Sampled-data control of LTI systems with relays: a convex optimization approach ⋆

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Abstract: We consider a generalized class of relay controllers where the system input may take values in a finite set of constant vectors. A simple continuous-time design method is proposed for linear time invariant (LTI) systems. Furthermore, it is used in the sampled-data case in order to guarantee (locally) the practical stabilization to a bounded ellipsoid containing the origin. The sampling intervals may be unknown and time-varying in a given interval. Simple linear matrix inequalities (LMIs) conditions are proposed for checking (local) practical stability.

Keywords: relay feedback control, sampled-data control, time-varying sampling, linear matrix inequalities.

1. INTRODUCTION

Relay feedback control is well known in a wide range of technical domains (Tzypkin [1955]). Due to its simplicity, it may be found in various forms (on-off control systems, bang-bang servo-mechanisms, etc.) and it has received a great attention from the robust control community. Relay feedback control represents the key component in variable structure systems (Utkin [1992], Emel'Yanov [1967], Edwards and Spurgeon [1998]) and has very interesting robustness properties faced to matched uncertainties and disturbances. However, in practical sampled-data implementations with a finite sampling frequency, relay actuators may induce oscillations and even instability. It is well known that in this case (local) asymptotic stability is no longer possible, but only convergence to a limit cycle or some bounded compact set containing the origin. For recent techniques on sampled-data control of LTI systems, we refer to the discrete-time approach in (Oishi and Fujioka [2010], Hetel et al. [2011]), the input-delay approach (Fridman [2010], Mirkin [2007], Seuret [2012]) and the impulsive system method in (Naghshtabrizi et al. [2008]). Very few articles have studied the robust sampled-data relay control problem in a formal quantitative manner.

This paper studies the sampled-data implementation of relay feedback controllers for the case of linear time invariant systems (LTI). We consider the case of multiple input systems. For the sake of generality, we assume that the system input may take values in a finite set of constant vectors, which includes as a particular case the classical relay control generated by sign functions. We propose a simple continuous-time design method based on the existence of a stabilizing state feedback and we show how it may be used in the sampled-data case in order to guarantee (locally) the practical stabilization to a bounded ellipsoid containing the origin. The main idea of the design procedure is to use the existence of an exponentially stabilizing state feedback as a reference control to be emulated (locally) by a relay feedback. The method is inspired by convex combination techniques used for switched systems (Liberzon [2003], Hetel and Fridman [2012]) and LMIs techniques for systems with bounded controls and saturation (Boyd et al. [1994], Hu and Lin [2001], Hu et al. [2002], Hindi and Boyd [1998]). It is based on simple convex optimization arguments and does not need any computation of normal forms. LMI conditions are proposed for dealing with robustness aspects as well as for estimating the maximum sampling interval that ensures (local) practical stabilization to a set.

This study may be related to works in (Polyakov [2010], Polyakov [2008], Shustin et al. [2008], Fridman et al. [1993],Han et al. [2012], Fridman et al. [2003]), where the effect of input delay has been studied for linear systems with relay feeback control, to (Nguyen et al. [2010]), where a sampled-data sliding mode control technique has been proposed, and to the simplex method stabilization in (Bartolini et al. [2004], Bartolini et al. [2011], Bajda and Izosimov [1985]).

The paper is structured as follows: Section II presents the system description and provides a simple continuous-time design method for relay feedback control constrained in a finite set of constant vectors. Section III is dedicated to sampled-data implementations of relay control laws. A numerical example is presented in Section IV.

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Notations : By I (or 0) we denote the identity (or the null) matrix with the appropriate dimension. $|\cdot|$ denotes the Euclidean vector norm and $|\cdot|_\infty$ the ∞-norm of a vector. For a square symmetric matrix, $M \succ 0$ ($M \prec 0$) indicates that M is positive (negative) definite. For a full rank square symmetric matrix M, M^{-1} denotes the inverse of M. M^T denotes the transpose of M. For a symmetric matrix,

$$M = \begin{bmatrix} A & B \\ * & C \end{bmatrix} \tag{1}$$

the symbol $*$ denotes a block B^T that may be inferred by symmetry. For a given set $\mathcal{S}$, the symbol conv $\{\mathcal{S}\}$ denotes the convex hull of the set. Int $\{\mathcal{S}\}$ denotes the interior of the set. For a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ and a positive scalar c we denote by $\mathcal{E}(P,c)$ the ellipsoid

$$\mathcal{E}(P,c) = \left\{x \in \mathbb{R}^n : x^T P x < c\right\}. \tag{2}$$

$\mathcal{B}(x,c)$ denotes the open ball centered on $x \in \mathbb{R}^n$ with radius $c > 0$:

$$\mathcal{B}(x,c) = \{y \in \mathbb{R}^n : |x-y| < c\}. \tag{3}$$

Let $\mathcal{S}$ be a bounded convex set. For a given positive scalar α, we denote $\alpha\mathcal{S} := \{\alpha x,\ x \in \mathcal{S}\}$.

Given a convex polytope $\mathcal{S}$ we denote by vert $\{\mathcal{S}\}$ its set of vertices.

Given a bounded set $\mathcal{X}$ and a continuous function $f : \mathcal{X} \to \mathbb{R}$ we denote

$$\arg\min_{x\in\mathcal{X}} f(x) = \{y \in \mathcal{X} : f(y) \le f(x), \forall x \in \mathcal{X}\}.$$

For a scalar y,

$$\text{sign}(y) \in \begin{cases} \{-1\}, & \text{if } y < 0, \\ \{-1,1\}, & \text{if } y = 0 \\ \{1\}, & \text{if } y > 0. \end{cases} \tag{4}$$

For a vector $y \in \mathbb{R}^n$, $y = (y_1, y_2, \ldots, y_n)^T$, $\text{sign}(y) = (\text{sign}(y_1), \text{sign}(y_2), \ldots, \text{sign}(y_n))^T$.

For a positive integer N, we denote by $\mathcal{I}_N$ the set $\{1, 2, \ldots, N\}$. By

$$\nabla_y V(x) := \lim_{\epsilon\to 0^+} \left(V(x+\epsilon y) - V(x)\right)\epsilon^{-1}$$

we denote the directional derivative of a function $V(x)$ along the direction y.

2. SYSTEM DESCRIPTION

Consider n, m, positive integers, matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$. Assume that the pair (A,B) is stabilizable and consider the system

$$\dot{x} = Ax + Bu, \tag{5}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control vector. We assume that the control u may only take values in a finite set of constant vectors $\mathcal{V} = \{v_1, v_2, \ldots, v_N\} \subset \mathbb{R}^m$, where N is a positive integer. We assume that conv $\{\mathcal{V}\}$ is a nonempty closed subset in $\mathbb{R}^m$ containing the null vector in its interior ($0_m \in \text{Int}\{\text{conv}\{\mathcal{V}\}\}$). Consider a relay feedback control law $u : \mathbb{R}^n \to \mathcal{V}$ of the form

$$u(x) \in \arg\min_{v\in\mathcal{V}} x^T \Gamma v \tag{6}$$

where $\Gamma \in \mathbb{R}^{n\times m}$ is a matrix to be designed, characterizing the switching hyperplanes. Note that for the case when the input u is a scalar constraint to the set $\mathcal{V} = \{-v, v\}$, with $v > 0$ a given constant, the control law (6) is reduced to the classical relay control $u(x) = -v\ \text{sign}(\Gamma x)$. This control structure may also be interpreted as a bounded simplex control with fixed constant vectors (Bartolini et al. [2004], Bartolini et al. [2011]).

To the closed-loop system (5),(6) we associate the differential inclusion (Filippov [1988])

$$\dot{x} \in F(x), \tag{7}$$

where $F(x)$ is a set-valued map

$$F(x) = \text{conv}\ \{Ax + B\tilde{u}, \tilde{u} \in \arg\min_{v\in\mathcal{V}} x^T\Gamma v\}. \tag{8}$$

Definition 2.1. Consider the closed-loop system (5), (6) and the differential inclusion (7). A *Filippov solution* of the closed-loop system (5), (6) on the interval $[t_a, t_b] \subset [0, \infty)$ is an absolutely continuous map $\phi : [t_a, t_b] \to \mathbb{R}^n$ such that

$$\dot{\phi}(t) \in F(\phi(t)) \tag{9}$$

for almost every $t \in [t_a, t_b]$.

The existence of a least one solution starting from each initial condition is guaranteed if for every $x \in \mathbb{R}^n$, $F(x)$ is locally bounded and takes non-empty, compact and convex values (Aubin and Cellina [1984]), which is the case for (8).

Definition 2.2. The differential inclusion (7) is said to be *locally exponentially stable* to the origin in a compact set Ω containing the origin if there exist positive scalars c_1, c_2 such that for any initial condition $x(0) \in \Omega$ and every possible solution $x(t)$ we have $|x(t)|^2 \le c_1 e^{-c_2 t}|x(0)|^2$.

3. SIMPLE DESIGN BASED ON LYAPUNOV FUNCTIONS

Under the assumption that the pair (A,B) is stabilizable, there exists a gain matrix K such that $A_{cl} = A + BK$ is a Hurwitz matrix. Furthermore, there exist a symmetric positive definite matrix P and a positive scalar δ such that

$$(A+BK)^T P + P(A+BK) \prec -2\delta P. \tag{10}$$

Then $V(x) = x^T P x$ satisfies

$$\frac{\partial V}{\partial x}(A+BK)x < -2\delta V(x), \forall x \neq 0, \tag{11}$$

i.e. it is a Lyapunov function for system (5) with the state-feedback control law Kx. Let us denote

$$\mathcal{C}_{\mathcal{V}}(K) := \{x \in \mathcal{R}^n : Kx \in \text{conv}\{\mathcal{V}\}\}. \tag{12}$$

Since conv $\{\mathcal{V}\}$ is a nonempty closed subset in $\mathbb{R}^m$ containing the null vector in its interior, there exists a level set described by $\gamma > 0$ such that

$$\mathcal{E}(P,\gamma) \subset \mathcal{C}_{\mathcal{V}}(K). \tag{13}$$

As follow we show how to use the Lyapunov inequality (10) in order to design a control law of the form (6) that ensures locally the exponential stabilization.

Let us denote $\Omega_0 := \mathcal{E}(P,\gamma)$. Then for any $x \in \Omega_0$ there exist N scalars $\alpha_j(x) \ge 0$, $\forall j \in \mathcal{I}_N$ with $\sum_{j=1}^N \alpha_j(x) = 1$ such that

$$Kx = \sum_{j=1}^N \alpha_j(x) v_j. \tag{14}$$

From (13) and (14) we have

$$\sum_{j=1}^N \alpha_j(x) \frac{\partial V}{\partial x}(Ax + Bv_j) < -2\delta V(x), \tag{15}$$

for all $x \in \Omega_0 \setminus \{0\}$. Considering that $\alpha_j(x) \geq 0, j \in \mathcal{I}_N$, there must be at least one $j \in \mathcal{I}_N$ such that

$$\frac{\partial V}{\partial x}(Ax + Bv_j) < -2\delta V(x),\ \forall x \in \Omega_0 \setminus \{0\}. \quad (16)$$

Since Ω_0 represents a sub-level set of $V(x)$, local stabilization in Ω_0 with a relay control is ensured by choosing the control $u(x)$ with the steepest decay of the Lyapunov function

$$u(x) \in \arg\min_{v \in \mathcal{V}} x^T PBv \quad (17)$$

which leads to setting $\Gamma = PB$ in (6). Note that if there are several minimizers v in $\arg\min_{v\in\mathcal{V}} x^T PBv$, they all ensure the decay of V. We arrive to the following:

Proposition 1. Consider system (5) with a control law (6). Assume that the pair (A, B) is stabilizable. Then there exits a function $V(x) = x^T Px$, with P a symmetric positive definite matrix, and scalars $\delta, \gamma > 0$ such that for $\Gamma = PB$

$$\frac{\partial V}{\partial x}(Ax + B\tilde{u}) < -2\delta V(x), \quad (18)$$

for all $\tilde{u} \in u(x), x \in \Omega_0 \setminus \{0\}$ where $\Omega_0 = \mathcal{E}(P, \gamma)$.

Remark 2. Inequality (18) implies that the function $V(x) = x^T Px$ satisfies

$$\max_{y \in F(x)} \nabla_y V(x) < -2\delta V(x), \forall x \in \Omega_0 \setminus \{0\}, \quad (19)$$

that is $\forall x(0) \in \Omega_0$ the solutions $x(t)$ of (7) with $\Gamma = PB$ satisfy $V(x(t)) < e^{-2\delta t} V(x(0))$ which is sufficient for local exponential stability in Ω_0.

Remark 3. Note that if the pair (A, B) is fully controllable, then for any chosen decay rate δ there exists a gain matrix K such that the inequality (10) is satisfied. Moreover, the design of K may be expressed as a classical linear matrix inequality (LMI) problem. The inequality (10) is satisfied if and only if (Boyd et al. [1994]) there exists $Q = Q^T \succ 0$ and $\lambda > 0$ such that

$$AQ + QA^T - \lambda BB^T \prec -2\delta Q. \quad (20)$$

The Lyapunov and gain matrices are given by $P = Q^{-1}$ and $K = -\frac{\lambda}{2} B^T Q^{-1}$, respectively.

4. SAMPLED-DATA IMPLEMENTATION

We consider that the system state x is available at sample times $\{t_k\}_{k\in\mathbb{N}}$, with $t_0 = 0$, $t_k < t_{k+1}$, $\forall\ k \in \mathbb{N}$, and we denote $x_k = x(t_k)$. We assume that the sampling interval $T_k := t_{k+1} - t_k$ is time-varying, with $0 < T_k \leq \overline{T}$ where $\overline{T}$ is a known bound on the sampling interval. Moreover we consider that the sequence of sampling times t_k does not admit any accumulation points, i.e. $\lim_{k\to\infty} t_k = \infty$.

Let $P \succ 0$, K and $\delta > 0$ satisfying (10). Consider a sampled-data implementation of the control law (6) with $\Gamma = PB$:

$$u(x_k) \in \arg\min_{v\in\mathcal{V}} x_k^T PBv. \quad (21)$$

With a sample-and-hold implementation of the control, the system input is constant between two sampling instants, that is

$$\dot{x}(t) = Ax(t) + Bu(x_k), \forall t \in [t_k, t_{k+1}). \quad (22)$$

Similarly to the continuous-time case, global stabilization cannot be provided, and the control law is effective only locally, in an ellipsoidal region $\Omega_0 = \mathcal{E}(P, \gamma)$, with γ such that $\mathcal{E}(P, \gamma) \subset \mathcal{C}_{\mathcal{V}}(K)$.

However, with a sampled-data implementation, even for small sampling intervals, the closed-loop system cannot be driven to the equilibrium point $x = 0$, but only to some bounded region containing the equilibrium that we denote Ω_∞, with $\Omega_\infty \subset \Omega_0$. Moreover the size of the set Ω_∞ grows according to the sampling interval $\overline{T}$. Here we are interested in regions Ω_∞ that are characterized by a sub-level set $\mathcal{E}(P, C)$ of the function $V(x) = x^T Px$ where the positive constant $C < \gamma$ is of the order of $\overline{T}$.

To provide an estimation of the domain of attraction, we may use the geometry of the convex set described by control vectors, $\operatorname{conv}\{\mathcal{V}\}$. Note that this set may be characterized by limiting hyperplanes. For any set $\mathcal{V}$ there exists a finite number n_h of vectors $h_i \in \mathbb{R}^{1\times m}, i = 1\ldots, n_h$ such that

$$\operatorname{conv}\{\mathcal{V}\} = \{y \in \mathbb{R}^m : \ h_i y \leq 1, i \in \mathcal{I}_{n_h}\} \quad (23)$$

which leads to a set $\mathcal{C}_{\mathcal{V}}(K)$ of the form

$$\mathcal{C}_{\mathcal{V}}(K) = \{x \in \mathbb{R}^n : \ h_i Kx \leq 1, i \in \mathcal{I}_{n_h}\}. \quad (24)$$

As follows we provide conditions that ensure that all solution of the closed-loop system (22), (21) with initial condition $x(0)$ in an ellipsoid Ω_0 are converging exponentially towards a smaller ellipsoid $\Omega_\infty = \mathcal{E}(P, C) \subset \Omega_0$ for small enougth $\overline{T}$.

Proposition 4. Consider system (22), (21) with $t_{k+1} - t_k \leq \overline{T}$, the control set $\mathcal{V}$ and the description of $\operatorname{conv}\{\mathcal{V}\}$ in (23). Assume that the pair (A, B) is stabilizable and consider a gain matrix K such that $A_{cl} = A + BK$ is Hurwitz. Given tuning parameters δ, γ, let there exist symmetric positive definite matrices P, U, and a positive scalar β such that:

a)
$$\begin{bmatrix} I & h_i K \\ * & \frac{P}{\gamma} \end{bmatrix} \succ 0, \forall i \in \mathcal{I}_{n_h}; \quad (25)$$

b) $\beta < 2\gamma\delta\overline{T}^{-1}$;

c) the set of LMIs

$$\begin{bmatrix} A_{cl}^T P + PA_{cl} + 2\delta P + \overline{T} A^T UA & \overline{T} A^T UBv \\ * & \overline{T}\left(v^T B^T UBv - \beta\right) \end{bmatrix} \prec 0, \quad v \in \mathcal{V} \quad (26)$$

$$\begin{bmatrix} A_{cl}^T P + PA_{cl} + 2\delta P & 0 & -(PBK)^T\,\overline{T} \\ * & -\beta\overline{T} & (PBv)^T\,\overline{T} \\ * & * & -\overline{T} Ue^{-2\delta\overline{T}} \end{bmatrix} \prec 0, v \in \mathcal{V} \quad (27)$$

is satisfied.

Then any solution $x(t)$ with initial condition $x(0) \in \Omega_0 = \mathcal{E}(P, \gamma)$ converges exponentially to $\Omega_\infty = \mathcal{E}(P, C)$ as $t \to \infty$, with $C = (2\delta)^{-1}\beta\overline{T}$.

Proof. In order to characterize the set inclusion $\Omega_0 = \mathcal{E}(P, \gamma) \subset \mathcal{C}_{\mathcal{V}}(K)$, we apply classical convex optimization procedures from (Hindi and Boyd [1998]). Let us remark that for $\Omega_0 = \mathcal{E}(P, \gamma) \subset \mathcal{C}_{\mathcal{V}}(K)$ it is necessary and sufficient that none of the hyperplanes $h_i Kx = 1$, $i \in \mathcal{I}_{n_h}$, crosses the ellipsoid $\mathcal{E}(P, \gamma)$. This leads to the following condition

$$\gamma \leq \min_{i\in\mathcal{I}_{n_h}} \left(h_i KP^{-1}K^T h_i^T\right)^{-1}. \quad (28)$$

From (28), it can be seen using the Schur complement lemma that condition a) implies that $\mathcal{E}(P,\gamma) \subset \mathcal{C}_{\mathcal{V}}(K)$. Condition b) guarantees that

$$\Omega_\infty = \mathcal{E}\left(P, \frac{\beta\overline{T}}{2\delta}\right) \subset \Omega_0 = \mathcal{E}(P,\gamma).$$

To show that the additional LMI constraints (26), (27), guarantee that any solution $x(t)$ with initial condition $x(0) \in \Omega_0$ converges exponentially to Ω_∞ as $t \to \infty$, consider a continuous function $W : \mathbb{R}^+ \to \mathbb{R}, k \in \mathbb{N}$, differentiable over $[t_k, t_{k+1})$, with $W(t_k) = 0$ and $W(t) \geq 0, \forall t \in (t_k, t_{k+1}), \forall\ k \in \mathbb{N}$, satisfying the following condition:

$$\dot{V}(x(t)) + \dot{W}(t) + 2\delta\left(V(x(t)) + W(t)\right) < \beta\overline{T}, \tag{29}$$

$\forall t \in [t_k, t_{k+1}),\ x(t) \in \Omega_0 \setminus \{0\}$. By the comparison principle, for $x(0) \in \Omega_0 \setminus \{0\}$, (29) yields $V(x(t)) < e^{-2\delta t}\left(V(x(0)) + W(0) - \frac{\beta}{2\delta}\overline{T}\right) + \frac{\beta}{2\delta}\overline{T} - W(t),\ \forall\ t > 0$, i.e.

$$V(x(t)) < e^{-2\delta t}V(x(0)) + \frac{\beta}{2\delta}\overline{T},\ \forall t > 0, \tag{30}$$

which means that $x(t)$ exponentially converges to the attractive ellipsoid $\Omega_\infty = \mathcal{E}\left(P, (2\delta)^{-1}\beta\overline{T}\right)$.

We will show that the LMIs (26), (27), imply (29) with $V(x(t)) = x^T(t)Px(t)$ and $W(t)$ given by

$$W(t) = (t_{k+1} - t_k - \tau(t)) \int_{t_k}^{t} e^{2\delta(s-t)}\dot{x}^T(s)U\dot{x}(s)ds \geq 0, \tag{31}$$

$\forall t \in [t_k, t_{k+1})$ with $U = U^T \succ 0$ and $\tau(t) := t - t_k, \forall t \in [t_k, t_{k+1})$.

Let us remark that $u(x_k) = v_i$, for some $i \in \mathcal{I}_N$, if

$$x_k^T PB(v_j - v_i) \geq 0, \forall j \in \mathcal{I}_N. \tag{32}$$

Furthermore, for all $x(t) \in \Omega_0$, there exists N scalars $\alpha_j(x(t)) \geq 0,\ \forall j \in \mathcal{I}_N$ with $\sum_{j=1}^{N}\alpha_j(x(t)) = 1$ such that

$$Kx(t) = \sum_{j=1}^{N}\alpha_j(x(t))\,v_j. \tag{33}$$

Multiplying (32) by the appropriate α_j and summing leads to

$$x_k^T PB(Kx(t) - v_i) \geq 0. \tag{34}$$

Consider the notation $\eta(t) = (x(t) - x_k)\,\tau^{-1}(t)$. Then $u(x_k) = v_i$ when

$$2\left(x(t) - \tau(t)\eta(t)\right)^T PB(Kx(t) - v_i) \geq 0. \tag{35}$$

Consider $W(t)$ as in (31) for $u(x_k) = v_i$. Furthermore, using Jensen inequality (Gu et al. [2003]),

$$\int_{t_k}^{t}\dot{x}^T(s)U\dot{x}(s)ds \geq \frac{1}{\tau(t)}(x(t) - x_k)^T U (x(t) - x_k) \geq \tau(t)\eta^T(t)U\eta(t). \tag{36}$$

Then

$$\begin{aligned}&\dot{W}(t) + 2\delta W(t) = \\ &(T_k - \tau(t))\,\dot{x}^T(t)U\dot{x} - \int_{t_k}^{t} e^{2\delta(s-t)}\dot{x}^T(s)U\dot{x}(s)ds \leq \\ &(T_k - \tau(t))\left(Ax(t) + Bv_i\right)^T U\left(Ax(t) + Bv_i\right) \\ &-\tau(t)\eta^T(t)U\eta(t)e^{-2\delta\overline{T}}.\end{aligned}$$

Therefore, (29) holds with $0 < t_{k+1} - t_k \leq \overline{T}$ and $u(x_k) = v_i$ if

$$\begin{aligned}&2x^T PAx + x^T\left(2\delta P + (\overline{T} - \tau)A^T UA\right)x \\ &+2x^T\left(PBv_i + (\overline{T} - \tau)A^T UBv_i\right) \\ &+(\overline{T} - \tau)v_i^T B^T UBv_i \\ &-\beta\overline{T} - \tau\eta^T Ue^{-2\delta\overline{T}}\eta < 0\end{aligned} \tag{37}$$

for all $\tau \in [0, \overline{T}]$, where $x = x(t), \eta = \eta(t)$.

Adding the left side of (35) to the left side of (37) we arrive to

$$z^T\Theta_i(\tau)z < 0,\ \forall\tau \in [0, \overline{T}] \tag{38}$$

where $z = \begin{bmatrix}x^T & 1 & \eta^T\end{bmatrix}^T$,

$$\Theta_i(\tau) = \begin{bmatrix}\Theta^1(\tau) & \Theta_i^2(\tau) & -\tau(PBK)^T \\ * & \Theta_i^3(\tau) & \tau(PBv_i)^T \\ * & * & -\tau Ue^{-2\delta\overline{T}}\end{bmatrix}$$

with

$$\begin{aligned}\Theta^1(\tau) &= A_{cl}^T P + PA_{cl} + 2\delta P + (\overline{T} - \tau)A^T UA, \\ \Theta_i^2(\tau) &= (\overline{T} - \tau)A^T UBv_i, \\ \Theta_i^3(\tau) &= (\overline{T} - \tau)v_i^T B^T UBv_i - \beta\overline{T}.\end{aligned}$$

Since $\Theta(\tau)$ is linear in τ, it is sufficient to verify that

$$\begin{bmatrix}\Theta^1(0) & \Theta^2(0)_i \\ * & \Theta_i^3(0)\end{bmatrix} \prec 0 \tag{39}$$

and $\Theta_i(\overline{T}) \prec 0$, for all $i \in \mathcal{I}_N$, which leads to (26), (27).

□

Remark 5. Note that the size of the obtained attracting ellipsoid is of the order of $\overline{T}$. Moreover, for $\overline{T} \to 0$, the set of LMIs (26), (27) are reduced to the LMI $A_{cl}^T P + PA_{cl} + 2\delta P \prec 0$ which guarantees the exponential stability of the continuous-time system.

5. NUMERICAL EXAMPLE

Consider a linear time-invariant system with

$$A = \begin{bmatrix}1 & -1 \\ 1 & 1\end{bmatrix},\ B = \begin{bmatrix}1 \\ 1\end{bmatrix}. \tag{40}$$

The A matrix has unstable eigenvalues $1 \pm i$. Consider that the control is constrained to the set $\mathcal{V} = \{-v, v\}$ with $v = 25$. The pair (A, B) is fully controllable. Then for $\delta = 0.23$, the LMI (20) is feasible with $\lambda = 500$ and leads to the Lyapunov matrix

$$P = 10^{-2}\begin{bmatrix}0.66 & -0.78 \\ -0.78 & 1.91\end{bmatrix}, \tag{41}$$

a state feedback gain

$$K = -\frac{1}{2}\cdot\lambda\cdot B^T P = [0.3125 \quad -2.8125].$$

and a switching matrix $\Gamma = PB$. In order to maximize the size of the attraction set for the continuous-time closed-loop system, we used classical optimization procedures in (Boyd et al. [1994]) to maximize the sum of squares of the major axes of $\mathcal{E}(P,1)$ under the constraint

$$\mathcal{E}(P,1) \subset \mathcal{C}\left(-\frac{1}{2}\lambda B^T Px\right) = \left\{x \in \mathbb{R}^n : |\frac{1}{2}\lambda B^T Px| \le v\right\}.$$

This is done by solving the optimization problem

$$\max\ \text{trace}\ (Q)$$

s.t. there exists $Q \succ 0, \lambda > 0$, satisfying (20) and

$$\begin{bmatrix} v^2 & \frac{1}{2}\lambda B^T \\ * & Q \end{bmatrix} \succ 0.$$

Then the continuous-time system (5) with the control law (6) is locally exponentially stable in $\Omega_0 = \mathcal{E}(P,\gamma)$. Using the obtained values of δ, P and K it is possible to analyze the sampled-data implementation of the control law. For this set of parameters, with U and β as decision variables, the conditions of Theorem 4 are feasible for $T_{max} \le 1.9 \cdot 10^{-2}$. In particular, for $T_{max} = 10^{-3}$, the LMIs are found feasible with $\beta = 15.63$ which leads to $\Omega_\infty = \mathcal{E}(P, 0.068)$. A numerical illustration is shown in Figure 1.

6. CONCLUSION

This article studied the sampled-data implementation of relay feedback controllers for the case of multiple input linear time invariant (LTI) systems. The system input is a relay that may take values in a finite set of constant vectors. A simple continuous-time design method has been proposed based on the existence of a stable state feedback. The method is extended to the sampled-data case in order to guarantee (locally) the practical stabilization to a bounded ellipsoid containing the origin. Simple LMI conditions have been proposed for checking (local) practical stability. The obtained relay feedback controllers are robust to variations of the sampling interval. The techniques are illustrated by a numerical example.

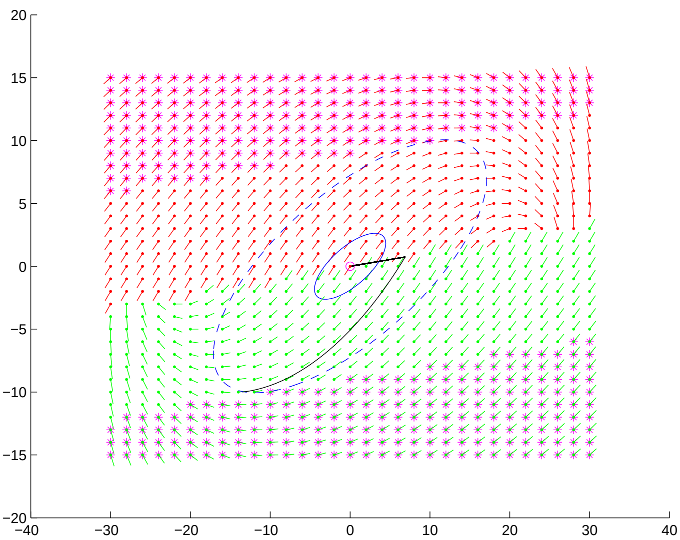

Figure 1. Illustration of evolution in the state space for a constant sampling interval $T = 10^{-3}$. Green – $u = v$, red – $u = -v$, ellipsoid in dashed line – domain of attraction Ω_0, ellipsoid in solid line – attractive set for $t \to \infty$, Ω_∞, magenta – state space zone $R^2 \setminus \mathcal{C}(K)$, black line – trajectory from the initial condition $x_0 = [-13.5 \quad -10]^T$.

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