rate related to diseases, ε is the rate at which the exposed population become infective, p is the rate of the offspring from exposed population, q is the rate of the offspring from infectious population and r is the rate at which the infected individuals are recovered.

The total population is given by

$$N = S + E + I + R \tag{2}$$

Throughout this paper, we will use the normalized model of (1), by setting : $x_1 = \frac{S}{N}$, $x_2 = \frac{E}{N}$, $x_3 = \frac{I}{N}$, $x_4 = \frac{R}{N}$. The equation (2) becomes

$$x_1 + x_2 + x_3 + x_4 = 1 \tag{3}$$

A straightforward calculation gives the normalized model

$$\dot{x}_1 = b - bx_1 + \gamma_1 x_1 x_3 - pbx_2 - qbx_3 \tag{4}$$
$$\dot{x}_2 = \beta x_1 x_3 + \gamma_2 x_2 + \delta x_2 x_3 + qbx_3 \tag{5}$$
$$\dot{x}_3 = \varepsilon x_2 + \gamma_3 x_3 + \delta x_3^2 \tag{6}$$
$$\dot{x}_4 = rx_3 - bx_4 + \delta x_3 x_4 \tag{7}$$
$$y = x_3 \tag{8}$$

where $\gamma_1 = -(\beta-\delta)$, $\gamma_2 = -(b+\varepsilon-pb)$ and $\gamma_3 = -(r+\delta+b)$. For the normalized model, $y = \frac{I}{N} := x_3$, is considered as output. To estimate x_2, and x_1 we use an observer, whereas x_4 is unobservable, then will be deduce form the algebraical equation (3) such that $x_4 = 1 - x_1 - x_2 - x_3$.

Later, we will show the both dynamical system (4-6) and the output $y = x_3$ does not fulfil any geometrical conditions existing in the literature which allow to transform it into observer normal form. However, as it will be showed below, if we add the following auxiliary dynamics

$$\dot{w} = -b + \gamma_1 x_3 \tag{9}$$

then, the diffeomorphism $z = \phi(x)$ given by

$$z_1 = \varepsilon\beta x_1 e^{-w} + bp\beta x_3 e^{-w}$$
$$z_2 = \varepsilon x_2 e^{-w} - (b+\gamma_2)x_3 e^{-w} - \frac{1}{2}(\delta-\gamma_1)x_3^2 e^{-w}$$
$$z_3 = x_3 e^{-w}$$
$$\xi = w$$

transforms both the dynamical system (4-6) and the auxiliary dynamics (9) into Extended Output Depending observer For

$$\begin{cases} \dot{z}_1 = B_1(w,y) \\ \dot{z}_2 = yz_1 + B_2(w,y) \\ \dot{z}_3 = z_2 + B_3(w,y) \\ \dot{\xi} = B_4(w,y) \\ \overline{y} = z_3 = ye^{-w} \end{cases} \tag{10}$$

where $\xi \in \mathbb{R}$, $w \in \mathbb{R}$, B_i are given in the last section, and $i = 1:3$.

Remark 1. The auxiliary variable w is considered as an extra output.

According to Busawon et al. [1998], if we have a dynamical system in the form

$$\begin{cases} \dot{z} = A(y)z + B(w,y) \\ \dot{\xi} = B_{n+1}(w,y) \\ \overline{y} = Cz \end{cases} \tag{11}$$

where $z \in \mathbb{R}^n$, $y \in \mathbb{R}$, $\xi \in \mathbb{R}$, $w \in \mathbb{R}$, $C = [0,...,0,1]$ and

$$A = \begin{pmatrix} 0 & ... & ... & ... & 0 \\ \alpha_2(y) & 0 & ... & ... & 0 \\ 0 & \alpha_3(y) & ... & ... & ... \\ 0 & ... & ... & ... & ... \\ 0 & ... & ... & \alpha_n(y) & 0 \end{pmatrix} \tag{12}$$

then we can use the high gain observer proposed in Busawon et al. [1998].

$$\dot{\widehat{z}} = A(y)\widehat{z} + B(w,y) - \Gamma^{-1}(y)R_\rho^{-1}C^T(C\widehat{z} - y)$$
$$0 = \rho R_\rho + G^T R_\rho - R_\rho G + C^T C$$

where G, Γ and R_ρ are a parameters defined, respectively by

$$G = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

$$\Gamma(y) = diag[\prod_{i=2}^{n}\alpha_i(y), \prod_{i=3}^{n}\alpha_i(y), \cdots, \alpha_n(y), 1]$$

$$R_\rho(n+1-i, n+1-j) = \frac{(-1)^{i+j}C_{i+j-2}^{j-1}}{\rho^{i+j-1}} \quad \text{for } 1 \leq i,j \leq n.$$

It can be shown that the observation error dynamics is governed by the following dynamics

$$\dot{e} = \dot{\widehat{z}} - \dot{z} = (A(y) - \Gamma^{-1}(y)R_\rho^{-1}C^T C)e$$

If y and w are bounded, then the observation error dynamics is exponentially stable by well choosing ρ.

3. SOME NONLINEAR OBSERVABILITY NORMAL FORMS

This section is firstly to recall some technical background of observer forms and at the same time to show the insufficiency of those existing results when studying SEIR model.

3.1 First Observability Normal Forms

Consider a single output nonlinear dynamical system in the following form

$$\begin{cases} \dot{x} = f(x) \\ y = h(x) \end{cases} \tag{13}$$

where $x \in U \subseteq \mathbb{R}^n$ is the state and $y \in \mathbb{R}$ is the output. We assume the pair (h, f) is smooth and satisfies the observability rank condition in the neighborhood of 0. Thus, the 1-forms

$$\theta_1 = dh$$
$$\theta_i = dL_f^{i-1}h \text{ for } 1 \leq i \leq n$$

are independent, where $L_f^k h$ is the k^{th} Lie derivative of h along f.

Now, we will construct the Kerner & Isodori frame (Krener and Isidori [1983]) $\tau = [\tau_1, ..., \tau_n]$ where the first vectors field τ_1 is given by the following algebraic equations

$$\begin{cases} \theta_i(\tau_1) = 0 \text{ for } 1 \leq i \leq n-1 \\ \theta_n(\tau_1) = 1 \end{cases}$$

and by induction, for $i = 2 : n$ we define

$$\tau_i = -ad_f \tau_{i-1} = [\tau_{i-1}, f]$$

where $[,]$ denotes the Lie bracket. It is well-known from Krener and Isidori [1983] that if

$$[\tau_i, \tau_j] = 0, \quad \text{for} \quad 1 \leq i, j \leq n \tag{14}$$

then the dynamical system (13) can be transformed, by means of diffeomorphism $z = \phi(x)$ into the following observer normal form

$$\begin{cases} \dot{z} = Az + B(y) \\ y = Cz \end{cases} \tag{15}$$

where $C = [0, ..., 0, 1]$ and A is a Brunovsky matrix. Let us show that conditions (14) are not satisfied by the SEIR epidemic dynamical system (4-6).

Example 1. We consider the SEIR epidemic model (4-6). A simple calculation gives its observability 1-forms as follows

$$\theta_1 = dx_3$$
$$\theta_2 = \varepsilon dx_2 + (\gamma_3 + 2\delta x_3)dx_3$$
$$\theta_3 = \varepsilon\beta x_3 dx_1 + \varepsilon(\gamma_2 + \gamma_3 + 3\delta x_3)dx_2 + Q_1 dx_3$$

where

$$Q_1 = \varepsilon\beta x_1 + 3\varepsilon\delta x_2 + \varepsilon qb + \gamma_3^2 + 6\delta\gamma_3 x_3 + 6\delta^2 x_3^2$$

and the Krener & Isidori associated frame is given by

$$\begin{aligned} \tau_1 &= \frac{1}{\varepsilon\beta x_3}\frac{\partial}{\partial x_1} \\ \tau_2 &= u\tau_1 + \frac{1}{\varepsilon}\frac{\partial}{\partial x_2} \quad , \quad u = (\gamma_3 - b) + (\delta + \gamma_1)x_3 + \varepsilon\frac{x_2}{x_3} \\ \tau_3 &= -\frac{pb}{\varepsilon}\frac{\partial}{\partial x_1} - (L_f u)\,\tau_1 + u\tau_2 + \frac{\gamma_2 + \delta x_3}{\varepsilon}\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} \end{aligned} \tag{16}$$

A direct calculation gives

$$\begin{aligned} [\tau_1, \tau_2] &= [\tau_1, \tau_3] = 0 \\ [\tau_2, \tau_3] &= Q_2\tau_1 + \frac{1}{x_3}\tau_2 \end{aligned} \tag{17}$$

where

$$Q_2 = -(3\delta + 2\gamma_1 + \frac{2\gamma_2 - \gamma_3}{x_3} - 3\varepsilon\frac{x_2}{x_3^2})$$

Therefore, conditions (14) are not satisfied. In the next, we consider an other observer normal form.

3.2 ***Second Observability Normal Form***

In the case where the above commutativity conditions (14) are not satisfied but the less restrictive following conditions are fulfilled

$$\begin{cases} [\tau_1, \tau_n] = \lambda_1(y)\tau_1 \\ [\tau_k, \tau_n] = \lambda_k(y)\tau_k \text{ modulo span}\{\tau_1, ..., \tau_{k-1}\} \\ \qquad \text{for} \quad k = 2 : n-2 \end{cases} \tag{18}$$

then according to Respondek et al. [2004], Zheng et al. [2007] or Wang and Lynch [2009] we can determine non vanishing functions of the output $\alpha_2(y), ..., \alpha_n(y)$ and construct a new frame as follows

$$\begin{cases} \overline{\tau}_1 = \pi\tau_1 \\ \overline{\tau}_i = \dfrac{1}{\alpha_i}[\overline{\tau}_{i-1}, f] \end{cases} \tag{19}$$

where $\pi = \alpha_2\alpha_3....\alpha_n$ is the product of α_i for $i = 2 : n$. If the new frame (19) commutes i.e:

$$[\overline{\tau}_i, \overline{\tau}_j] = 0, \quad \text{for} \quad 1 \leq i, j \leq n \tag{20}$$

then the dynamical system (13) can be transformed by means of diffeomorphism $z = \phi(x)$ into the following nonlinear observability normal form

$$\begin{cases} \dot{z} = A(y)z + B(y) \\ y = Cz \end{cases}$$

where the matrix $A(y)$ is defined as follows

$$A(y) = \begin{pmatrix} 0 & ... & ... & ... & 0 \\ \alpha_2(y) & 0 & ... & ... & 0 \\ 0 & \alpha_3(y) & ... & ... & ... \\ 0 & ... & ... & ... & ... \\ 0 & ... & ... & \alpha_n(y) & 0 \end{pmatrix} \tag{21}$$

Remarks *1.*

- This result has been reported by Respondek et al. [2004] for the case where $\alpha_2(y) = \cdots = \alpha_n(y)$ and by Zheng et al. [2007] for distinct $\alpha_i(y)$.
- We can assume that $\alpha_n(y) = 1$. In fact, if we set $z_n = \overline{y} = \int_0^y \frac{ds}{\alpha_n(s)}$ then $\dot{z}_n = z_{n-1}$. This enables us to compute only $\alpha_2(y), \cdots, \alpha_{n-1}(y)$.

Let us show that the condition (20) are not satisfied by the studied epidemic dynamical system (4-6).

Example 2. From (17), we notice that τ_i for $1 \leq i \leq 3$ computed in (16) satisfy the condition (18), where by identification between (17) and (18), we have $\lambda_1(y) = 0$ and $\lambda_2(y) = \frac{1}{x_3}$. Thus according to Zheng et al. [2007], using the second point of Remark 1, a simple calculation gives

$$\alpha_2 = x_3 \text{ and } \alpha_3 = 1$$

Thus, $\pi = x_3$ and from (19) we have

A straightforward calculation gives

$$\begin{cases} [\overline{\tau}_1, \overline{\tau}_2] = [\overline{\tau}_1, \overline{\tau}_3] = 0 \\ [\overline{\tau}_2, \overline{\tau}_3] = -2\dfrac{b}{x_3^2}\overline{\tau}_1 \end{cases} \tag{22}$$

which means that conditions (20) is not satisfied. In the next section, we introduce a new observability normal form to overcome this limitation.

4. EXTENDED OUTPUT DEPENDING NONLINEAR OBSERVABILITY NORMAL FORMS

This section is devoted to deducing sufficient geometrical conditions which guarantees the existence of an auxiliary dynamics $\dot{w} = \eta(y, w)$ such as the extended dynamical system of (13) defined as follows

$$\dot{x} = f(x) \quad (23)$$

$$\dot{w} = \eta(w, y) \quad (24)$$

$$y = h(x) \quad (25)$$

can be transformed by means of diffeomorphism $(z^T, \xi)^T = \phi(x, w)$ into the following nonlinear extended normal form

$$\dot{z} = A(y)z + B(w, y) \quad (26)$$

$$\dot{\xi} = B_{n+1}(w, y) \quad (27)$$

$$\overline{y} = Cz \quad (28)$$

where $z \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, $\overline{y} \in \mathbb{R}$, $\xi \in \mathbb{R}$, $w \in \mathbb{R}$ and $A(y)$ is given in (12). The case where the matrix $A(y)$ is constant was widely studied in Jouan [2003], Back et al. [2006], Noh et al. [2004], Boutat [2007], Boutat (2011).

Furthermore, if we assume that the conditions (18) are fulfilled, then we can determine functions $\alpha_2(y), ..., \alpha_n(y)$ and the frame $\overline{\tau}$ defined in (19). Moreover, if the conditions in (20) are not satisfied, then we can define a new frame as follows

$$\sigma_1 = l(w)\overline{\tau}_1$$

$$\sigma_k = \frac{1}{\alpha_k}[\sigma_{k-1}, F] \quad \text{for} \quad 2 \leq k \leq n$$

where function $l(w) \neq 0$ to be determined and $F = f + \eta(w, y)\frac{\partial}{\partial w}$ is the vector field of the extended dynamics.

***Theorem** 1.* If there exist a function $l(w)$ of the auxiliary variable w such that $[\sigma_i, \sigma_j] = 0$ for $1 \leq i, j \leq n$, then, there exist a coordinates change $(z^T, \xi)^T = \phi(x, w)$ which transforms the extended dynamical system (23-25) into the normal form (26-28)

***Proof** 1.* Assuming that there exists $l(w)$ such that $[\sigma_i, \sigma_j] = 0$ for $1 \leq i, j \leq n$.

Now, let σ_{n+1} be a vector field independent of σ_i for all $0 \leq i \leq n$, such that $[\sigma_i, \sigma_{n+1}] = 0$ for $1 \leq i \leq n$ and $dw(\sigma_{n+1}) = 1$.

To give the change of coordinates, we consider the multi-functions matrix $\Lambda = (\Lambda_{i,j})_{1 \leq i,j \leq n+1}$ defined as the evaluation of the 1-forms $\theta_i = dL_F^{i-1}h$ for $0 \leq i \leq n$ and $\theta_{n+1} = dw$ on the frame $\sigma = [\sigma_1, ..., \sigma_n, \sigma_{n+1}]$. Thus, we have $\Lambda_{i,j} = \theta_i(\sigma_j)$ for $1 \leq i, j \leq n+1$. This matrix has the following form

$$\Lambda = \theta\sigma = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & l & * \\ 0 & 0 & 0 & 0 & l\alpha_n & * & \vdots \\ 0 & \vdots & 0 & \ldots\vdots & * & \vdots & * \\ \vdots & 0 & \frac{l\pi}{\alpha_2\alpha_3} & * & \vdots & * & * \\ 0 & \frac{l\pi}{\alpha_2} & * & \cdots & * & * & * \\ l\pi & * & * & \cdots & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

It is clear that Λ is invertible, thus one can define the following multi 1-forms

$$\omega = \Lambda^{-1}\sigma = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \cdots \\ \omega_{n+1} \end{pmatrix} \quad (29)$$

we have $\omega_i\sigma_j = \delta_i^j$.

Let X and Y two vectors fields in $\{\sigma_1, ..., \sigma_n, \sigma_{n+1}\}$ we have

$$d\omega_i(X, Y) = L_Y\omega_i(X) - L_X\omega_i(Y) - \omega_i[X, Y]$$

As $\omega_i(X)$ and $\omega_i(Y)$ are constant, we have

$$d\omega_i(X, Y) = -\omega_i[X, Y].$$

Therefore, we have $[X, Y] = 0$ for all X, Y if and only if $d\omega_i = 0$ for all $1 \leq i \leq n+1$. Thus, by Poincaré's lemma there exist

$$\phi = (\phi_1, ..., \phi_{n+1})^T$$

such that

$$\omega = D\phi := \phi_*.$$

Let us set $\phi_*(\sigma_i) = \frac{\partial}{\partial z_i}$ for $i = 1 : n$ and $\phi_*(\sigma_{n+1}) = \frac{\partial}{\partial \xi}$.

Now, we will see how ϕ_* transforms the vector field F.

We have $\sigma_{i+1} = \frac{1}{\alpha_i}[\sigma_i, F]$, then for $0 \leq i \leq n$, we have

$$\left[\frac{\partial}{\partial z_i}, \phi_*(F)\right] = [\phi_*(\sigma_i), \phi_*(F)]$$

$$= \phi_*[\sigma_i, F] = \alpha_i\phi_*\sigma_{i+1}$$

$$= \alpha_i\frac{\partial}{\partial z_{i+1}}$$

Thus, by integration we obtain $\dot{z}_i = \alpha_i(y)z_i + \beta_i(y, w)$

We finish this section by some remarks.

***Remarks** 2.*

- A difeomorphism is given by the integration of (29) such that $z_i = \phi_i(x) = \int \omega_i$.
- If the vector field σ_{n+1} is obtained by induction as $\sigma_{n+1} = [\sigma_n, F]$, then the normal form become $\dot{z} = A(y)z + B(w)$, where the second term $B(w)$ depends only on the auxiliary variable w.
- The determination of non vanishing functions α_i is clearly explained in Zheng et al. [2007].
- We can adopt algorithm of Back et al. [2006], Boutat [2007],Yang et al. [2011] ,or Boutat and Busawon [2011] to compute the function $l(w)$ and $\eta(w, y)$.

5. SEIR MODEL CONTINUED

In this section, we will use the above result to compute the diffeomorphism transforming the SEIR model (4-6) into the observer normal form (30).

Based on the calculations in Example 1 and 2, one has

$$\alpha_2 = x_3, \ \alpha_3 = 1$$

and with

$$\begin{cases} [\overline{\tau}_1, \overline{\tau}_2] = [\overline{\tau}_1, \overline{\tau}_3] = 0 \\ [\overline{\tau}_2, \overline{\tau}_3] = -2\frac{b}{x_3^2}\overline{\tau}_1 \end{cases}$$

Then, we will seek an auxiliary $\dot{w} = \eta(w, y)$ and a non zero function $l(w)$ which fulfill the condition of the theorem 1. For this, set

$$\sigma_1 = l(w)\overline{\tau}_1$$

which gives

$$\sigma_1 = \frac{l}{\varepsilon\beta}\frac{\partial}{\partial x_1}$$

Now, we have

$$\sigma_2 = \frac{1}{x_3}[\sigma_1, F] = \frac{1}{x_3}(lH - \eta l')\sigma_1 + \frac{l}{\varepsilon}\frac{\partial}{\partial x_2}$$

where $H = -b + \gamma_1 x_3$.

Afterwards, we have

$$\sigma_3 = -\frac{pb}{\varepsilon} l \frac{\partial}{\partial x_1} + \frac{1}{\varepsilon}(lH - \eta l')\sigma_2 - \frac{G}{\varepsilon l}\sigma_1 + \frac{1}{\varepsilon}(l(\gamma_2 + qx_3) - \eta l')\frac{\partial}{\partial x_2} + l\frac{\partial}{\partial x_3}$$

where

$G = \varepsilon^2 bl \frac{x_2}{x_3} - \varepsilon^2 l' x_2 \left(\frac{\eta' x_3 - \eta}{x_3^2}\right) + \eta\left(\frac{\varepsilon}{x_3}(l'H - \eta l'')\right)\sigma_1$.

Finally, we obtain

$$[\sigma_2, \sigma_3] = 2(\varepsilon^2 l' l \frac{\eta' x_3 - \eta}{x_3} - \frac{b\varepsilon^2 l^2}{x_3^2})\sigma_1$$

$$[\sigma_2, \sigma_3] = 0 \Longleftrightarrow l' l(\eta' x_3 - \eta) - bl^2 = 0$$

The solutions of the above equation are

$$l = e^w \quad \text{and} \quad \eta = \gamma_1 x_3 - b$$

Thus, the extended dynamical system is given by

$$F = \begin{pmatrix} \dot{x}_1 = b - bx_1 + \gamma_1 x_1 x_3 - pbx_2 - qbx_3 \\ \dot{x}_2 = \beta x_1 x_3 + \gamma_2 x_2 + \delta x_2 x_3 + qbx_3 \\ \dot{x}_3 = \varepsilon x_2 + \gamma_3 x_3 + \delta x_3^2 \\ \dot{w} = \gamma_1 x_3 - b \\ y = x_3 \end{pmatrix}$$

By applying the algorithm previously described, we obtain

$$\sigma_1 = \frac{e^w}{\varepsilon\beta}\frac{\partial}{\partial x_1}, \quad \sigma_2 = \frac{e^w}{\varepsilon}\frac{\partial}{\partial x_2}$$

$$\sigma_3 = -\frac{e^w}{\varepsilon} pb \frac{\partial}{\partial x_1} + \frac{e^w}{\varepsilon}(\gamma_2 + b + (\delta - \gamma_1)x_3)\frac{\partial}{\partial x_2} + e^w \frac{\partial}{\partial x_3}$$

consequently, we can check that

$$[\sigma_1, \sigma_2] = 0, \quad [\sigma_1, \sigma_3] = 0, \text{ and } [\sigma_2, \sigma_3] = 0$$

To complete the dimension of the frame, we should find σ_4 that commutes with σ_1, σ_2 and σ_3. For this, we choose it as follows

$$\sigma_4 = \frac{\partial}{\partial w} + x_1 \frac{\partial}{\partial x_1} + (x_2 + \frac{\delta - \gamma_1}{2\varepsilon} x_3^2)\frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}$$

It is easy to calculate the observability maps θ_e, where

$$\theta_e = [dh, dL_F h, \ldots .dL_F^{n-1}, dw]^T$$

then

$$\Lambda = \begin{pmatrix} 0 & 0 & e^w & x_3 \\ 0 & e^w & \Lambda_{23} & \Lambda_{24} \\ x_3 e^w & e^w(\gamma_2 + \delta x_3) & \Lambda_{33} & \Lambda_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$\Lambda_{23} = e^w(b + \gamma_2 + x_3(\delta - \gamma_1))$$

$$\Lambda_{24} = \varepsilon x_2 + (\delta - \gamma_1)\frac{x_3^2}{2}$$

$$\Lambda_{33} = e^w(\gamma_2 + \delta x_3)(b + \gamma_2 + x_3(\delta - \gamma_1)) + \varepsilon e^w(\beta x_1 + \delta x_2 + bq) - bp\beta x_3 e^w$$

$$\Lambda_{34} = \varepsilon x_3(\beta x_1 + \delta x_2 + bq) + \varepsilon(\gamma_2 + \delta x_3)\left(x_2 + \frac{1}{2\varepsilon}x_3^2(\delta - \gamma_1)\right) + \beta\varepsilon x_1 x_3$$

consequently

$$\omega = \Lambda^{-1}\theta_e = \phi_* = dz$$

$$= d\begin{pmatrix} \varepsilon\beta x_1 e^{-w} + bp\beta x_3 e^{-w} \\ \varepsilon x_2 e^{-w} - (b + \gamma_2)x_3 e^{-w} - \frac{1}{2}(\delta - \gamma_1)x_3^2 e^{-w} \\ x_3 e^{-w} \\ w \end{pmatrix}$$

Finally, the normal form of system (4-6) become as follow

$$\begin{cases} \dot{z}_1 = B_1(w, y) \\ \dot{z}_2 = yz_1 + B_2(w, y) \\ \dot{z}_3 = z_2 + B_3(w, y) \\ \dot{w} = \gamma_1 y - b \\ \overline{y} = ye^{-w} \end{cases} \tag{30}$$

where

$$B_1(w, y) = b\beta(\varepsilon + (p(b + \gamma_3) - q\varepsilon)y + p(\delta - \gamma_1)y^2)e^{-w}$$

$$B_2(w, y) = -\beta bpy + (\delta^2 + \frac{\gamma_1}{2}(\gamma_1 - 3\delta))e^{-w}y^3 - (\delta(\gamma_2 + \gamma_3 + \frac{3}{2}b) - 2\gamma_1\gamma_3 - \frac{3}{2}b\gamma_1)e^{-w}y^2 - (b(\gamma_2 + \gamma_3 + b - q\varepsilon) + \gamma_2\gamma_3)e^{-w}y$$

$$B_3(w, y) = (3b + \gamma_2 + \gamma_3)e^{-w}y + \frac{3}{2}(\delta - \frac{5}{3}\gamma_1)e^{-w}y^2$$

For the simulation, we use the same parameters as in Li and Muldowney [1996] where $N = 141$, $b = 0.221176/N$, $d = 0.002$, $p = 0.8$, $q = 0.95$, $\beta = 0.05$, $\varepsilon = 0.05$, $r = 0.003$. $S(0) = 140$, $E(0) = 0.01$, $I(0) = 0.02$, $N(0) = 141$. The high gain observer Busawon et al. [1998] is used for the states estimation. The results are presented in Fig. 1-3, which highlight the feasibility of the proposed approach.

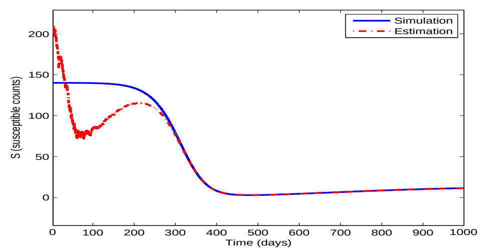

Fig. 1. Evolution of suspect population

6. CONCLUSION

Since the existing results on observer normal forms are not satisfied when studying SEIR model, this paper introduced a new form, which mixes both the extended and the output

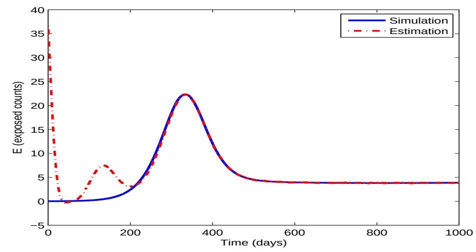

Fig. 2. Evolution of exposed population

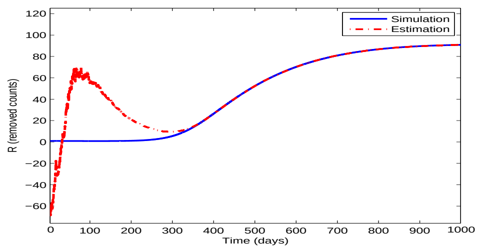

Fig. 3. Evolution of removed population

depending normal form. This last enables to design a high gain observer. Sufficient conditions was given in order to guarantee the existence of a diffeomorphism allowing to transform the SEIR model into the proposed observer normal form. Finally, the proposed result was successfully applied to the SEIR model to estimate the evolution of different populations during a contagious disease.

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