

Control Lyapunov function based feedback design for quasi-polynomial systems

Attila Magyar* Katalin M. Hangos**

* *Department of Electrical Engineering and Information Systems,
University of Pannonia, Veszprém, Hungary (e-mail:
magyar.attila@virt.uni-pannon.hu).*

** *Process Control Research Group, Computer and Automation
Research Institute HAS, Budapest, Hungary (e-mail:
hangos@scl.sztaki.hu).*

Abstract: The aim of this work is to present an entropy-like Lyapunov function based dynamic feedback design technique for quasi-polynomial and Lotka-Volterra systems. It is shown, that the dynamic feedback design problem is equivalent to the feasibility of a bilinear matrix inequality. The problem is also formulated as a control Lyapunov function based feedback design when the Lyapunov function parameters are given, the solution of this problem can be obtained by solving a linear matrix inequality.

The developed method is illustrated on a simple numerical example.

Keywords: Lyapunov stability, nonlinear systems, stabilizing feedback, dynamic feedback, optimization problems

1. INTRODUCTION

A wide range of systems can only be tackled using nonlinear techniques (Isidori (1995)). Most of them are applicable only for a small class of nonlinear systems, while the more generally applicable methods suffer from computational complexity problems. One possible way of balancing between general applicability and computational feasibility is to find nonlinear system classes with good descriptive power but well characterized structure, and utilize this structure when developing control design methods.

The class of quasi-polynomial (QP) systems plays an important role in the theory of nonlinear and nonnegative dynamical systems because nonlinear systems with smooth nonlinearities can be transformed into quasi-polynomial form (Hernández-Bermejo and Fairén (1995)). This means, that any applicable method for quasi-polynomial systems can be regarded as a general technique for nonlinear systems.

QP-systems are invariant under quasi-monomial transformation (Figueiredo et al. (2000)), this enables to partition them into equivalence classes represented by a Lotka-Volterra (LV) system. Dynamic similarity can also be defined for them (see Hangos and Szederkényi (2012)) that shows the dynamic relationship between QP and chemical reaction networks with mass action law kinetics in a simple and transparent way.

Previous work in the field of quasi-polynomial systems include the paper of Figueiredo et al. (2000), which proves, that the global stability analysis of quasi-polynomial systems is equivalent to the feasibility of a linear matrix inequality (LMI). It has been shown in Magyar et al. (2008), that the globally stabilizing state feedback design

for quasi-polynomial systems is equivalent to a bilinear matrix inequality, although the solution of a bilinear matrix inequality is an NP hard problem an iterative LMI algorithm could be used. A summary of linear and bilinear matrix inequalities and the available software tools for solving them can be found in VanAntwerp and Braatz (2000). Another control synthesis algorithm for polynomial systems is presented in Tong et al. (2007).

Bliman (2004) shows that a system of LMIs depending continuously on some scalar parameters admits a branch of solutions polynomials in the parameters. This result points to some further directions of robust/optimal control formalized for quasi-polynomial systems.

The above earlier attempts to design controllers for (quasi-)polynomial systems have resulted in computationally hard optimization problems by using static feedback structures. The aim of this paper is to propose a computationally feasible alternative by utilizing the more rich structure of a dynamic QP-feedback.

2. BASIC NOTIONS

2.1 Quasi-polynomial and Lotka-Volterra models

Quasi-polynomial models are systems of ODEs of the following form

$$\dot{x}_i = x_i \left(\lambda_i + \sum_{j=1}^m A_{i,j} \prod_{k=1}^n x_k^{B_{j,k}} \right), \quad i = 1, \dots, n. \quad (1)$$

where $\mathbf{x} = [x_1, x_2, \dots, x_n]^T \in \text{int}(\mathbb{R}_+^n)$, $\mathbf{A} \in \mathbb{R}^{n \times m}$, $\mathbf{B} \in \mathbb{R}^{m \times n}$, $\boldsymbol{\lambda} \in \mathbb{R}^n$. The equilibrium point of interest of (1) is denoted by $\mathbf{x}^* = [x_1^* \ x_2^* \ \dots \ x_n^*]^T$. Without the

loss of generality it can be assumed that $\text{rank}(\mathbf{B}) = n$ and $m \geq n$ (see Hernández-Bermejo et al. (1998)).

A special property of systems in the form (1) is that the products $\mathbf{M} = \mathbf{B} \mathbf{A}$ and $\mathbf{N} = \mathbf{B} \boldsymbol{\lambda}$ are invariant for groups of models. This way the class of quasi-polynomial models can be partitioned according to the values of these products. The *Lotka-Volterra form* known from the field of population biology (see Lotka (1925), Volterra (1926)), gives the representative elements of these classes of equivalence. If $\text{rank}(\mathbf{B}) = n$, then the set of ODEs in (1) can be embedded into the following m -dimensional set of equations, the so called Lotka-Volterra model:

$$\dot{z}_j = z_j \left(N_j + \sum_{i=1}^m M_{j,i} z_i \right), \quad j = 1, \dots, m \quad (2)$$

where

$$\mathbf{M} = \mathbf{B} \mathbf{A}, \quad \mathbf{N} = \mathbf{B} \boldsymbol{\lambda},$$

and each z_j represents a so called *quasi-monomial*:

$$z_j = \prod_{k=1}^n x_k^{B_{j,k}}, \quad j = 1, \dots, m. \quad (3)$$

2.2 Input-affine quasi-polynomial system models

A general input-affine nonlinear state equation

$$\dot{x}_i = f_i(x) + \sum_{l=1}^p g_i(x) u_l, \quad i = 1, \dots, n, l = 1, \dots, p \quad (4)$$

is in quasi-polynomial form if all of the functions f_i , and g_i are quasi-polynomial functions. Then the general form of the state equation of an input-affine quasi-polynomial system model with p -inputs is:

$$\dot{x}_i = x_i \left(\lambda_{0i} + \sum_{j=1}^m A_{0,i,j} \prod_{k=1}^n x_k^{B_{j,k}} \right) + \quad (5)$$

$$+ \sum_{l=1}^p x_i \left(\lambda_{li} + \sum_{j=1}^m A_{l,i,j} \prod_{k=1}^n x_k^{B_{j,k}} \right) u_l$$

where $i = 1, \dots, n$, $\mathbf{A}_0, \mathbf{A}_l \in \mathbb{R}^{n \times m}$, $\mathbf{B} \in \mathbb{R}^{m \times n}$,

$$\lambda_{0i}, \lambda_{li} \in \mathbb{R}^n, \quad l = 1, \dots, p.$$

The equilibrium point of interest of (5) is denoted by $\mathbf{x}^* = [x_1^* \ x_2^* \ \dots \ x_n^*]^T$ and the corresponding equilibrium inputs are denoted as $\mathbf{u}^* = [u_1^* \ u_2^* \ \dots \ u_p^*]^T$.

The corresponding input-affine Lotka-Volterra model is in the form

$$\dot{z}_j = z_j \left(N_{0j} + \sum_{k=1}^m M_{0,j,k} z_k \right) + \quad (6)$$

$$+ \sum_{l=1}^p z_j \left(N_{lj} + \sum_{k=1}^m M_{l,j,k} z_k \right) u_l$$

where $j = 1, \dots, m$, $\mathbf{M}_0, \mathbf{M}_l \in \mathbb{R}^{m \times m}$, $\mathbf{N}_0, \mathbf{N}_l \in \mathbb{R}^m$, $l = 1, \dots, p$, and the parameters can be obtained from

the input-affine quasi-polynomial system's ones in the following way

$$\begin{aligned} \mathbf{M}_0 &= \mathbf{B} \mathbf{A}_0 \\ \mathbf{N}_0 &= \mathbf{B} \boldsymbol{\lambda}_0 \\ \mathbf{M}_l &= \mathbf{B} \mathbf{A}_l \\ \mathbf{N}_l &= \mathbf{B} \boldsymbol{\lambda}_l \end{aligned} \quad l = 1, \dots, p. \quad (7)$$

For the sake of simplicity in the sequel, the following non-autonomous quasi-polynomial form is defined:

$$\dot{x}_i = x_i \left(\lambda_i + \sum_{j=1}^m A_{i,j} \prod_{k=1}^n x_k^{B_{j,k}} \prod_{l=1}^p u_l^{B_{u_{j,k}}} \right), \quad i = 1, \dots, n \quad (8)$$

where B_u is a coefficient matrix containing the input-dependent quasi-monomial coefficients. It can be checked, that input-affine quasi-polynomial systems of the form (5) can be expressed in the form (8), and the input coefficient matrix \mathbf{B}_u consists of zero and one entries.

2.3 Global stability analysis of quasi-polynomial systems

This section reformulates the time-decreasing condition of a class of Lyapunov functions for Lotka-Volterra systems so that widespread numerical solvers can be used for their global stability analysis.

Henceforth it is assumed that \mathbf{x}^* is a positive equilibrium point, i.e. $\mathbf{x}^* \in \text{int}(\mathbb{R}_+^n)$ in the quasi-polynomial case and similarly $\mathbf{z}^* \in \text{int}(\mathbb{R}_+^m)$ is a positive equilibrium point in the Lotka-Volterra case. For LV systems there is a well known candidate Lyapunov function family (Hernández-Bermejo (2002), Figueiredo et al. (2000)), which is in the form:

$$V(\mathbf{z}) = \sum_{i=1}^m c_i \left(z_i - z_i^* - z_i^* \ln \frac{z_i}{z_i^*} \right), \quad (9)$$

$$c_i > 0, \quad i = 1 \dots m,$$

where $\mathbf{z}^* = [z_1^* \ \dots \ z_m^*]^T$ is the equilibrium point corresponding to the equilibrium \mathbf{x}^* of the original quasi-polynomial system (1). The time derivative of the Lyapunov function (9) is:

$$\dot{V}(\mathbf{z}) = \frac{1}{2} (\mathbf{z} - \mathbf{z}^*) (\mathbf{C} \mathbf{M} + \mathbf{M}^T \mathbf{C}) (\mathbf{z} - \mathbf{z}^*) \quad (10)$$

where $\mathbf{C} = \text{diag}(c_1, \dots, c_m)$ and \mathbf{M} is the invariant characterizing the Lotka-Volterra form (2). Therefore the non-increasing nature of the Lyapunov function is equivalent to a feasibility problem over the following set of *linear matrix inequality* (LMI) constraints (see Boyd et al. (1994) or Scherer and Weiland (2000)):

$$\begin{aligned} \mathbf{C} \mathbf{M} + \mathbf{M}^T \mathbf{C} &\leq \mathbf{0} \\ \mathbf{C} &> \mathbf{0} \end{aligned} \quad (11)$$

where the unknown matrix is \mathbf{C} , which is diagonal and contains the coefficients of (9).

It is important to note that the strict positivity constraint on c_i can be relaxed since if the equations of the model (1) are ordered in such a way that the first n rows of \mathbf{B} are linearly independent, then $c_i > 0$ for $i = 1, \dots, n$ and $c_j = 0$ for $j = n + 1, \dots, m$ still guarantee global stability (Figueiredo et al. (2000)).

It is proved in Figueiredo et al. (2000) and Hernández-Bermejo (2002), that the global stability of (2) with Lyapunov function (9) implies the boundedness of solutions

and global stability of the original quasi-polynomial system (1).

Note, that the possibilities to find a Lyapunov function that proves the global asymptotic stability of a quasi-polynomial system can be increased by using time-reparametrization (see Szederkényi et al. (2005) for details).

The derivation of global stability analysis for nonautonomous quasi-polynomial systems from the autonomous case is straightforward. The Lyapunov function (10) also depends on the equilibrium value of the input (\mathbf{u}^*) and has the form

$$\dot{V}(\mathbf{z}) = \frac{1}{2}(\mathbf{z} - \mathbf{z}^*)(\mathbf{C}\tilde{\mathbf{M}} + \tilde{\mathbf{M}}^T\mathbf{C})(\mathbf{z} - \mathbf{z}^*) \quad (12)$$

where $\tilde{\mathbf{M}}$ depends on the coefficient matrices of the input-affine Lotka-Volterra model (6).

$$\tilde{\mathbf{M}} = \mathbf{M}_0 + \sum_{l=1}^p \mathbf{M}_l u_l^*.$$

The corresponding LMI feasibility problem to be solved in order to check global asymptotic stability is

$$\begin{aligned} \mathbf{C}\tilde{\mathbf{M}} + \tilde{\mathbf{M}}^T\mathbf{C} &\leq \mathbf{0} \\ \mathbf{C} &> \mathbf{0} \end{aligned} \quad (13)$$

3. STABILIZING CONTROL OF QUASI-POLYNOMIAL SYSTEMS

First, the static situation with the most general nonlinear QP-structure is revisited, when the numerical solution is computationally challenging. Afterwards, a special case with a simple structure is shown, when a given Lyapunov function is prescribed and a dynamical feedback is used.

3.1 The general static QP-feedback controller design

The globally stabilizing quasi-polynomial static state feedback design problem for QP systems has been formulated in Magyar et al. (2008) as follows. Consider arbitrary quasi-polynomial inputs in the form:

$$u_l = \sum_{i=1}^r k_{il} \hat{q}_i, \quad l = 1, \dots, p \quad (14)$$

where $\hat{q}_i = \hat{q}_i(x_1, \dots, x_n)$, $i = 1, \dots, r$ are arbitrary quasi-polynomial functions of the state variables of (5) and k_{il} is the constant gain of the state-monomial function \hat{q}_i in the l -th input u_l . The closed loop system will also be a QP system with matrices

$$\hat{\mathbf{A}} = \mathbf{A}_0 + \sum_{l=1}^p \sum_{i=1}^r k_{il} \mathbf{A}_{il}, \quad \hat{\mathbf{B}}, \quad (15)$$

$$\hat{\mathbf{L}} = \mathbf{L}_0 + \sum_{l=1}^p \sum_{i=1}^r k_{il} \mathbf{L}_{il}, \quad (16)$$

i.e. $\hat{\mathbf{A}}$ and $\hat{\mathbf{L}}$ are affine functions of the control parameters k_{il} . The matrices and vectors \mathbf{A}_{il} and \mathbf{L}_{il} are defined as the coefficient matrices of the control parameter k_{il} in the closed loop system with control law (14).

Note that the number of quasi-monomials in the closed-loop system (i.e. the dimension of the matrices) together with the matrix $\hat{\mathbf{B}}$ may significantly change depending

on the choice of the feedback structure, i.e. on the quasi-monomial functions \hat{q}_i .

Furthermore, the closed loop Lotka-Volterra coefficient matrix $\hat{\mathbf{M}}$ can also be expressed in the form:

$$\hat{\mathbf{M}} = \hat{\mathbf{B}} \hat{\mathbf{A}} = \mathbf{M}_0 + \sum_{l=1}^p \sum_{i=1}^r k_{il} \mathbf{M}_{il}.$$

Then the global stability analysis of the closed loop system with unknown feedback gains k_{il} leads to the following bilinear matrix inequality

$$\hat{\mathbf{M}}^T \mathbf{C} + \mathbf{C} \hat{\mathbf{M}} \leq \mathbf{0}, \quad (17)$$

i.e.

$$\mathbf{M}_0^T \mathbf{C} + \mathbf{C} \mathbf{M}_0 + \sum_{l=1}^p \sum_{i=1}^r k_{il} \left(\mathbf{M}_{il}^T \mathbf{C} + \mathbf{C} \mathbf{M}_{il} \right) \leq \mathbf{0}. \quad (18)$$

The variables of the BMI are the $p \times r$ k_{il} feedback gain parameters and the c_j , $j = 1, \dots, m$ parameters of the Lyapunov function. If the BMI above is feasible then there exists a globally stabilizing feedback with the selected structure.

It is important to note that the solution of a bilinear matrix inequality is an NP-hard problem. Therefore, an iterative LMI algorithm has been developed and used to iteratively solve the controller design problem.

3.2 Dynamic feedback controller design

The general QP-structure of the static feedback (14) enables to include any quasi-monomials - possibly others, than that of the open-loop system - to the closed-loop system structure, that helps in finding a solution of the controller design BMI problem. At the same time, the design of the general QP-structure itself, i.e. the selection of the suitable quasi-monomials, remains an open problem that can be solved only in a heuristic trial-and-error way.

In what follows, a dynamic QP-feedback controller is supposed that uses only the already available quasi-monomials in the open-loop system, that is, the control law is obtained as the solution of the differential equation (19).

$$\dot{u}_\ell = u_\ell \left(\kappa_\ell + \sum_{j=1}^m k_{\ell j} \prod_{k=1}^n x_k^{B_{j,k}} \prod_{l=1}^p u_l^{B_{u_l,l}} \right), \quad \ell = 1, \dots, p, \quad (19)$$

In order that the number of monomials (and the size of the problem) not increase, the input monomial coefficient matrix \mathbf{B}_u has only 0 and 1 entries depending on if the given input is present in the given monomial or not.

Since the differential equations (19) generating the control law are also in quasi-polynomial form, the closed loop system will also be a quasi-polynomial system with the following form

$$\dot{\hat{x}}_i = \hat{x}_i \left(\hat{\lambda}_i + \sum_{j=1}^{(p+1)m+p} \hat{A}_{i,j} \prod_{k=1}^{n+p} \hat{x}_k^{\hat{B}_{j,k}} \right), \quad i = 1, \dots, n+p \quad (20)$$

where

$$\hat{\mathbf{x}} = \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \in \mathbb{R}^{n+p}$$

The quasi-polynomial parameter matrices of the closed loop system (20) can be expressed using the parameter matrices of the input-affine quasi-polynomial model (5) and the dynamic feedback (19) as follows.

$$\hat{B} = \begin{bmatrix} B & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ B & \mathbf{1} & \mathbf{0} & \dots & \mathbf{0} \\ B & \mathbf{0} & \mathbf{1} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B & \mathbf{0} & \mathbf{0} & \dots & \mathbf{1} \\ \mathbf{0}^T & \mathbf{1} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0}^T & \mathbf{0} & \mathbf{1} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}^T & \mathbf{0} & \mathbf{0} & \dots & \mathbf{1} \end{bmatrix} \in \mathbb{R}^{((p+1)m+p) \times (n+p)}$$

where

$$\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^{m \times 1}, \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{m \times 1}$$

and

$$\mathbf{0}^T = [0 \ 0 \ \dots \ 0] \in \mathbb{R}^{1 \times n}.$$

Moreover,

$$\hat{A} = \begin{bmatrix} A_0 & A_1 & A_2 & \dots & A_p & \lambda_1 & \dots & \lambda_p \\ \mathbf{0} & \mathbf{k}_1 & \mathbf{0} & \dots & \mathbf{0} & 0 & \dots & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{k}_2 & \dots & \mathbf{0} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{k}_p & 0 & \dots & 0 \end{bmatrix}$$

where A_i , $i = 0, \dots, p$ are the parameter matrices of the input-affine QP model (5),

$$\mathbf{k}_i \in \mathbb{R}^{1 \times m} \quad i = 1, \dots, p, \quad \text{and} \quad \mathbf{0} = [0 \ 0 \ \dots \ 0] \in \mathbb{R}^{1 \times m}$$

and $\hat{A} \in \mathbb{R}^{(n+p) \times ((p+1)m+p)}$. Furthermore,

$$\hat{\lambda} = [\lambda_1 \ \dots \ \lambda_n \ \kappa_1 \ \dots \ \kappa_l]^T \in \mathbb{R}^{n+p}.$$

The coefficient matrix of the resulting Lotka-Volterra system is

$$\mathbb{R}^{((p+1)m+p) \times ((p+1)m+p)} \ni \hat{M} = \hat{B} \hat{A} = \begin{bmatrix} BA_0 & BA_1 & \dots & BA_p & B\lambda_1 & \dots & B\lambda_p \\ BA_0 & BA_1 + \mathbf{1}k_1 & \dots & BA_p & B\lambda_1 & \dots & B\lambda_p \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ BA_0 & BA_1 & \dots & BA_p + \mathbf{1}k_p & B\lambda_1 & \dots & B\lambda_p \\ \mathbf{0}^T & \mathbf{k}_1 & \dots & \mathbf{0}^T & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0}^T & \mathbf{0}^T & \dots & \mathbf{k}_p & 0 & \dots & 0 \end{bmatrix}$$

the resulting matrix depends linearly on the controller parameter vectors $(\mathbf{k}_1, \dots, \mathbf{k}_p)$.

Suppose, that the Lyapunov function of the system is in the form (10) with the unknown parameter matrix $\hat{C} \in \mathbb{R}^{((p+1)m+p) \times ((p+1)m+p)}$ (i.e. the original diagonal matrix with additional elements in the main diagonal). The negative definiteness of the time derivative of the Lyapunov function

$$\hat{M}^T \hat{C} + \hat{C} \hat{M} \leq 0, \quad \hat{C} > 0 \quad (21)$$

is again a BMI. The variables of the bilinear matrix inequality (21) are the controller parameters and the diagonal elements of \hat{C} . The solution of such problems are

far from being trivial as bilinear matrix inequalities belong to the NP-hard class (see VanAntwerp and Braatz (2000)), however, there are iterative methods (see, e.g. Cao et al. (1998) and Magyar et al. (2008)).

The problem can also be formulated as a control Lyapunov function based dynamic feedback design. The Lyapunov function candidate of the system is still in the form (10) with the given diagonal parameter matrix $\hat{C} \in \mathbb{R}^{((p+1)m+p) \times ((p+1)m+p)}$, but the negative definiteness of (21) becomes a LMI with the controller parameters being the variables.

It is important to note that the *design parameter* of the presented control Lyapunov function based dynamic feedback design LMI problem is the diagonal parameter matrix \hat{C} of the Lyapunov function (10).

The above BMI problem can be used to check the possibility of stabilization of a given controller structure, after one have found a feasible solution for the BMI, the control design for the given control Lyapunov function can be performed by solving the LMI version of the problem.

Note, that the parameter vector $\kappa = [\kappa_1 \ \kappa_2 \ \dots \ \kappa_l]$ does not influence the global stability of the system, however, they are useful in the steady state design for the closed loop system as it is described in Magyar et al. (2008) by substituting the desired equilibrium point into

$$0 = x_i^* \left(\hat{\lambda}_i(\kappa) + \sum_{j=1}^{(p+1)m+p} \hat{A}_{i,j} \prod_{k=1}^{n+p} \hat{x}_k^{*\hat{B}_{j,k}} \right), \quad (22)$$

$$i = 1, \dots, n+p$$

and solving it for κ .

4. SIMPLE NUMERICAL EXAMPLE

In order to demonstrate the capabilities of the proposed method, a simple numerical example is presented in the sequel.

4.1 Open loop system

Consider the two dimensional quasi-polynomial dynamics (23)

$$\begin{aligned} \dot{x}_1 &= x_1 \left(2 + 1.5x_1^{-2}x_2u - 1.75x_1^{\frac{3}{2}}x_2^5u - x_2^{-4} \right) \\ \dot{x}_2 &= x_2 \left(-1 + x_2^{-4} \right) \end{aligned} \quad (23)$$

The above system has one equilibrium point in the positive orthant at

$$\mathbf{x}^* = \begin{bmatrix} 1.0334 \\ 1 \end{bmatrix}$$

with $u^* = 2.3046$, for which global asymptotic stability cannot be proven using the linear matrix inequality based method discussed in Section 2.3, but numerical simulations show (see Figure 1) that \mathbf{x}^* is a stable equilibrium point.

4.2 Static QP-feedback

In order to demonstrate the advantage of dynamic feedback over the static one, we use the same set of quasi-monomials, that are the same as in the open-loop system, i.e. $x_1^{-2}x_2$, $x_1^{\frac{3}{2}}x_2^5$ and x_2^{-4} .

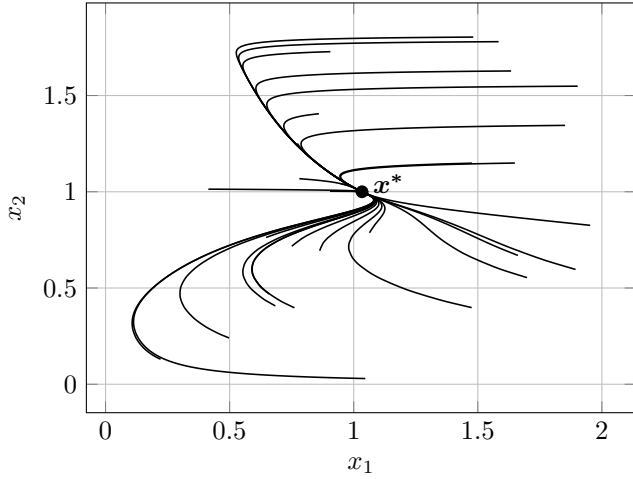


Fig. 1. Some trajectories of the open loop system (23) running to the steady state \mathbf{x}^* with $u^* = 2.3046$. Global asymptotic stability cannot be proven for \mathbf{x}^* .

The control law applied to the input is given as follows:

$$u = k_1 x_1^{-2} x_2 + k_2 x_1^{\frac{3}{2}} x_2^5 + k_3 x_2^{-4}$$

The QP matrices of the closed loop system are

$$\hat{\mathbf{A}} = \begin{bmatrix} 1.5k_1 & -1.75k_1 & 1.5k_2 & -1.75k_2 & 1.5k_3 & -1.75k_3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\hat{\mathbf{B}} = \begin{bmatrix} -4 & 2 \\ -\frac{1}{2} & 2 \\ -\frac{1}{2} & 6 \\ \frac{2}{3} & 6 \\ -2 & -3 \\ \frac{3}{2} & -3 \\ 0 & -4 \end{bmatrix}$$

The bilinear matrix inequality to be solved in order to find a globally stabilizing static state feedback has the form (17) where the BMI variables are the three control gain parameters (k_1, k_2, k_3) and the seven Lyapunov function parameters (c_1, \dots, c_7).

The available BMI solvers (one based on Cao et al. (1998)) could not find a feasible solution for the above problem.

4.3 Dynamic feedback

Consider the following dynamical controller being in quasi-polynomial form

$$\dot{u} = u \left(\kappa + k_1 x_1^{-2} x_2 u + k_2 x_1^{\frac{3}{2}} x_2^5 u + k_3 x_2^{-4} \right) \quad (24)$$

It can be seen, that the overall (i.e. closed loop) system can be described by the three dimensional quasi-polynomial system (23)-(24).

4.4 Control Lyapunov function based controller design

The matrices $\hat{\mathbf{B}}$, and $\hat{\mathbf{A}}$ describing the quasi-monomials and their linear combination is

$$\hat{\mathbf{B}} = \begin{bmatrix} -2 & 1 & 0 \\ 1.5 & 5 & 0 \\ 0 & -4 & 0 \\ -2 & 1 & 1 \\ 1.5 & 5 & 1 \\ 0 & -4 & 1 \end{bmatrix}$$

$$\hat{\mathbf{A}} = \begin{bmatrix} 0 & 0 & 0 & 1.5 & -1.75 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & k_1 & k_2 & k_3 \end{bmatrix}$$

Since the coefficients of three of the monomials defined in $\hat{\mathbf{B}}$ are zero, the Lotka-Volterra coefficient matrix can be simplified to the following matrix

$$\hat{\mathbf{M}}_{\text{red}} = \begin{bmatrix} -3 + k_1 & 3.5 + k_2 & 3 + k_3 \\ 2.25 + k_1 & -2.625 + k_2 & 3.5 + k_3 \\ 0 & 0 & -4 \end{bmatrix}$$

Suppose, that the control Lyapunov function (9) parameters are chosen to be

$$\begin{aligned} c_1 &= 1 \\ c_2 &= 1 \\ c_3 &= 1 \end{aligned} \quad (25)$$

The linear matrix inequality to be solved in order to find the stabilizing feedback is in the form

$$\begin{bmatrix} -6 + 2k_1 & 5.75 + k_1 + k_2 & 3 + k_3 \\ 5.75 + k_1 + k_2 & -5.25 + 2k_2 & 3.5 + k_3 \\ 3 + k_3 & 3.5 + k_3 & -8 \end{bmatrix} \leq 0 \quad (26)$$

It can be shown (e.g. using The MathWorks (2007) Matlab), that a feasible solution of the above LMI is

$$\begin{aligned} k_1 &= -1 \\ k_2 &= -2 \\ k_3 &= -3 \end{aligned} \quad (27)$$

It means, that the closed loop system is globally asymptotically stable using the dynamic feedback parameters (27) and the parameters of the Lyapunov function of the form (9) that proves global asymptotic stability are (25).

With a wise selection of κ using (22) (described in Magyar et al. (2008)), the steady state of the x dynamics in the closed loop system can be reset to the original one. Using $\kappa = 10$, the unique equilibrium $\bar{\mathbf{x}}^* = [x_1^*, x_2^*, u^*]^T$ of the closed loop system is at

$$\begin{aligned} x_1^* &= 1.0334 \\ x_2^* &= 1.0 \\ u^* &= 2.3046 \end{aligned} ,$$

which is globally asymptotically stable.

5. CONCLUSION AND FURTHER WORK

An optimization based globally stabilizing dynamic controller synthesis technique has been presented for nonnegative and nonlinear systems in quasi-polynomial form. The dynamic feedback structure is also in quasi-polynomial form with only the quasi-monomials present in the open-loop system model therefore the feedback design problem with known control Lyapunov function parameters is equivalent to a linear matrix inequality feasibility problem with the controller parameters as variables.

Of course, the global asymptotic stability is a very strong property and several nonlinear systems does not meet its conditions. Thus, the application of the presented

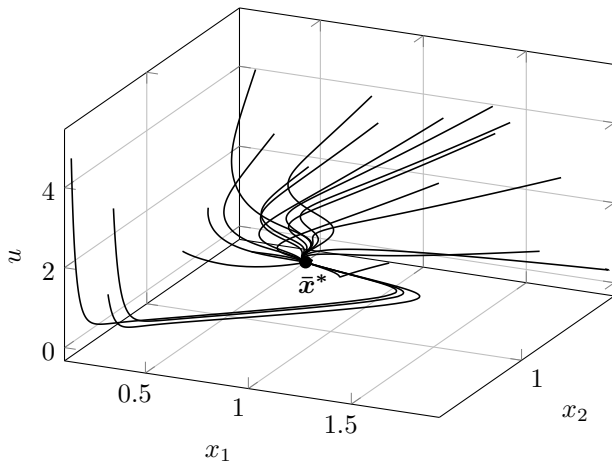


Fig. 2. Some trajectories of the closed loop system with dynamical QP feedback (23, 24) running to the steady state \bar{x}^* . Global asymptotic stability is proven in this case.

technique for controller design for a given stability region is a possible future direction of research.

Further work is directed to find structural conditions for the existence of globally stabilizing feedback controllers in QP-form.

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