Stabilisation of a Nonlinear Flexible Beam in Port-Hamiltonian Form

Alessandro Macchelli

University of Bologna, Department of Electrical, Electronic, and Information Engineering "Guglielmo Marconi"
email: alessandro.macchelli@unibo.it

Abstract: The aim of this paper is to present a simple extension of the theory of linear, distributed, port-Hamiltonian systems to the nonlinear scenario. More precisely, an algebraic nonlinear skew-symmetric term has now been included in the PDE. It is then shown that the system can be equivalently written in terms of the scattering variables, and that these variables are strictly related with the Riemann invariants that appear in quasi-linear hyperbolic PDEs. For this class of PDEs, several results about the existence of solutions, and asymptotic stability of equilibria have already been presented in literature. Here, these results have been extended and applied within the port-Hamiltonian framework, where are suitable of a nice physical interpretation. The final scope is the boundary asymptotic stabilisation of a nonlinear flexible beam with a free-end, and full actuation on the other side.

Keywords: Passivity, stabilisation, nonlinear distributed parameter systems

1. INTRODUCTION

In Macchelli et al. [2007, 2009], a nonlinear model of a flexible link able to describe finite deformations in the 3-D space has been presented. Following Simo [1985], the configuration in space of the link cross section is described by an homogeneous transformation, and the resulting model is written as time evolution of elements in $se(3) \times se^*(3)$. The model has been written in distributed port-Hamiltonian form (see van der Schaft and Maschke [2002], Macchelli and Maschke [2009]), since the geometric hypotheses behind the model itself naturally define a StokesDirac structure, the fundamental geometric structure behind each port-Hamiltonian system, as discussed e.g. in Maschke and van der Schaft [1992], van der Schaft [2000], Duindam et al. [2009]. Due to the presence of a nonlinear, skew-symmetric, algebraic coupling between rotational and translational motion, most of the tools already presented in literature are not suitable for studying the existence of solution for a given set of (possibly) time-varying boundary conditions, and the stability of equilibrium configuration. The main reason is that most of the research activity has been focused on the linear case, for which several results about existence of solution (Le Gorrec et al. [2005], Jacob and Zwart [2012]), energy-based methods (Macchelli and Melchiorri [2004, 2005], Macchelli [2012b,a]), and stability analysis (Villegas et al. [2005, 2009]), are now available.

In this paper, a simple class of non-linear, hyperbolic, port-Hamiltonian systems has been introduced, and boundary conditions have been selected in a simple way to have the resulting system in impedance form. Due to the fact that nonlinearity appears just in a skew-symmetric algebraic terms, and the Hamiltonian is still quadratic, these distributed port-Hamiltonian systems will be called "almost linear" in this paper. Furthermore, it is also verified that such class of systems is, at the end, a particular case of the so-called quasi-linear hyperbolic systems, for which several results about existence of solutions for a given set of time varying boundary conditions, together with tools for the stability analysis, have been already presented in literature. In this paper, we refer mostly to Ta-Tsien [1994], Prieur et al. [2008], and we show that there always exists a coordinate transformation that maps an almost linear distributed port-Hamiltonian systems into a quasi-linear hyperbolic systems. So, it turns out that this class of infinite dimensional port-Hamiltonian systems is just a particular case of quasi-linear hyperbolic systems. This transformations, however, has the advantage of pointing out that the Riemann invariants are equivalent to the scattering variables, and the formulation of the system dynamics in terms of these invariants equivalent to the scattering formulation of a port-Hamiltonian system. More precisely, the scattering variables follow from the scattering representation of the Dirac structure, once the metric defined by the energy function is adopted to perform the scattering decomposition. This is in line with what has been presented in Macchelli et al. [2002], but only as far the boundary terms are concerned.

For the given class of almost linear distributed port-Hamiltonian systems, a simple result on the existence of solutions for a given algebraic input-output mapping is presented. It is the relation between Riemann invariants and scattering variables that allows for an intuitive physical interpretation of the result itself. Moreover, an immediate consequence is a tool for analysing, or achieving, asymptotic stability of a constant equilibrium by boundary control. These results are quite general, and have been successfully applied to the nonlinear flexible link model, more precisely, when the end of the link is free, and full actuation is on the other side. In this case, the boundary conditions have been selected in order to have solutions in

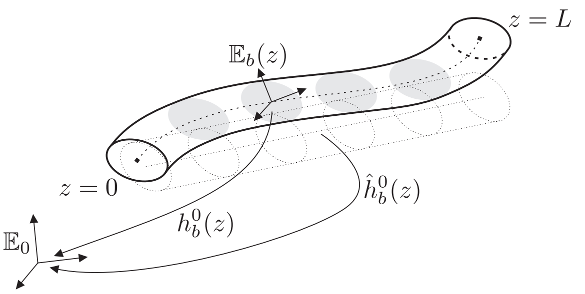

Fig. 1. Flexible link in the deformed and unstressed configurations, Macchelli et al. [2007, 2009].

closed-loop, together with local asymptotic stability of the unstressed configuration. Once mapped to the link model in impedance form, the stabilising relation is imposing nothing else than full boundary dissipation at one side of the link. In this respect, this is a generalisation to the nonlinear case of Villegas et al. [2009], Macchelli [2012b,a] that are valid in the linear scenario. Finally, it is worth nothing that with this paper it is shown that Riemann invariants and scattering representation of port-Hamiltonian systems could be a valuable tool for tackling nonlinearities. The next step is to deal with distributed port-Hamiltonian systems characterised not only by a nonlinear Dirac structure, but also by a nonlinear energy function.

The paper is organised as follows. The nonlinear model of the flexible link in port-Hamiltonian form is briefly recalled in Sect. 2. Then, in Sect. 3, the class of nonlinear distributed port-Hamiltonian systems is introduced, and the main results about the existence of solution and stability analysis are provided. Such results are applied to the boundary stabilisation of the flexible beam in Sect. 4, while in Sect. 5 conclusions and idead for future researches are given.

2. FLEXIBLE LINK MODEL

In Macchelli et al. [2007, 2009], the following model of a flexible beam in port-Hamiltonian form has been introduced:

$$\begin{cases} \dfrac{\partial q}{\partial t} = \dfrac{\partial}{\partial z}\dfrac{\delta H}{\delta p} + \mathrm{ad}_{(q+\hat{n})}\dfrac{\delta H}{\delta p} \\ \dfrac{\partial p}{\partial t} = \dfrac{\partial}{\partial z}\dfrac{\delta H}{\delta q} - \mathrm{ad}^*_{(q+\hat{n})}\dfrac{\delta H}{\delta q} + p \wedge \dfrac{\delta H}{\delta p} \end{cases} \tag{1}$$

As reported in Fig. 1, if L is the length of the link, for all $z \in Z \equiv [0, L]$ position and orientation of the cross section with respect to an inertial reference $\mathbb{E}_0$ is given by $h_b^0(z) \in SE(3)$, where the subscript "b" denotes the body reference $\mathbb{E}_b$ attached to the cross-section, Simo [1985]. The unstressed configuration, which is *not* required to be a straight line, is denoted by $\hat{h}_b^0(z)$.

In (1), $q(t, z)$ and $p(t, z)$ denote the infinitesimal deformation and momentum of the cross-section, that are mathematically described by C^1 functions from Z to $se(3)$ and $se^*(3)$, respectively. Moreover, given $z \in Z$, $\hat{n}(z)$ represents the "direction" along which the unstressed configuration "evolves." All these quantities are expressed in body frame, i.e. $\mathbb{E}_b(z)$. The function H is the Hamiltonian (energy) function given by the integration on Z of the sum of a kinetic energy density $\mathcal{K}(p) = 1/2\,\langle p \mid p\rangle_Y$, and a potential elastic energy density $\mathcal{W}(q) = 1/2\,\langle q \mid q\rangle_{C^{-1}}$:

$$H(q,p) = \frac{1}{2}\int_Z \Big(\langle p \mid p\rangle_Y + \langle q \mid q\rangle_{C^{-1}}\Big)\mathrm{d}z \tag{2}$$

Here, Y denotes the inverse of the inertia tensor I_ρ of the cross-section, i.e. $Y = I_\rho^{-1}$, which defines a quadratic form on $se^*(3)$, while C is the compliance tensor describing the (supposed linear) elastic behaviour of the link, whose inverse C^{-1} defines a quadratic form on $se(3)$. Moreover, $\langle\cdot \mid \cdot\rangle$ is the inner product defined by a proper metric, i.e. by Y on $se^*(3)$ and by C^{-1} on $se(3)$. In (1), δ denotes the variational derivative (see van der Schaft and Maschke [2002]), $p \wedge {}^{\delta H}/_{\delta p} \equiv \mathrm{ad}^*_{\delta_p H} p$ (see Stramigioli [2001]), while

$$\hat{n}(z) = \Big(\hat{h}_b^0(z)\Big)^{-1}\frac{\partial \hat{h}_b^0}{\partial z}(z) \in se(3)$$

The flexible link exchanges power with the environment through a couple of power ports (i.e. a pair twist/wrench) defined in $z = 0$ and $z = L$:

$$\begin{aligned} \big(T_0(t), W_0(t)\big) &= \left(\frac{\delta H}{\delta p}(t,0), -\frac{\delta H}{\delta q}(t,0)\right) \\ \big(T_L(t), W_L(t)\big) &= \left(\frac{\delta H}{\delta p}(t,L), \frac{\delta H}{\delta q}(t,L)\right) \end{aligned} \tag{3}$$

These quantities are expressed in body frame, i.e. in $\mathbb{E}_b(0)$ and $\mathbb{E}_b(L)$, and represent the boundary conditions of the distributed parameter system. Clearly $T_0, T_L \in se(3)$ and $W_0, W_L \in se^*(3)$. Since no dissipative effect is considered, (1) satisfies the following energy balance condition:

$$\frac{\mathrm{d}H}{\mathrm{d}t}(t) = \langle W_L(t), T_L(t)\rangle + \langle W_0(t), T_0(t)\rangle \tag{4}$$

where $\langle\cdot,\cdot\rangle$ is the dual product on $se(3) \times se^*(3)$.

3. "ALMOST LINEAR" HYPERBOLIC PORT-HAMILTONIAN SYSTEMS

In coordinates, and if a matrix representation of the group operations is adopted, the distributed port-Hamiltonian system (1) with boundary conditions (3) belongs to the following class of systems:

$$\frac{\partial x}{\partial t}(t,z) = P_1\frac{\partial}{\partial z}\big(\mathcal{L}(z)x(t,z)\big) + P_0(x,z)\mathcal{L}(z)x(t,z) \tag{5}$$

Such class generalises what has been presented in Le Gorrec et al. [2005] in the linear case. Here, $x \in \mathbb{R}^n$ and $z \in Z \equiv [a, b]$. Moreover, $P_1 = P_1^{\mathrm{T}} > 0$, and $P_0(\cdot,\cdot) = -P_0^{\mathrm{T}}(\cdot,\cdot)$, while $\mathcal{L}(\cdot)$ is a bounded continuously differentiable matrix-valued function such that $\mathcal{L}(z) = \mathcal{L}^{\mathrm{T}}(z)$ and $\mathcal{L}(z) \geq \kappa I$, with $\kappa > 0$ for all $z \in Z$. For simplicity, $\mathcal{L}(z)x(t,z) \equiv (\mathcal{L}x)(t,z)$. Note that the entries in P_0 can be non-linear.

Differently from Le Gorrec et al. [2005] where the state space is $L_2([a, b]; \mathbb{R}^n)$, we assume that the state space is $X = C^1([a, b]; \mathbb{R}^n)$. This hypothesis is necessary since we want to rely directly on Prieur et al. [2008] as far as the existence of solutions and their stability analysis is concerned, i.e. on Theorem 3.1. The distributed port-Hamiltonian system (5) is characterised by the following Hamiltonian function

$$H(x(t,z)) = \frac{1}{2}\int_a^b x^{\mathrm{T}}(t,z)\mathcal{L}(z)x(t,z)\,\mathrm{d}z$$

Similarly to the finite dimensional case, X is also called the space of energy variables, and $\mathcal{L}x$ is the co-energy variable. This class is quite general and includes models of flexible structures, traveling waves, heat exchangers, and bioreactors among others (if also dissipative effects are included, Villegas et al. [2009]).

To define a distributed port-Hamiltonian system, the PDE (5) has to be "completed" by a boundary port. More precisely, given $\mathcal{L}x \in C^1([a, b]; \mathbb{R}^n)$, the boundary port variables associated to (5) are the vectors $f_\partial, e_\partial \in \mathbb{R}^n$ defined by

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix} \begin{pmatrix} (\mathcal{L}x)(b) \\ (\mathcal{L}x)(a) \end{pmatrix} \tag{6}$$

The boundary port variables are just a linear combination of the restriction on the boundary of the spatial domain of the co-energy variables, and simple integration by parts shows that

$$\begin{aligned} \frac{\mathrm{d}H}{\mathrm{d}t}(x(t)) &= \frac{1}{2} \begin{pmatrix} (\mathcal{L}x)(t, b) \\ (\mathcal{L}x)(t, a) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} P_1 & 0 \\ 0 & -P_1 \end{pmatrix} \begin{pmatrix} (\mathcal{L}x)(t, b) \\ (\mathcal{L}x)(t, a) \end{pmatrix} \\ &= e_\partial^{\mathrm{T}}(t) f_\partial(t) \end{aligned} \tag{7}$$

Inputs and outputs have to be defined in order to have a so-called boundary control system, Curtain and Zwart [1995]. From Le Gorrec et al. [2005], a simple procedure to have system (5) in impedance form is the following. Let $\hat{W}$ and $\tilde{W}$ a pair of $n \times 2n$ full rank real matrices, such that $\begin{pmatrix} \hat{W}^{\mathrm{T}} & \tilde{W}^{\mathrm{T}} \end{pmatrix}$ is invertible, and

$$\hat{W} \Sigma \hat{W}^{\mathrm{T}} = 0 \qquad \hat{W} \Sigma \tilde{W}^{\mathrm{T}} = I \qquad \tilde{W} \Sigma \tilde{W}^{\mathrm{T}} = 0 \tag{8}$$

being

$$\Sigma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{9}$$

The (boundary) input u and output y can be defined as

$$u(t) = \hat{W} \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} \qquad y(t) = \tilde{W} \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} \tag{10}$$

and it is easy to prove that the following energy balance equation is satisfied:

$$\frac{\mathrm{d}}{\mathrm{d}t} H(t) = y^{\mathrm{T}}(t) u(t) \tag{11}$$

Proposition 3.1. There always exists a coordinate change that puts (5) in the following form:

$$\frac{\partial \xi}{\partial t}(t, z) = \Lambda(z) \frac{\partial \xi}{\partial z}(t, z) + M(\xi, z)\xi(t, z) \tag{12}$$

where $\Lambda(z)$ is diagonal, and the vector function $M(\xi, z)\xi$ that groups the nonlinear terms is of class $C^1([a, b]; \mathbb{R}^n)$.

Proof. Denote by $\sqrt{\mathcal{L}}(z)$ the symmetric square root of $\mathcal{L}(z)$, i.e. $\mathcal{L} = \sqrt{\mathcal{L}}\sqrt{\mathcal{L}}$, and by $\Phi(z)$ the unitary matrix, i.e. $\Phi\Phi^{\mathrm{T}} = I$, that diagonalizes the symmetric matrix $\sqrt{\mathcal{L}} P_1 \sqrt{\mathcal{L}}$. This means that

$$\sqrt{\mathcal{L}}(z) P_1 \sqrt{\mathcal{L}}(z) = \Phi^{\mathrm{T}}(z) \Lambda(z) \Phi(z) \tag{13}$$

where

$$\Lambda(z) = \begin{pmatrix} \Lambda_-(z) & 0 \\ 0 & -\Lambda_+(z) \end{pmatrix} \tag{14}$$

In (14), we can chose Φ is such a way that Λ_- contains the positive eigenvalues of Λ, while $-\Lambda_+$ the negative ones. Clearly, the entries of both Λ_- and Λ_+ are positive. Then, simple computations show that

$$\xi(t, z) = \Phi(z) \sqrt{\mathcal{L}}(z) x(t, z) \tag{15}$$

is the coordinate change that maps the PDE (5) into (12).

Remark 3.1. From (15), the total energy can be written as

$$\bar{H}(t) = \frac{1}{2} \int_a^b \xi^{\mathrm{T}}(t, z) \xi(t, z) \, \mathrm{d}z \tag{16}$$

Moreover, let us now write $\xi = (\xi_-, \xi_+)$ with the dimensions chosen having the block partition of Λ in (14) in mind. Then, (12) satisfies the following energy-balance relation:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \bar{H} = \frac{1}{2} \Big[&\xi_-^{\mathrm{T}}(b) \Lambda_-(b) \xi_-(b) - \xi_+^{\mathrm{T}}(b) \Lambda_+(b) \xi_+(b) \Big] + \\ &+ \frac{1}{2} \Big[\xi_+^{\mathrm{T}}(a) \Lambda_+(a) \xi_+(a) - \xi_-^{\mathrm{T}}(a) \Lambda_-(a) \xi_-(a) \Big] \end{aligned} \tag{17}$$

The quantity ξ_+ is associated to an amount of power "flowing" from $z = a$ to $z = b$, while ξ_- is going in the opposite directions. It is then natural to define this new set of boundary terms

$$\begin{aligned} \big(s_{+,a}(t), s_{-,a}(t)\big) &= \big(\xi_+(t, a), \xi_-(t, a)\big) \\ \big(s_{+,b}(t), s_{-,b}(t)\big) &= \big(\xi_+(t, b), \xi_-(t, b)\big) \end{aligned} \tag{18}$$

that realises the scattering decomposition of the boundary port (f_∂, e_∂), see Macchelli et al. [2002]. In this respect, (12) provides the dynamics in term of the scattering variables.

Remark 3.2. In the context of quasi-linear hyperbolic PDEs, (12) is the dynamics in terms of the Riemann invariants or coordinates, Ta-Tsien [1994]. Note that, if $M(\xi, z) = 0$, then (12) is equivalent to the following set of PDEs:

$$\frac{\partial \xi_i}{\partial t}(t, z) = \lambda_i(z) \frac{\partial \xi_i}{\partial z}(t, z), \quad i = 1, \ldots, n \tag{19}$$

Moreover, it is easy to prove that $\dot{\xi}_i(t, z(t)) = 0$, i.e. each ξ_i is constant, along the "line" $\dot{z}(t) + \lambda_i(z(t)) = 0$. Clearly, z is increasing if $\lambda_i < 0$, decreasing if $\lambda_i > 0$. This explains why ξ_+ is related to some amount of power flowing in the positive direction of the spatial domain, while ξ_- to power travelling in the opposite direction. In case $M(\xi, z) \neq 0$, this property is lost, but it is still important in the study of the existence of solutions of (12), and in the stability analysis under the hypothesis that this term is bounded in a neighbourhood of the equilibrium. See Theorem 3.1 for more details.

Remark 3.3. Assume that $\mathcal{L}$ and P_0 in (5) do not depend on z. Then, it is possible to recover for (12) a port-Hamiltonian representation, where now the state variables are precisely the scattering variables:

$$\begin{aligned} \frac{\partial}{\partial t} \begin{pmatrix} \xi_-(t, z) \\ \xi_+(t, z) \end{pmatrix} &= \begin{pmatrix} \Lambda_- & 0 \\ 0 & -\Lambda_+ \end{pmatrix} \frac{\partial}{\partial z} \begin{pmatrix} \xi_-(t, z) \\ \xi_+(t, z) \end{pmatrix} \\ &\quad + \bar{P}_0(\xi_-(t, z), \xi_+(t, z)) \begin{pmatrix} \xi_-(t, z) \\ \xi_+(t, z) \end{pmatrix} \end{aligned} \tag{20}$$

with $\bar{P}_0 = \sqrt{\mathcal{L}} \Phi^{\mathrm{T}} P_0 \Phi \sqrt{\mathcal{L}}$. It turns out that the constant symmetric and positive definite matrix $\mathcal{L}$ is the metric that is employed in the scattering decomposition of the Stokes–Dirac structure of the distributed port-Hamiltonian system (5), see Stramigioli et al. [2002], van der Schaft [2009]. For this linear port-Hamiltonian system, a "physical" choice for boundary inputs and outputs can be the following:

$$u_\xi(t) = \begin{pmatrix} s_{-,b}(t) \\ s_{+,a}(t) \end{pmatrix} \qquad y_\xi(t) = \begin{pmatrix} s_{-,a}(t) \\ s_{+,b}(t) \end{pmatrix} \tag{21}$$

The input u is associated to the "incoming" power flow, while the output y to the outgoing one. Under mild assumptions, the distributed port-Hamiltonian system (20) or, equivalently, (5) turns out to be a boundary control system in the sense of Curtain and Zwart [1995], as it can be verified by applying the techniques presented in Le Gorrec et al. [2005].

Inputs and outputs have to be specified also in the proposed almost linear scenario (12) or (5), and in spite of Remark 3.2 and (17) the choice (21) is still valid. Simple computations show that for (5)

$$\begin{pmatrix} u_\xi \\ y_\xi \end{pmatrix} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \end{pmatrix} \times$$

$$\times \begin{pmatrix} \left(\Phi\sqrt{\mathcal{L}^{-1}}\right)(b) & 0 \\ 0 & \left(\Phi\sqrt{\mathcal{L}^{-1}}\right)(a) \end{pmatrix} \begin{pmatrix} (\mathcal{L}x)(b) \\ (\mathcal{L}x)(a) \end{pmatrix} \tag{22}$$

and, from (22), it is it easy to verify that there exists an invertible mapping between (u_ξ, y_ξ) and (u, y) defined in (10), so particular choices in the boundary conditions in the scattering representation can be mapped into the impedance description, and vice-versa.

With the choice (21), existence of solutions together with some considerations on the stability of the zero equilibrium for (12) are discussed later, with reference to Prieur et al. [2008]. In this respect, assume that the following static relation holds:

$$u_\xi(t) = g(y_\xi(t)) \tag{23}$$

Let us assume that the function g is continuously differentiable, is defined is a neighbourhood of the origin, and satisfies $g(0) = 0$. The first step is to investigate under which hypotheses, given an initial condition $\xi^\sharp \in C^1([a, b]; \mathbb{R}^n)$ that satisfies the compatibility conditions

$$\begin{gathered} \begin{pmatrix} \xi^\sharp_-(b) \\ \xi^\sharp_+(a) \end{pmatrix} = g\begin{pmatrix} \xi^\sharp_-(a) \\ \xi^\sharp_+(b) \end{pmatrix} \\ \begin{pmatrix} \Lambda_-(b)\dfrac{\partial \xi^\sharp_-}{\partial z}(b) + M_-(\xi^\sharp(b), b)\xi^\sharp(b) \\ \Lambda_+(a)\dfrac{\partial \xi^\sharp_+}{\partial z}(a) + M_+(\xi^\sharp(a), a)\xi^\sharp(a) \end{pmatrix} = \\ = \nabla g\begin{pmatrix} \xi^\sharp_-(a) \\ \xi^\sharp_+(b) \end{pmatrix} \begin{pmatrix} \Lambda_-(a)\dfrac{\partial \xi^\sharp_-}{\partial z}(a) + M_-(\xi^\sharp(a), a)\xi^\sharp(b) \\ \Lambda_+(b)\dfrac{\partial \xi^\sharp_+}{\partial z}(b) + M_+(\xi^\sharp(b), b)\xi^\sharp(a) \end{pmatrix} \end{gathered} \tag{24}$$

system (12) admits a solution. A second important point is when the zero equilibrium is asymptotically stable, at least locally. Moreover, from Proposition 3.1 and (22), it is clear that (23) can be transformed into a relation involving u and y defined in (10), and vice-versa: consequently, the results dealing with the existence of solutions and stability are also valid for the initial distributed port-Hamiltonian system (5).

Before stating the main result, some further definitions are necessary, see Prieur et al. [2008]:

- Given $\xi \in \mathbb{R}^n$, we define
$$|\xi| := \max(|\xi_i|, \ i = 1, \dots, n)$$
and we denote by $B(\epsilon)$ the ball centered in $0 \in \mathbb{R}^n$ and radius $\epsilon > 0$.
- Given $f_0 \in C^0([a, b]; \mathbb{R}^n)$ and $f_1 \in C^1([a, b]; \mathbb{R}^n)$, we denote
$$|f_0|_{C^0(a,b)} = \max_{z\in[a,\, b]} |f_0(z)|$$
$$|f_1|_{C^1(a,b)} = |f_1|_{C^0(a,b)} + |f_1'|_{C^0(a,b)}$$
- $B_C(\epsilon)$ denotes the set of functions $\xi^\sharp \in C^1([a, b]; \mathbb{R}^n)$ that satisfy the compatibility conditions (24), and such that $|\xi^\sharp|_{C^1(a,b)} \leq \epsilon$.
- Given a matrix $A = (a_{ij})$, $\rho(A)$ denotes its spectral radius, and $\mathrm{abs}(A) = (|a_{ij}|)$.

Theorem 3.1. (Prieur et al. [2008]). Let us consider (12), with boundary input and output given by (21), and such that (23) holds. Given $\epsilon_0 > 0$ and $M > 0$, if

$$\rho\left(\mathrm{abs}\left(\nabla g(0)\right)\right) < 1 \tag{25}$$

and

$$\left|\nabla\left(M(\xi)\xi\right)\right|_{\xi=0} \leq M \tag{26}$$

then there exists $0 < \epsilon_1 < \epsilon_0$, $\mu > 0$ and $C > 0$ such that, for all continuously differentiable $\xi^\sharp \in B_C(\epsilon_1)$, there exists an unique function $\xi \in C^1([0, L] \times [0, +\infty]; \mathbb{R}^n)$ satisfying (12), the boundary conditions (23), and the initial condition $\xi(0, z) = \xi^\sharp(z)$, $\forall z \in [0, L]$. Moreover, this function satisfies

$$|\xi(\cdot, t)|_{C^1(0,L)} \leq Ce^{-\mu t}|\xi^\sharp|_{C_1(0,L)}, \ \forall t \geq 0 \tag{27}$$

As discussed in the next section, condition (25) is equivalent to have full boundary dissipation at least on one side of the domain, in accordance with Villegas et al. [2009], Macchelli [2012a,b] in the linear scenario.

Proposition 3.2. Under the hypothese of Theorem 3.1, the 0 equilibrium of (12) is asymptotically stable if

$$u_\xi = \begin{pmatrix} 0 & k_b \\ k_a & 0 \end{pmatrix} y_\xi \tag{28}$$

with $k_a, k_b \in \mathbb{R}$ and $|k_a k_b| < 1$.

Proof. Relation (28) means that

$$s_{+,a} = k_a\, s_{-,a} \qquad s_{-,b} = k_b\, s_{+,b} \tag{29}$$

The results follows since from (29):

$$\rho(\mathrm{abs}(\nabla g(0))) = \sqrt{|k_a k_b|}$$

Remark 3.4. Relation (29), together with the condition $|k_a k_b| < 1$, means that the power flowing into the system is lower than the one flowing outside. The meaning of (29) in terms of the inputs and outputs (10), i.e. when the distributed port-Hamiltonian system is considered in impedance form, will be clear in the next section. The extension to the case in which k_a and k_b are replaced by e.g. symmetric matrices K_a and K_b is trivial.

4. APPLICATION TO THE NON-LINEAR FLEXIBLE BEAM EQUATION

The PDE (1) can be written in coordinates by assuming that q and p are vectors in $\mathbb{R}^6$, Stramigioli [2001]. Then, with (5) in mind, we have that

$$x(t, z) = \begin{pmatrix} q(t, z) \\ p(t, z) \end{pmatrix} \quad \mathcal{L} = \begin{pmatrix} C^{-1} & 0 \\ 0 & I_\rho^{-1} \end{pmatrix} \quad P_1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{30}$$

where, with some abuse in notation, C and I_ρ are the matrix formulation of the compliance and of the inertia tensors respectively. Clearly, in accordance with (3), for all $z \in [0, L]$, we have that

$$(\mathcal{L}x)(t,z) = \begin{pmatrix} C^{-1}q(t,z) \\ I_\rho^{-1}p(t,z) \end{pmatrix} = \begin{pmatrix} W(t,z) \\ T(t,z) \end{pmatrix} \tag{31}$$

defines a pair twist and wrench along the spatial domain of the link; T is the twist (i.e., linear and angular velocity), of the cross section, while W the applied wrench (i.e., force and torque), due to deformation, both expressed in body frame $\mathbb{E}_b$. From (3) and (6), we have that

$$f_\partial = \frac{1}{\sqrt{2}} \begin{pmatrix} T_L - T_0 \\ W_L + W_0 \end{pmatrix} \quad e_\partial = \frac{1}{\sqrt{2}} \begin{pmatrix} W_L - W_0 \\ T_L + T_0 \end{pmatrix} \tag{32}$$

which means that, to have the system in impedance form and with effort-in causality at both sides it is necessary to have

$$\hat{W} = \frac{\sqrt{2}}{2} \begin{pmatrix} 0 & I & 0 & -I \\ 0 & I & 0 & I \end{pmatrix} \quad \tilde{W} = \frac{\sqrt{2}}{2} \begin{pmatrix} -I & 0 & I & 0 \\ I & 0 & I & 0 \end{pmatrix} \tag{33}$$

which leads to

$$u = \begin{pmatrix} W_0 \\ W_L \end{pmatrix} \qquad y = \begin{pmatrix} T_0 \\ T_L \end{pmatrix} \tag{34}$$

as desired.

Now, denote by Γ the (positive) eigenvalues of $\sqrt{C^{-1}}\sqrt{I_\rho^{-1}}$, and by Ψ a coordinate change such that

$$\sqrt{C^{-1}}\sqrt{I_\rho^{-1}} = \Psi^{\mathrm{T}}\Gamma\Psi$$

with $\Psi\Psi^{\mathrm{T}} = I$. Then,

$$\Lambda = \begin{pmatrix} \Gamma & 0 \\ 0 & -\Gamma \end{pmatrix} \tag{35}$$

and the coordinate change (15) is given by

$$\begin{aligned} \xi_-(t,z) &= \frac{1}{\sqrt{2}}\Psi \left(\sqrt{C^{-1}}q(t,z) + \sqrt{I_\rho^{-1}}p(t,z)\right) \\ \xi_+(t,z) &= \frac{1}{\sqrt{2}}\Psi \left(\sqrt{C^{-1}}q(t,z) - \sqrt{I_\rho^{-1}}p(t,z)\right) \end{aligned} \tag{36}$$

From (31), relations (36) can be equivalently written as

$$\begin{aligned} \xi_-(t,z) &= \frac{1}{\sqrt{2}}\Psi \left(\sqrt{C}W(t,z) + \sqrt{I}_\rho T(t,z)\right) \\ \xi_+(t,z) &= \frac{1}{\sqrt{2}}\Psi \left(\sqrt{C}W(t,z) - \sqrt{I}_\rho T(t,z)\right) \end{aligned} \tag{37}$$

that clearly shows that the ξ variables are nothing else than the scattering variable associated to the Stokes-Dirac structure, once the metric defined by the total energy of the system is used to perform the decomposition. Finally, from (37), it easily follows that

$$\begin{aligned} (s_{+,0}, s_{-,0}) &= \frac{-\Psi}{\sqrt{2}} \left(\sqrt{C}W_0 + \sqrt{I_\rho}T_0, \sqrt{C}W_0 - \sqrt{I_\rho}T_0\right) \\ (s_{+,L}, s_{-,L}) &= \frac{\Psi}{\sqrt{2}} \left(\sqrt{C}W_L - \sqrt{I_\rho}T_L, \sqrt{C}W_L + \sqrt{I_\rho}T_L\right) \end{aligned} \tag{38}$$

which gives the "new" inputs and outputs in the scattering formulation (20). Note that, with the help of (22), the mapping between input and output in the scattering and impedance formulation can be immediately computed.

Let us assume that $z = L$ is the free-end. The unstressed configuration $q(t,z) = 0$ and $p(t,z) = 0$ can be made locally asymptotically stable by introducing a full boundary dissipation at $z = 0$. In other words, the unstressed configuration is asymptotically stable if

$$u = \begin{pmatrix} W_0 \\ W_L \end{pmatrix} = -\begin{pmatrix} K_D & 0 \\ 0 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} T_0 \\ T_L \end{pmatrix}}_{=y} \tag{39}$$

with $K_D = K_D^{\mathrm{T}} > 0$. In terms of the scattering variables at the extremities of the domain, the boundary conditions (39) can be written as

$$\begin{aligned} s_{+,0} &= \left(\sqrt{C}K_D - \sqrt{I}_\rho\right)\left(\sqrt{C}K_D + \sqrt{I}_\rho\right)^{-1} s_{-,0} \\ s_{-,L} &= -s_{+,L} \end{aligned} \tag{40}$$

This relation is in the form (28), and satisfies condition (25) provided that $K_D > 0$.

Under the hypothesis of Theorem 3.1, which basically means that q and p are C^1 functions, the proof of the existence of solution is a straightforward consequence of the same theorem. On the other hand, asymptotic stability follows from Proposition 3.2. Note that condition (26) is satisfied since the term $M(\xi)\xi$ is quadratic in the ξ variables. Finally, it is not difficult to prove that this result generalises Macchelli [2012a], i.e. the case in which a constant force is applied at one extremity and full-boundary dissipation is present at the other side, to the nonlinear scenario, since it can be verified that the linearization of (1) around the unstressed configuration provides the Timoshenko beam equation.

5. CONCLUSIONS AND FUTURE WORK

In this paper, the class of "almost linear" hyperbolic port-Hamiltonian systems is presented as a simple extension to the nonlinear scenario of the theory of linear, distributed port-Hamiltonian systems. Having in mind a nonlinear flexible link model that belongs to this class, it is shown that such systems can be written in terms of the scattering variables, that are nothing else than the Riemann invariants appearing in the theory of quasi-linear hyperbolic systems. By relying on established results in this field, conditions for the existence of solutions for this novel class of infinite dimensional port-Hamiltonian systems are given, together with a simple tool for studying and achieving asymptotic stability (at least, locally) of constant equilibrium configurations. Such results have been applied to the nonlinear model of the flexible beam.

Future work deals with the extension of these results to more complex situations, namely the case in which the Hamiltonian is no longer quadratic in the energy variables, as discussed e.g. in Prieur et al. [2008], but also in De Halleux et al. [2003] and related works. An important goal is also the generalisation to the nonlinear case of energy-based control techniques (e.g., the control by interconnection and energy shaping via Casimir generation, or control by state-modulated source), already successfully applied in the linear case, see e.g. Macchelli [2013].

REFERENCES

R. F. Curtain and H. J. Zwart. *An introduction to infinite dimensional linear systems theory.* Springer–Verlag, New York, 1995.

J. De Halleux, C. Prieur, J.-M. Coron, B. D'Andrea-Novel, and G. Bastin. Boundary feedback control in networks of open channels. *Automatica*, 39(8):1365–1376, Aug. 2003.

V. Duindam, A. Macchelli, S. Stramigioli, and H. Bruyninckx, editors. *Modeling and Control of Complex Physical Systems. The Port-Hamiltonian Approach*. Springer, 2009.

B. Jacob and H. J. Zwart. *Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces*, volume 223 of *Operator Theory: Advances and Applications*. Birkhäuser, Basel, 2012.

Y. Le Gorrec, H. Zwart, and B. M. Maschke. Dirac structures and boundary control systems associated with skew-symmetric differential operators. *SIAM Journal on Control and Optimization*, 44(5):1864–1892, 2005.

A. Macchelli. Asymptotic stability of forced equilibria for distributed port-Hamiltonian systems. In *Decision and Control (CDC 2012). Proceedings of the 51st IEEE Conference on*, Maui, HI, Dec. 10-13 2012a.

A. Macchelli. Boundary energy shaping of linear distributed port-Hamiltonian systems. In B. M. Maschke, C. Melchiorri, and A. J. van der Schaft, editors, *Lagrangian and Hamiltonian Methods for Nonlinear Control (LHMNLC 2012). Proceedings of the 4th IFAC Workshop on*, Bertinoro, Italy, Aug. 29-31 2012b. University of Bologna.

A. Macchelli. Dirac structures on Hilbert spaces and boundary control of distributed port-Hamiltonian systems. In *Control of Systems Governed by Partial Differential Equations (CPDE 2013). Proceeding of the 1st IFAC Workshop on*, Paris, France, Sep. 25-27 2013.

A. Macchelli and B. M. Maschke. *Modeling and Control of Complex Physical Systems. The Port-Hamiltonian Approach*, chapter Infinite-Dimensional Port-Hamiltonian Systems, pages 211–271. In Duindam et al. [2009], 2009.

A. Macchelli and C. Melchiorri. Modeling and control of the Timoshenko beam. The distributed port Hamiltonian approach. *SIAM Journal on Control and Optimization*, 43(2):743–767, 2004.

A. Macchelli and C. Melchiorri. Control by interconnection of mixed port Hamiltonian systems. *Automatic Control, IEEE Transactions on*, 50(11):1839–1844, Nov. 2005.

A. Macchelli, S. Stramigioli, A. J. van der Schaft, and C. Melchiorri. Scattering for infinite dimensional port Hamiltonian systems. In *Decision and Control (CDC 2002). Proceedings of the 41st IEEE Conference on*, volume 4, pages 4581–4586, Dec. 2002.

A. Macchelli, C. Melchiorri, and S. Stramigioli. Port-based modeling of a flexible link. *Robotics, IEEE Transactions on*, 23(4):650–660, Aug. 2007.

A. Macchelli, C. Melchiorri, and S. Stramigioli. Port-based modeling and simulation of mechanical systems with rigid and flexible links. *Robotics, IEEE Transactions on*, 25(5):1016–1029, Oct. 2009.

B. M. Maschke and A. J. van der Schaft. Port controlled Hamiltonian systems: modeling origins and system theoretic properties. In *Nonlinear Control Systems (NOLCOS 1992). Proceedings of the 3rd IFAC Symposium on*, pages 282–288, Bordeaux, France, Jun. 1992.

C. Prieur, J. Winkin, and G. Bastin. Robust boundary control of systems of conservation laws. *Mathematics of Control, Signals, and Systems*, 20(2):173–197, 2008.

J. C. Simo. A strain beam formulation. The three-dimensional dynamic problem. Part I. *Computer Methods in Applied Mechanics and Engineering*, 49:55–70, 1985.

S. Stramigioli. *Modeling and IPC control of interactive mechanical systems: a coordinate free approach*. Lecture Notes in Control and Information Sciences. Springer, London, 1st edition, 2001.

S. Stramigioli, A. J. van der Schaft, B. M. Maschke, and C. Melchiorri. Geometric scattering in robotic telemanipulation. *Robotics and Automation, IEEE Transactions on*, 18(4):588–596, Aug. 2002.

Li Ta-Tsien. *Global classical solutions for quasilinear hyperbolic systems*. Number 32 in RAM: Research in Applied Mathematics. John Wiley & Sons, 1994.

A. J. van der Schaft. *L_2-Gain and Passivity Techniques in Nonlinear Control*. Communication and Control Engineering. Springer–Verlag, 2000.

A. J. van der Schaft. *Modeling and Control of Complex Physical Systems. The Port-Hamiltonian Approach*, chapter Port-Hamiltonian Systems, pages 53–130. In Duindam et al. [2009], 2009.

A. J. van der Schaft and B. M. Maschke. Hamiltonian formulation of distributed parameter systems with boundary energy flow. *Journal of Geometry and Physics*, 42 (1-2):166–194, May 2002.

J. A. Villegas, H. Zwart, Y. Le Gorrec, B. M. Maschke, and A. J. van der Schaft. Stability and stabilization of a class of boundary control systems. In *Decision and Control and European Control Conference (CDC-ECC 2005). Proceedings of the 44th IEEE Conference on*, pages 3850–3855, Dec. 12-15 2005.

J. A. Villegas, H. Zwart, Y. Le Gorrec, and B. M. Maschke. Exponential stability of a class of boundary control systems. *Automatic Control, IEEE Transactions on*, 54 (1):142–147, Jan. 2009.

