Spatio-temporal symmetries in control systems: an application to formation control

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Abstract: With the aim of addressing the stabilization problem of periodic trajectories in systems composed of identical interconnected subsystems, we introduce the class of "spatio-temporally symmetric" nonlinear systems. We address in detail the linear, time-varying case and present conditions for the synthesis of a static and a dynamic stabilizing controller. We show that linear spatio-temporally symmetric systems can be reduced to hybrid systems, described by a periodic linear system with periodic state jumps. As an application example, we present the stabilization of a formation of unicycle robots in cyclic pursuit.

Keywords: Nonlinear Cooperative Control; Stabilization; Robotics.

1. INTRODUCTION

Various biological and human made systems are composed of identical interconnected subsystems in which, normally, each component reproduces the same periodic behavior with a phase difference. Following the terminology used in Golubitsky and Stewart [2003], we say that the state trajectory of these systems has a property of spatio-temporal symmetry. Some examples are animal locomotion(Buono and Golubitsky [2001], Golubitsky et al. [1999]), hearth rhythm generation (Karma and Robert F. Gilmour [2007]), formation control for mobile robots (Marshall et al. [2004], El-Hawwary and Maggiore [2012]).

In this paper, we introduce a class of systems which has a property of spatio-temporal symmetry and we propose a method for locally stabilizing an assigned spatio-temporal symmetric trajectory, using the same time-varying control law for each subsystem. To understand the main idea, consider the case of a system composed of 4 identical components with a T-periodic reference solution $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4)$ where $\tilde{x}_i$ represents the reference state of the i-th subsystem and suppose that the following property holds: $\tilde{x}_{i+1}(t) = \tilde{x}_i(t + \frac{T}{4})$, $i = 1, \ldots, 4$, where the indexes are considered modulo 4. In other words, each subsystem follows the same trajectory with a different delay (see Figure 1). Let Γ be the permutation that assigns to each subsystem the state of the subsequent one (i.e. x_1 becomes x_2, x_2 becomes x_3 and so on). Then, the reference trajectory verifies the property $\Gamma\tilde{x}(t) = \tilde{x}(t+\frac{T}{4})$, hence the permutation Γ of the states corresponds to an anticipation of $\frac{T}{4}$ in the reference trajectory, in other words the trajectory $\tilde{x}$ has a spatio-temporal symmetry. We now define the following state transformation. Assume that at time $\frac{T}{4}$ the inverse permutation Γ^{-1} is applied to the system state x. Figure 1 shows the effect of this operation. Consider for instance the first subsystem with state $\tilde{x}_1$. Just before time $\frac{T}{4}$, the state $\tilde{x}_1$ reaches the initial state $\tilde{x}_2(0)$ of the second subsystem, then the inverse permutation Γ^{-1}, applied at time $\frac{T}{4}$ brings x_1 back to the initial state $\tilde{x}_1(0)$. Following this observation, define a periodic hybrid system

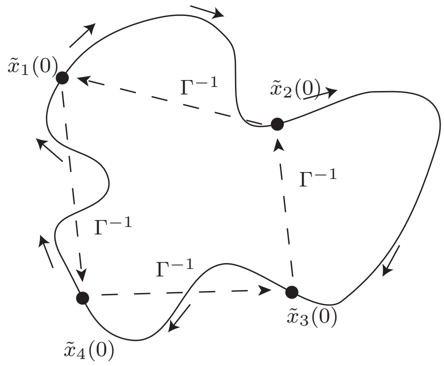


Fig. 1. The action of the permutation Γ^{-1} at time $\frac{T}{4}$.

with state ξ, that satisfies the same differential equation as the original system with state x, with the difference that, at multiples of $\frac{T}{4}$, the permutation Γ^{-1} is applied to ξ. The corresponding transformed trajectory $\tilde{\xi}$ of $\tilde{x}$ becomes $\frac{T}{4}$-periodic, discontinuous at times multiples of $\frac{T}{4}$. In this way, the problem of designing a control that stabilizes the T-periodic trajectory $\tilde{x}$ is reformulated as the problem of stabilizing the $\frac{T}{4}$-periodic reference $\tilde{\xi}$. We will show that any feedback stabilizing control law formulated in the new coordinates ξ, has a property of spatio-temporal symmetry when rewritten in the original coordinates x. That is, every subsystem uses the same feedback control with a different delay.

To address the problem of local asymptotic stabilization of an assigned spatio-temporally symmetric trajectory, we consider the system linearization, which is given by a linear, time-varying spatio-temporally symmetric system. We address in detail this linear case and present conditions for the synthesis of a static and a dynamic stabilizing controller. In particular, we show that, with the change of coordinates previously described, linear spatio-temporally symmetric systems are equivalent to hybrid periodic systems, described by a periodic linear systems with periodic state jumps. As an application example, we present the stabilization of a formation of unicycle robots in cyclic pursuit.

Notations: Let n, m, p be positive integers. In the paper we will suppose that τ is a positive real number and that $\Gamma \in \mathbb{R}^{n\times n}$, $\Theta \in \mathbb{R}^{m\times m}$, $\Sigma \in \mathbb{R}^{p\times p}$ are invertible matrices such that there exists a constant c:

$$\|\Gamma^k\|, \|\theta^k\|, \|\Sigma^k\| \le c, \ \forall k \ge 0 .$$

Set $\tau\mathbb{Z} = \{\tau i | i \in \mathbb{Z}\}$, $\mathbb{R}\backslash\tau\mathbb{Z} = \{t \in \mathbb{R} | t \notin \tau\mathbb{Z}\}$. If Ω is an open subset of $\mathbb{R}$, we denote by $\mathcal{C}(\Omega, \mathbb{R}^n)$ the set of continuous functions defined on Ω with values in $\mathbb{R}^n$, by $\mathcal{C}_p(\Omega, \mathbb{R})$ the set of piecewise continuous functions on Ω and bounded on bounded subset of Ω with values in $\mathbb{R}^n$ and by $\mathcal{C}_+(\Omega, \mathbb{R}^n)$ the set of bounded right-continuous functions defined on Ω with values on $\mathbb{R}^n$. We denote by $\mathcal{C}^1(\Omega, \mathbb{R}^n)$ the $\mathcal{C}^1$ functions on Ω with values in $\mathbb{R}^n$.

2. STABILIZATION OF SPATIO-TEMPORALLY SYMMETRIC TRAJECTORIES

In this section, we introduce the notions of spatio-temporally symmetric control systems and spatio-temporally symmetric trajectories.

***Definition** 1.* Consider the nonlinear control system

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)) \\ y(t) = h(t, x(t), u(t)) , \end{cases} \tag{1}$$

where $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $h : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ are continuous on $t \in \mathbb{R}$ and locally Lipschitz on $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$. We say that system (1) is $(\Gamma, \Theta, \Sigma, \tau)$-symmetric if, $\forall (x, u, t) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$

$$\begin{aligned} \Gamma f(t, x, u) &= f(t + \tau, \Gamma x, \Theta u) \\ \Sigma h(t, x, u) &= h(t + \tau, \Gamma x, \Theta u) . \end{aligned} \tag{2}$$

Similarly, we say that the autonomous system

$$\begin{cases} \dot{x}(t) = f(t, x(t)) \\ y(t) = h(t, x(t)) , \end{cases} \tag{3}$$

is (Γ, Σ, τ)-symmetric if $\Gamma f(t, x) = f(t+\tau, \Gamma x)$, $\Sigma h(t, x) = h(t + \tau, \Gamma x)$.

***Remark** 1.* If system (1) is not time-varying, conditions (2) reduce to

$$\begin{aligned} \Gamma f(x, u) &= f(\Gamma x, \Theta u) \\ \Sigma h(x, u) &= h(\Gamma x, \Theta u) . \end{aligned} \tag{4}$$

In this case, we simply say that system (4) is (Γ, Θ, Σ)-symmetric. If the system is autonomous, this case corresponds to a particular case of an equivariant system (for a discussion on equivariant systems, see for instance Chossat and Lauterbach [2000] or Golubitsky and Stewart [2003]).

***Remark** 2.* Suppose that the control system (1) is linear in x and u, that is

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t) , \end{aligned} \tag{5}$$

where $A \in \mathcal{C}(\mathbb{R}, \mathbb{R}^{n\times n})$, $B \in \mathcal{C}(\mathbb{R}, \mathbb{R}^{n\times m})$, $C \in \mathcal{C}(\mathbb{R}, \mathbb{R}^{p\times n})$, $D \in \mathcal{C}(\mathbb{R}, \mathbb{R}^{p\times m})$. Then system (5) is $(\Gamma, \Theta, \Sigma, \tau)$-symmetric if and only if

$$\Gamma A(t) = A(t + \tau)\Gamma, \tag{6a}$$
$$\Gamma B(t) = B(t + \tau)\Theta, \tag{6b}$$
$$\Sigma C(t) = C(t + \tau)\Gamma, \tag{6c}$$
$$\Sigma D(t) = D(t + \tau)\Theta . \tag{6d}$$

***Definition** 2.* If system (5) is $(\Gamma, \Theta, \Sigma, \tau)$-symmetric, we also say that the quadruple (A, B, C, D) is $(\Gamma, \Theta, \Sigma, \tau)$-symmetric. Similarly, we say that A is (Γ, τ)-symmetric if (6a) is verified, that couple (A, B) is (Γ, Θ, τ)-symmetric if (6a), (6b) hold and that the couple (A, C) is (Γ, Σ, τ)-symmetric if (6a), (6c) hold.

***Remark** 3.* Linear spatio-temporally symmetric systems are related to patterned systems, introduced in Hamilton and Broucke [2012a] and Hamilton and Broucke [2012b]. In fact, if $\Gamma = \Sigma = \Theta$ and A,B,C,D are constant, conditions (6a)–(6d) imply that A, B, C, D commute with Γ. Moreover, if Γ has distinct eigenvalues, A, B, C, D can be expressed as a polynomial function of Γ (see for instance chapter 3.1 of Zhang [1999]) and define a patterned system.

***Example** 1.* (A cyclic formation of unicycles). As a motivating example, consider a cyclic formation of n nonholonomic vehicles which move with constant unitary speed, described by the following system, for $i = 1, \dots, k - 1$

$$\begin{cases} \dot{z}_i(t) = \cos\theta_i(t) \\ \dot{w}_i(t) = \sin\theta_i(t) \\ \dot{\theta}_i(t) = \omega_i(t) . \end{cases} \tag{7}$$

Vector $(z_i, w_i)^T \in \mathbb{R}^2$ is the position of the i-th robot and $\theta_i \in S^1$ is its direction. The angular velocities $u = (\omega_1, \omega_2, \dots, \omega_k)^T$ are the control inputs. Let $x_i = (z_i, w_i, \theta_i)^T$ be the state of the i-th robot and $x = (x_1, x_2, \dots, x_k)^T$ the state of the formation. As output function we choose $y(x) = \begin{bmatrix} a(x) \\ d(x) \end{bmatrix}$, with $d(x) = (d_1(x), d_2(x), \dots, d_k(x))^T$ where $a(x) = \frac{1}{k}\sum_{i=0}^{k-1} \begin{pmatrix} x_i \\ y_i \\ \theta_i \end{pmatrix}$ is the average of the positions and the angles of the robots and $d_i(x) = \left\| \begin{pmatrix} z_i \\ w_i \end{pmatrix} - \begin{pmatrix} z_{i+1} \\ w_{i+1} \end{pmatrix} \right\|$, is the distance between the i-th and the $i+1$-th robot, where the indexes are computed modulo k (i.e. if $i = k - 1$, $i + 1 = 0$). Let P be the cyclic permutation matrix, defined as

$$P = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 & 0 \end{bmatrix},$$

and define $\Sigma = \text{blkdiag}\,(I_3, P)$, $\Gamma = P \otimes I_3$, $\Theta = P$, where I_3 denotes the 3 by 3 identity matrix and blkdiag denotes a block diagonal matrix. Then, system (7) with the output function y is (Σ, Γ, Θ)-symmetric. In this example, the symmetry is due to the fact that every vehicle is described by the same equation and the permutation Γ of the order of the subsystems leaves unchanged the output function y.

***Definition** 3.* Let $\tilde{u} \in \mathcal{C}(\mathbb{R}, \mathbb{R}^m)$, $\tilde{x} \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^n)$ be a reference input and state trajectory such that (1) is verified, then $(\tilde{x}, \tilde{u})$ is (Γ, Θ, τ)-symmetric if, $\forall t \ge 0$,

$$\begin{aligned} \Gamma\tilde{x}(t) &= \tilde{x}(t + \tau) \\ \Theta\tilde{u}(t) &= \tilde{u}(t + \tau) . \end{aligned} \tag{8}$$

We will consider the following two control problems, consisting in designing a static or a dynamic controller that locally stabilize system (1) on the reference trajectory $\tilde{x}$.

***Problem** 1.* (Static feedback controller). Design a static state-feedback controller of the form

$$u(t) = l(t, x(t)) \tag{9}$$

such that

1) local asymptotical exact tracking is achieved for the closed-loop system (1)+(9), that is, there exists a neigh-

borhood U of $\tilde{x}(0)$ such that, if $x(0) \in U$, the solution of (1)+(9) satisfies $\lim_{t\to\infty}(\tilde{x}(t) - x(t)) = 0$,

2) the controller (9) satisfies, $\forall (t, x) \in \mathbb{R} \times \mathbb{R}^n$,

$$\Theta l(t, x) = l(t + \tau, \Gamma x)\,. \tag{10}$$

***Problem** 2.* (Dynamic feedback controller). Design a dynamic controller of the form

$$\begin{cases} \dot{e}(t) = g(t, e(t), y(t)) \\ u(t) = l(t, e(t)) \end{cases} \tag{11}$$

with $e(t) \in \mathbb{R}^n$, such that

1) local asymptotical exact tracking is achieved for the closed-loop system (1)+(11), that is, there exists a neighborhood U of $\tilde{x}(0)$ and a neighborhood V of 0, such that, if $x(0) \in U$ and $e(0) \in V$, the solution of (1)+(11) satisfies $\lim_{t\to\infty}(\tilde{x}(t) - x(t)) = 0\,,$

2) the controller (11) is $(\Gamma, \Theta, \Sigma, \tau)$-symmetric, that is, $\forall (t, e, y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p$,

$$\begin{aligned} \Gamma g(t, e, y) &= g(t + \tau, \Gamma e, \Sigma y) \\ \Theta l(t, e) &= l(t + \tau, \Gamma e)\,. \end{aligned} \tag{12}$$

The following two propositions shows that conditions (10) and (12) guarantee the spatio-temporal symmetry of the closed-loop system in the static and dynamic feedback cases.

***Proposition** 1.* If conditions (10) are satisfied, then the closed loop system (1)+(9) is (Γ, Σ, τ)-symmetric.

Proof.

Setting $\hat{f}(t, x) = f(t, x, l(t, x))$, the closed loop system satisfies the equation $\dot{x}(t) = \hat{f}(t, x(t))$ and

$$\begin{aligned} \Gamma \hat{f}(t, x) = \Gamma f(t, x, l(t, x) &= f(t + \tau, \Gamma x, l(t + \tau, \Gamma x)) \\ &= \hat{f}(t + \tau, \Gamma x)\,. \end{aligned} \qquad \square$$

***Proposition** 2.* If conditions (12) are satisfied, then the closed loop system (1)+(11), with state $\begin{bmatrix} x \\ e \end{bmatrix} \in \mathbb{R}^{2n}$ and output y is $(\hat{\Gamma}, \Theta, \Sigma, \tau)$-symmetric, where $\hat{\Gamma} =$ blkdiag (Γ, Γ).

Proof. Set $z = \begin{bmatrix} x \\ e \end{bmatrix}$, $\hat{f}(t, z) = \begin{bmatrix} f(t, x, l(t, e)) \\ g(t, e, h(t, x, l(t, e))) \end{bmatrix}$, $\hat{h}(t, z) = h(t, x)$, then

$$\begin{aligned} \hat{\Gamma}\hat{f}(t, z) &= \begin{bmatrix} \Gamma f(t, x(t), l(t, e(t))) \\ \Gamma g(t, e(t), h(t, x(t), l(t, e(t)))) \end{bmatrix} \\ &= \begin{bmatrix} f(t + \tau, \Gamma x(t), \Theta l(t, e(t))) \\ g(t + \tau, \Gamma e(t), \Sigma h(t, x(t), l(t, e(t)))) \end{bmatrix} \\ &= \begin{bmatrix} f(t + \tau, \Gamma x(t), l(t + \tau, \Gamma e(t))) \\ g(t + \tau, \Gamma e(t), h(t + \tau, \Gamma x(t), l(t + \tau, \Gamma e(t)))) \end{bmatrix} = \hat{f}(t + \tau, \hat{\Gamma} z)\,, \end{aligned}$$

moreover $\Sigma \hat{h}(t, z) = h(t + \tau, \Gamma x) = \hat{h}(t + \tau, \hat{\Gamma} z)$. $\square$

***Example** 2.* (The cyclic formation of unicycles, continued). Let $\gamma \in \mathcal{C}^2(\mathbb{R}, \mathbb{R}^2)$ be a L-periodic function that represents a closed curve in $\mathbb{R}^2$ such that $\|\dot{\gamma}(t)\| = 1$, $\forall t \in \mathbb{R}$. Set $x_r : \mathbb{R} \to \mathbb{R}^2 \times S^1$ such that $x_r(t) = (\gamma(t), \arg\dot{\gamma}(t))$ and $u_r : \mathbb{R} \to \mathbb{R}$ such that $u_r(t) = \frac{d}{dt}\arg\dot{\gamma}(t)$. Set $\tilde{x}(t) = (x_r(t), x_r(t + L/k), \ldots, \tilde{x}_r(t + L\frac{k-1}{k}))$, $\tilde{u}(t) = (u_r(t), u_r(t + L/k), \ldots, u_r(t + L\frac{k-1}{k}))$. Then $\tilde{x}$, with control $\tilde{u}$ is a solution of (7). Note that, by construction, $\Gamma\tilde{x}(t) = \tilde{x}(t + \tau)$ and $\Theta\tilde{u}(t) = \tilde{u}(t + \tau)$, with $\tau = \frac{L}{k}$. For instance, for $k = 4$, if γ is the unit-speed reparameterization of the parametric curve $\bar{\gamma}(s) = (3\cos(s/3), \sin(s))^T$, $\forall s \in \mathbb{R}$, the initial configuration of the vehicles $\tilde{x}(0)$ is represented in figure 2.

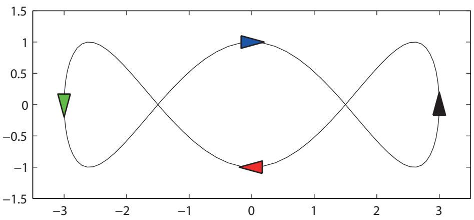


Fig. 2. The initial reference configuration $\tilde{x}(0)$.

Consider the linearization of the $(\Gamma, \Theta, \Sigma, \tau)$-symmetric system (1) along the trajectory $\tilde{x}$, $\tilde{u}$

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t)\,, \end{aligned} \tag{13}$$

where

$$\begin{aligned} A(t) = \partial_x f(t, x, u)|_{x=\tilde{x}(t), u=\tilde{u}(t)}\,, B(t) = \partial_u f(t, x, u)|_{x=\tilde{x}(t), u=\tilde{u}(t)}\,, \\ C(t) = \partial_x h(t, x, u)|_{x=\tilde{x}(t), u=\tilde{u}(t)}\,, D(t) = \partial_u h(t, x, u)|_{x=\tilde{x}(t), u=\tilde{u}(t)}\,. \end{aligned} \tag{14}$$

***Proposition** 3.* The quadruple (A, B, C, D) defined in (14) is $(\Gamma, \Theta, \Sigma, \tau)$-symmetric, that is, it verifies properties (6a)-(6d).

Proof.

$$\begin{aligned} \Gamma A(t) = \Gamma \partial_x f(t, x, u)|_{x=\tilde{x}(t), u=\tilde{u}(t)} &= \partial_x \Gamma f(t, x, u)|_{x=\tilde{x}(t), u=\tilde{u}(t)} \\ = \partial_x f(t + \tau, \Gamma x, \Theta u)|_{x=\tilde{x}(t), u=\tilde{u}(t)} &= \partial_x f(t + \tau, x, u)|_{x=\Gamma\tilde{x}(t), u=\Theta\tilde{u}(t)}\Gamma \\ = \partial_x f(t + \tau, x, u)|_{x=\tilde{x}(t+\tau), u=\tilde{u}(t+\tau)}\Gamma &= A(t + \tau)\Gamma, \end{aligned}$$

the proof for $B(t), C(t), D(t)$ is analogous. $\square$

***Proposition** 4.* Consider the linear time-varying state feedback

$$u(t) = F(t)x\,, \tag{15}$$

if

$$\Theta F(t) = F(t + \tau)\Gamma \tag{16}$$

and the closed-loop system (13)+(15) is exponentially stable, then the controller

$$u(t) = l(t, x) = \tilde{u}(t) + F(t)(x - \tilde{x}(t))\,, \tag{17}$$

solves problem 1.

Proof. The linearization of the closed loop system (1)+(17) along the trajectory $\tilde{x}$ and the nominal input $\tilde{u}$ is given by (13)+(15). Hence (1)+(17) is exponentially stable if and only if (13)+(15) is exponentially stable. Moreover conditions (10) are satisfied since

$$\begin{aligned} \Theta l(t, x) = \Theta F(t)(x - \tilde{x}(t)) &= F(t + \tau)\Gamma(x - \tilde{x}(t)) \\ = F(t + \tau)(\Gamma x - x(t + \tau)) &= l(t + \tau, \Gamma x)\,. \end{aligned} \qquad \square$$

***Proposition** 5.* Consider the linear observer-based controller

$$\begin{aligned} \dot{e}(t) &= g(t, e(t), y(t)) \\ &= A(t)e(t) - K(t)(y(t) - C(t)e(t) - D(t)u(t)) \\ u(t) &= l(t, e(t)) = F(t)e(t)\,, \end{aligned} \tag{18}$$

if

$$\Gamma K(t) = K(t + \tau)\Sigma, \quad \Theta F(t) = F(t + \tau)\Gamma \tag{19}$$

and the closed-loop system (13)+(18) is exponentially stable, then the controller

$$\begin{aligned} \dot{e}(t) &= A(t)e(t) \\ &\quad - K(t)(y(t) - \tilde{y}(t) - C(t)e(t) - D(t)(u(t) - \tilde{u}(t))) \\ u(t) &= \tilde{u}(t) + F(t)e(t)\,, \end{aligned} \tag{20}$$

solves problem 2.

Proof. The linearization of the closed loop system (1)+(20) along the trajectory $\tilde{x}$ and the nominal input $\tilde{u}$ is given by (13)+(18). Hence (1)+(20) is exponentially stable if and only if (13)+(18) is exponentially stable. Moreover conditions (12) are satisfied since

$$\begin{aligned}\Gamma g(t,e(t),y(t)) &= \Gamma(A(t)e(t) - K(t)(y(t) - (C(t)e(t) + \tilde{y}(t))))\\ &= (A(t+\tau)\Gamma e(t) - K(t+\tau)(\Theta y(t) - C(t+\tau)\Gamma e(t) + \tilde{y}(t+\tau)))\\ &= g(t+\tau, \Gamma e(t), \Theta y(t)),\end{aligned}$$

and

$$\begin{aligned}\Theta l(t,e(t)) &= \Theta(\tilde{u}(t) + F(t)e(t))\\ &= \tilde{u}(t+\tau) + F(t+\tau)\Gamma e(t) = l(t+\tau, \Gamma e(t)).\end{aligned}$$ □

3. LINEAR SYSTEMS WITH SPATIO-TEMPORAL SYMMETRY

Consider the class of linear time-varying systems

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t)\\ y(t) &= C(t)x(t) + D(t)u(t)\end{aligned} \tag{21}$$

where $A \in \mathcal{C}(\mathbb{R}, \mathbb{R}^{n\times n})$, $B \in \mathcal{C}(\mathbb{R}, \mathbb{R}^{n\times m})$, $C \in \mathcal{C}(\mathbb{R}, \mathbb{R}^{p\times n})$, $D \in \mathcal{C}(\mathbb{R}, \mathbb{R}^{p\times m})$.

Definition 4. Set $\lfloor t\rfloor = \max\{i \in \mathbb{Z}, \mid i \le t\}$ as the integer part of t and denote by $\pi : \mathbb{R} \to [0,\tau)$ the map defined by $\pi(t) = t - \lfloor \frac{t}{\tau}\rfloor\tau$, in other words, $\pi(t)$ is the remainder of the division of t by τ.

The following proposition shows that any $(\Gamma,\Theta,\Sigma,\tau)$-symmetric quadruple (A,B,C,D) is uniquely determined by its value in the interval $[0,\tau)$.

Proposition 6. The quadruple (A,B,C,D) satisfies (6a)–(6d) if and only if, $\forall t \in \mathbb{R}$,

$$A(t) = \Gamma^{\lfloor \frac{t}{\tau}\rfloor} A(\pi(t))\Gamma^{-\lfloor \frac{t}{\tau}\rfloor} \tag{22a}$$

$$B(t) = \Gamma^{\lfloor \frac{t}{\tau}\rfloor} B(\pi(t))\Theta^{-\lfloor \frac{t}{\tau}\rfloor} \tag{22b}$$

$$C(t) = \Sigma^{\lfloor \frac{t}{\tau}\rfloor} C(\pi(t))\Gamma^{-\lfloor \frac{t}{\tau}\rfloor} \tag{22c}$$

$$D(t) = \Sigma^{\lfloor \frac{t}{\tau}\rfloor} D(\pi(t))\Theta^{-\lfloor \frac{t}{\tau}\rfloor}. \tag{22d}$$

Proof. We prove the first of (22a), the others are analogous. Applying $\lfloor \frac{t}{\tau}\rfloor$ times (6a), it follows that $\Gamma^{\lfloor \frac{t}{\tau}\rfloor}A(\pi(t)) = \Gamma^{\lfloor \frac{t}{\tau}\rfloor}A(t - \lfloor \frac{t}{\tau}\rfloor\tau) = A(t)\Gamma^{\lfloor \frac{t}{\tau}\rfloor}$, $\forall t \in \mathbb{R}$, from which (22a) follows, since Γ is invertible. Conversely, if (22a) holds,

$$\begin{aligned}\Gamma A(t) &= \Gamma^{\lfloor \frac{t}{\tau}\rfloor+1} A(\pi(t))\Gamma^{-\lfloor \frac{t}{\tau}\rfloor}\\ &= \Gamma^{\lfloor \frac{t}{\tau}\rfloor+1} A(\pi(t))\Gamma^{-\lfloor \frac{t}{\tau}\rfloor-1}\Gamma = A(t+\tau)\Gamma,\ \forall t \in \mathbb{R}.\end{aligned}$$ □

The following proposition shows that, if (A,B,C,D) is $(\Gamma,\Theta,\Sigma,\tau)$-symmetric, system (21) is equivalent, after a change of variables, to an hybrid periodic system (see equation (23) below).

Proposition 7. Suppose that (A,B,C,D) satisfies (6a)–(6d). Then if x, y, u satisfy system (21), functions

$$\xi(t) = \Gamma^{-\lfloor \frac{t}{\tau}\rfloor}x(t),\ \eta(t) = \Sigma^{-\lfloor \frac{t}{\tau}\rfloor}y(t), v(t) = \theta^{-\lfloor \frac{t}{\tau}\rfloor}u(t),$$

satisfy the system

$$\begin{cases}\dot{\xi}(t) = A(\pi(t))\xi(t) + B(\pi(t))v(t), \text{ if } t \in \mathbb{R}\backslash\tau\mathbb{Z}\\ \xi(t) = \lim_{s\to t^-} \Gamma^{-1}\xi(s), \text{ if } t \in \tau\mathbb{Z}\\ \eta(t) = C(\pi(t))\xi(t) + D(\pi(t))v(t)\ \forall t \in \mathbb{R}.\end{cases} \tag{23}$$

Conversely, if (ξ,η,v) is a solution of (23), then $x(t) = \Gamma^{\lfloor \frac{t}{\tau}\rfloor}\xi(t)$, $y(t) = \Sigma^{\lfloor \frac{t}{\tau}\rfloor}\eta$, $u = \Theta^{\lfloor \frac{t}{\tau}\rfloor}v$ is a solution of (21).

Proof. It follows from proposition 14 (see the Appendix) with $G(t) = B(t)u(t)$ and from (6b), (6c), (6d). □

We first consider the autonomous case of (21): $\dot{x}(t) = Ax(t)$, where A is (Γ,τ)-symmetric (i.e. it verifies (6a) and there exists $k \ge 1$ such that $\Gamma^k = I$). In the following, we denote the transition matrix of $A \in \mathcal{C}(\mathbb{R}, \mathbb{R}^{n\times n})$ by $\Phi(t)$, that is the solution of

$$\begin{cases}\dot{\Phi}(t) = A(t)\Phi(t)\\ \Phi(0) = I.\end{cases} \tag{24}$$

Proposition 8. Suppose that A is (Γ,τ)-symmetric. Then system $\dot{x}(t) = A(t)x(t)$ is asymptotically stable if and only if all the eigenvalues λ of $\Gamma^{-1}\Phi(\tau,0)$ are such that $|\lambda| < 1$.

Proof. The thesis follows from propositions 14 and 16 (see the Appendix) and the fact that $\Gamma^{\lfloor \frac{t}{\tau}\rfloor}$ is bounded $\forall t \ge 0$ (see the notations) □

Consider the controlled system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t). \tag{25}$$

The following proposition gives a condition under which the (Γ,Θ,τ)-symmetry of (25) is preserved after the application of the feedback law $u(t) = F(t)x(t) + r(t)$.

Proposition 9. If the couple (A,B) is (Γ,Θ,τ)-symmetric, then the couple $(A+BF,B)$ has the same property if

$$\Theta F(t) = F(t+\tau)\Gamma,\ \forall t \in \mathbb{R}. \tag{26}$$

Conversely, if (26) holds and $B(t)$ is full rank for all $t \in \mathbb{R}$, then $(A+BF,B)$ is (Γ,Θ,τ)-symmetric.

Proof. (Sufficiency) Assume that (26) holds, then

$$\begin{aligned}\Gamma(A(t) + B(t)F(t)) &= A(t+\tau)\Gamma + B(t+\tau)\Theta F(t)\\ &= (A(t+\tau) + B(t+\tau)F(t+\tau))\Gamma.\end{aligned}$$

(Necessity) If the closed-loop system is (Γ,Θ,τ)-symmetric if follows that

$$\Gamma(A(t) + B(t)F(t)) = (A(t+\tau) + B(t+\tau)F(t+\tau))\Gamma.$$

Moreover,

$$\Gamma(A(t) + B(t)F(t)) = A(t+\tau)\Gamma + B(t+\tau)\Theta F(t).$$

These two properties imply that

$$B(t+\tau)(\Theta F(t) - F(t+\tau)\Gamma) = 0.$$

Since $B(t+\tau)$ is full rank for every $t \in \mathbb{R}$, equation (26) follows. □

The following discussion on stabilizability is an extension to systems with spatio-temporal symmetry of the classical results for periodic system (see Bittanti and Bolzern [1985], Kano and Nishimura [1985]). In particular, the following definition is based on the notion of W-stabilizability.

Definition 5. The (Γ,Θ,τ)-symmetric couple (A,B) is (Γ,Θ,τ)-stabilizable if there exists a matrix function $F : \mathbb{R} \to \mathbb{R}^{m\times n}$ satisfying (26), such that system

$$\dot{x}(t) = (A(t) + B(t)F(t))x(t)$$

is asymptotically stable.

Definition 6. A complex number λ is called an uncontrollable eigenvalue of the (Γ,Θ,τ)-symmetric system (25) if there exists $\eta \in \mathbb{C}^n$ such that

$$\eta^T\Gamma^{-1}\Phi(\tau) = \lambda\eta^T \tag{27a}$$

$$\eta^T(\Phi^{-1}\Gamma)^i\Phi(s)^{-1}B(s) = 0,\ \forall s \in [0,\tau),\ \forall i = 0,\ldots,n-1. \tag{27b}$$

The following proposition characterizes the stabilizability of linear systems with spatio-temporal symmetries and presents a method for the synthesis of a stabilizing feedback gain matrix $F(t)$ that satisfies (26).

***Theorem** 1.* System (25) is (Γ,Θ,τ)-stabilizable if and only if all its uncontrollable eigenvalues λ are such that $|\lambda| < 1$. Moreover, if this condition holds, for any symmetric positive definite matrices $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$ the system is stabilized by the feedback control $u(t) = F(t)x(t)$, where

$$F(t) = -\Theta^{\lfloor \frac{t}{\tau} \rfloor} R^{-1} B^T(\pi(t)) S(t) \Gamma^{-\lfloor \frac{t}{\tau} \rfloor}, \tag{28}$$

where S is the unique τ-periodic symmetric and positive definite matrix solution of the following hybrid Riccati equation

$$\begin{cases} \dot S(t) - S(t)B(\pi(t))R^{-1}B^T(\pi(t))S(t) + S(t)A(\pi(t)) \\ \qquad +A^T(\pi(t))S(t) + Q = 0 \text{ if } t \in \mathbb{R}\backslash\tau\mathbb{Z} \\ S(t) = \lim_{s\to t^-} \Gamma^T S(s)\Gamma, \text{ if } t \in \tau\mathbb{Z}\,. \end{cases} \tag{29}$$

Finally, function F satisfies (26).

Proof. ($\Rightarrow$) Assume by contradiction that λ is an uncontrollable eigenvalue of $\Gamma^{-1}\Phi(\tau,0)$ such that $|\lambda| \geq 1$. Let η^T be the associated left eigenvector. Consider the solution of (23) with initial condition $\xi(0) = \eta$. By (37) and (36) setting $G(t) = B(\pi(t))F(\pi(t))\xi(t)$, it follows that $\xi(k\tau) = (\Gamma^{-1}\Phi(\tau))^k(\eta + \int_0^{k\tau}(\Phi^{-1}(\tau)\Gamma)^{\lfloor \frac{s}{\tau}\rfloor}\Phi^{-1}(\pi(s))$ $B(\pi(s))F(\pi(s))\Gamma^{\lfloor \frac{s}{\tau}\rfloor}\xi(s)ds)$. Therefore, by (27a),

$$\eta^T\xi(k\tau) = \lambda^k\eta^T\big(\eta + \int_0^{k\tau}(\Phi^{-1}(\tau)\Gamma)^{\lfloor \frac{s}{\tau}\rfloor}$$
$$\cdot\Phi^{-1}(\pi(s))B(\pi(s))F(\pi(s))\Gamma^{\lfloor \frac{s}{\tau}\rfloor}\xi(s)ds\big) = \lambda^k\eta^T\eta\,,$$

since, by the Hamilton-Cayley theorem, (27b) holds for any $i \in \mathbb{N}$. This implies that (25) is not asymptotically stable by proposition 14, since $|\lambda| > 1$.

($\Leftarrow$) This part of the proof is more complex and, due to space limitations, is not presented in this conference paper.

□

The use of theorem 1 and proposition 4 allows to give a sufficient condition for the solution of problem 1.

***Proposition** 10.* If the linearization (5) of the (Γ,Θ,Σ)-symmetric system (1) on the (Γ,Θ,τ)-symmetric couple $(\tilde x,\tilde u)$ has no uncontrollable eigenvalues λ such that $|\lambda| \geq 1$, then the controller (17), with F given by (28), solves problem 1.

Proof. It is a consequence of propositions 1 and 4. □

3.1 Detectability

In this section we design an asymptotic observer for system (21) of the form

$$\begin{aligned} \dot{\hat x}(t) &= A(t)\hat x(t) + B(t)u(t) + K(t)(\hat y(t) - y(t))) \\ \hat y(t) &= C(t)\hat x(t) + D(t)u(t) \end{aligned}$$

such that the observer gain matrix $K(t)$ satisfies (19). As one would expect, $K(t)$ can be designed by the same procedure of the feedback gain matrix $F(t)$ by considering an appropriate dual system.

The following result is analogous to proposition 9 and its proof is omitted.

***Proposition** 11.* If the couple (A,C) is (Γ,Σ,τ)-symmetric, then the couple $(A + KC, C)$ has the same property if

$$\Gamma K(t) = K(t+\tau)\Sigma,\ \forall t \in \mathbb{R}\,. \tag{30}$$

Conversely, if (30) holds and C is full rank for all $t \in \mathbb{R}$, then $(A + KC, C)$ is (Γ,Σ,τ)-symmetric.

The following definition is the counterpart of definition 5 related to detectability.

***Definition** 7.* The (Γ,Σ,τ)-symmetric couple (A,C) is (Γ,Σ,τ)-detectable if there exists a matrix function $K : \mathbb{R}\to\mathbb{R}^{m\times n}$ satisfying (30), such that system

$$\dot x(t) = (A(t) + K(t)C(t))x(t)$$

is asymptotically stable.

The following propositions follow from the duality result given in proposition 17 of the appendix.

***Proposition** 12.* If (A,C) is (Γ,Σ,τ)-symmetric, then system $\dot x = (A(t)+K(t)C(t))x(t)$ is asymptotically stable if and only if the dual system

$$\dot x(t) = (A^T(-t) + C^T(-t)K^T(-t))x(t)$$

is asymptotically stable.

***Proposition** 13.* The (Γ,Σ,τ)-symmetric couple (A,C) is (Γ,Σ,τ)-detectable if and only if the dual couple $(A^T(-t), C^T(-t))$ is (Γ,Θ,τ)-stabilizable.

***Remark** 4.* A stabilizing observer gain matrix can be obtained by applying the method presented in proposition 1 to the dual system.

The use of propositions 13 and 5 allows to give a sufficient condition for the solution of problem 2.

***Theorem** 2.* If the linearization (5) of the (Γ,Θ,Σ)-symmetric system (1) on the (Γ,Θ,τ)-symmetric couple $\tilde x$, $\tilde u$, has no uncontrollable eigenvalues λ such that $|\lambda| \geq 1$ and the dual system satisfies the same property, then the controller (20) where F is given by (28) and K is given by (28) applied to the dual system, solves problem 2.

Proof. It is a consequence of propositions 13 and 5. □

4. APPLICATION TO THE CONTROL OF A CYCLIC FORMATION OF MOBILE ROBOTS

We go back to the unicycle formation example. By construction $\Gamma\tilde x(t) = \tilde x(t+\tau)\Gamma$ and $\Theta\tilde u(t) = \tilde u(t+\tau)$, therefore, by proposition 3, the linearized system (13) is $(\Gamma,\Sigma,\Theta,L/n)$-symmetric. Since the hypotheses of proposition 20 are satisfied, we can locally stabilize the trajectory $\tilde x$ with the controller (20) if the matrix functions F, K stabilize the linearized system (13)+(18). To find these functions, we use the method presented in section 3. In particular K and F are obtained using the Riccati equation (29) for the linearized system and its dual, with matrices Q and R chosen as the identity. The vehicles' trajectories, together with the the norm of the tracking error and the observer error are reported in Figures 3, 4(a), 4(b).

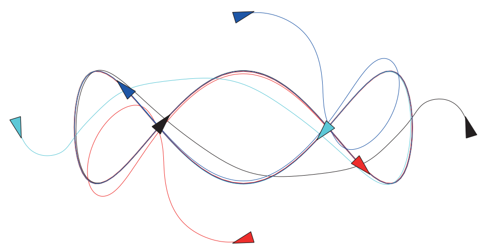
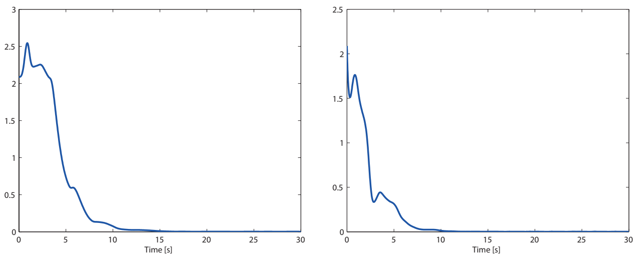


Fig. 3. The closed-loop trajectories $x(t)$.

APPENDIX

In the following, we suppose that $A \in \mathcal{C}(\mathbb{R},\mathbb{R}^{n\times n})$.



(a) Tracking error norm (b) Observer error norm

Fig. 4. Plot (a): the tracking error norm $\|x(t)-\tilde{x}(t)\|$. Plot (b): the observer error norm $\|x(t)-\tilde{x}(t)-e(t)\|$.

Remark 5. Let $M : [0,\tau) \to \mathbb{R}^{n\times n}$ be a continuous and bounded map and $N : \mathbb{R} \to \mathbb{R}^n$ a map bounded on $\mathbb{R}$ and continuous on $\mathbb{R}\setminus\tau\mathbb{Z}$. Then, $\forall t_0 \in \mathbb{R}$, $\forall x_0 \in \mathbb{R}^n$, there exists a unique $x \in \mathcal{C}^1(\mathbb{R}\backslash\tau\mathbb{Z},\mathbb{R}^n)\cap\mathcal{C}_+(\mathbb{R},\mathbb{R}^n)$, such that

$$\begin{cases} \dot{x}(t) = M(\pi(t)) + N(t), \text{if } t\in\mathbb{R}\backslash\tau\mathbb{Z} \\ x(t) = \lim_{s\to t^-}\Gamma^{-1}x(s), \text{if } t\in\tau\mathbb{Z} \\ x(t_0) = x_0\,. \end{cases}$$

Proposition 14. If A verifies (6a) and $G \in \mathcal{C}(\mathbb{R},\mathbb{R}^n)$, then

a) If $x \in \mathcal{C}^1(\mathbb{R},\mathbb{R}^n)$ is a solution of

$$\dot{x}(t) = A(t)x(t) + G(t),\ \forall t\in\mathbb{R} \qquad (31)$$

then the map $\xi(t) = \Gamma^{-\lfloor\frac{t}{\tau}\rfloor}x(t)$ is such that $\xi \in \mathcal{C}^1(\mathbb{R}\backslash\tau\mathbb{Z},\mathbb{R}^n)\cap\mathcal{C}_+(\mathbb{R},\mathbb{R}^n)$ and

$$\begin{cases} \dot{\xi}(t) = A(\pi(t))\xi(t) + \Gamma^{-\lfloor\frac{t}{\tau}\rfloor}G(t), \forall t\in\mathbb{R}\backslash\tau\mathbb{Z} \\ \xi(t) = \lim_{s\to t^-}\Gamma^{-1}\xi(s), \text{if } t\in\tau\mathbb{Z}\,. \end{cases} \qquad (32)$$

b) Conversely if $\xi \in \mathcal{C}^1(\mathbb{R}\backslash\tau\mathbb{Z},\mathbb{R}^n)\cap\mathcal{C}_+(\mathbb{R},\mathbb{R}^n)$ verifies (32) then

$$x(t) = \Gamma^{\lfloor\frac{t}{\tau}\rfloor}\xi(t),\ \forall\mathbb{R}\,, \qquad (33)$$

is such that $x \in \mathcal{C}^1(\mathbb{R},\mathbb{R}^n)$ and is a solution of system (31).

Proof omitted due to space limitations.

Proposition 15. Suppose that A verifies (6a), then the matrix solution of

$$\begin{cases} \dot{\Psi}(t) = A(\pi(t))\Psi(t), \text{ if } t\in\mathbb{R}\backslash\tau\mathbb{Z} \\ \Psi(t) = \lim_{s\to t^-}\Gamma^{-1}\Psi(s), \text{ if } t\in\tau\mathbb{Z} \\ \Psi(0) = I\,, \end{cases} \qquad (34)$$

is given by

$$\Psi(t) = \Phi(\pi(t))(\Gamma^{-1}\Phi(\tau))^{\lfloor\frac{t}{\tau}\rfloor},\ \forall t\in\mathbb{R}\,. \qquad (35)$$

Proof. Clearly $\Psi(0) = \Phi(0)(\Gamma^{-1}\Phi(\tau)^{\lfloor 0\rfloor} = I$ and if $t\in\tau\mathbb{Z}$, there exists $i\in\mathbb{Z}$ such that $t = i\tau$, then

$$\Psi(t) = \Phi(0)(\Gamma^{-1}\Phi(\tau))^i = \Gamma^{-1}\lim_{s\to i\tau^-}\Phi(s)(\Gamma^{-1}\Phi(\tau))^{i-1}$$

$$= \Gamma^{-1}\lim_{s\to i\tau^-}\Phi(\pi(s))(\Gamma^{-1}\Phi(\tau))^{i-1} = \lim_{s\to i\tau^-}\Gamma^{-1}\Psi(s)\,.$$

Moreover, being $\lfloor\frac{t}{\tau}\rfloor$ locally constant on $\mathbb{R}\backslash\tau\mathbb{Z}$, we have that, $\forall t\in\mathbb{R}\backslash\tau\mathbb{Z}$, $\dot{\Psi}(t) = \frac{d}{dt}\Phi(\pi(t))(\Gamma^{-1}\Phi(\tau))^{\lfloor\frac{t}{\tau}\rfloor}$ $= A(\pi(t))\Phi(\pi(t))(\Gamma^{-1}\Phi(\tau))^{\lfloor\frac{t}{\tau}\rfloor} = A(\pi(t))\Psi(t)$. The uniqueness is proved by induction on intervals $[i\tau,\tau+i\tau]$. □

Remark 6. By defining $\Psi(\tau,s) = \Psi(t)\Psi^{-1}(s)$, $\forall t,s\in\mathbb{R}$ it follows that

$$\Psi(t,s) = \Phi(\pi(t))(\Gamma^{-1}\Phi(\tau))^{\lfloor\frac{t}{\tau}\rfloor-\lfloor\frac{s}{\tau}\rfloor}\Phi(\pi(s))^{-1} \qquad (36)$$

is the solution of (32) with initial condition $\xi(\bar{t}) = \bar{\xi}$ is given by

$$\xi(t) = \Psi(t,\bar{t})\bar{\xi} + \int_{\bar{t}}^{t}\Psi(t,s)\Gamma^{-\lfloor\frac{t}{\tau}\rfloor}G(s)ds\,. \qquad (37)$$

Proposition 16. Suppose that A satisfies (6a), system

$$\begin{cases} \dot{\xi}(t) = A(\pi(t))\xi(t), \text{ if } t\in\mathbb{R}\backslash\tau\mathbb{Z} \\ \xi(t) = \lim_{s\to t^-}\Gamma^{-1}\xi(t),\ \forall t\in\tau\mathbb{Z} \end{cases} \qquad (38)$$

is asymptotically stable if and only if all the eigenvalues λ of $\Psi(\tau) = \Gamma^{-1}\Phi(\tau)$ are such that $|\lambda| < 1$.

Proof omitted due to space limitations.

The following proposition is a duality result that is used for the synthesis of an asymptotic observer.

Proposition 17. If $\Psi(t)$ is the solution of

$$\begin{cases} \dot{\Psi}(t) = A(\pi(t))\Psi(t),\ \forall t\in\mathbb{R} \\ \Psi(t) = \lim_{s\to t^-}\Gamma^{-1}\Psi(s),\ \forall t\in\tau\mathbb{Z} \\ \Psi(\tau) = I\,, \end{cases} \qquad (39)$$

and $\Psi_D(t)$ is the solution of

$$\begin{cases} \dot{\Psi}_D(t) = A^T(\pi(-t))\Psi_D(t),\ \forall t\in\mathbb{R} \\ \Psi_D(t) = \lim_{s\to t^+}\Gamma^T\Psi_D(s),\ \forall t\in\tau\mathbb{Z} \\ \Psi_D(0) = I\,, \end{cases} \qquad (40)$$

then $\Psi_D(t) = \Psi^{-T}(-t)$.

Proof omitted due to space limitations.

REFERENCES

Sergio Bittanti and Paolo Bolzern. Stabilizability and detectability of linear periodic systems. *Systems & Control Letters*, 6(2):141 – 145, 1985.

Pietro-Luciano Buono and Martin Golubitsky. Models of central pattern generators for quadruped locomotion i. primary gaits. *Journal of Mathematical Biology*, 42:291–326, 2001.

P. Chossat and R. Lauterbach. *Methods in Equivariant Bifurcations and Dynamical Systems.* Advanced Series in Nonlinear Dynamics. World Scientific, 2000.

M. El-Hawwary and M. Maggiore. Distributed circular formation stabilization for dynamic unicycles. *Automatic Control, IEEE Transactions on*, in press(99):1, 2012.

M. Golubitsky and I. Stewart. *The Symmetry Perspective.* Birkhäuser Verlag, 2003.

Martin Golubitsky, Ian Stewart, Pietro-Luciano Buono, and J. J. Collins. Symmetry in locomotor central pattern generators and animal gaits. *Nature*, 401:693–695, 1999.

Sarah C. Hamilton and Mireille E. Broucke. Patterned linear systems. *Automatica*, 48(2):263 – 272, 2012a.

S.C. Hamilton and M.E. Broucke. *Geometric Control of Patterned Linear Systems.* Lecture Notes in Control and Information Sciences. Springer, 2012b.

H. Kano and T. Nishimura. Controllability, stabilizability, and matrix riccati equations for periodic systems. *Automatic Control, IEEE Transactions on*, 30(11):1129 – 1131, nov 1985.

Alain Karma and Jr Robert F. Gilmour. Nonlinear dynamics of heart rhythm disorders. *Physics Today*, 60 (3):51 – 57, 2007.

J.A. Marshall, M.E. Broucke, and B.A. Francis. Formations of vehicles in cyclic pursuit. *Automatic Control, IEEE Transactions on*, 49(11):1963 – 1974, nov. 2004.

F. Zhang. *Matrix Theory: Basic Results and Techniques.* Universitext (1979). Springer, 1999.