

## Internal models for nonlinear output agreement and optimal flow control

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**Abstract:** This paper studies the problem of output agreement in networks of nonlinear dynamical systems under time-varying disturbances. Necessary and sufficient conditions for output agreement are derived for the class of incrementally passive systems. Following this, it is shown that the optimal distribution problem in dynamic inventory systems with time-varying supply and demand can be cast as a special version of the output agreement problem. We show in particular that the time-varying optimal distribution problem can be solved by applying an internal model controller to the dual variables of a certain convex network optimization problem.

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### 1. INTRODUCTION

Output agreement has evolved as one of the most important objectives in cooperative control. It has been studied in various contexts, ranging from distributed optimization (Tsitsiklis et al. [1986]) up to oscillator synchronization (Stan and Sepulchre [2007]). Adding up to these results, we discuss in this paper the output agreement problem in the context of optimal distribution control for inventory networks with time-varying supply. We generalize the results of De Persis [2013] (see van der Schaft and Wei [2012] for a special case of the problem with constant signals).

Internal model control tools have been used to handle output agreement problems in a variety of different formulations, see e.g., (Wieland et al. [2011]), (Bai et al. [2011]), (De Persis and Jayawardhana [2012a]). We consider here a different problem set-up, involving time-varying external disturbance signals, and solve the output agreement problem for the class of incrementally passive systems. Our derivations follow the trail opened by Pavlov and Marconi [2008].

The output agreement problem with time-varying external signals, studied in this paper, turns out to be of particular relevance for the routing control in inventory systems. We consider a simple inventory system, taking into account the storage levels and routing between the different inventories. This dynamics models, e.g., supply chains (Alessandri et al. [2011]) or data networks (Moss and Segall [1983]). A key challenge in inventory systems is to handle a time-varying supply/demand in an optimal way, using only a distributed control strategy.

The contributions of this paper are twofold. First, we present necessary and sufficient conditions for the *output agreement problem* under time-varying disturbances. We consider networks of nonlinear systems interacting according to an undirected network topology. Following an internal model control approach, we consider controllers placed on the edges of the network and provide necessary conditions for the output agreement problem to be feasible. Sufficient conditions for output

agreement in networks of *incrementally passive* dynamical systems are provided. As a second contribution, we show that the *optimal distribution problem* in inventory systems with time-varying supply and demand can be cast as an output agreement problem. The necessary conditions for the optimal distribution problem turn out to be specific representation of the regulator equations. Subsequently, we present controllers solving the optimal distribution problem, for either quadratic cost functions or constant supplies, generalizing the results of De Persis [2013]. The remainder of the paper is organized as follows. The basic problem formulation and necessary conditions for output agreement are presented in Section 2. Sufficient conditions for output agreement in networks of incrementally passive systems are discussed in Section 3. The time-varying optimal distribution problem is introduced in Section 4, where also necessary conditions are discussed. We present then the solution to the problem for linear-quadratic problems in Section 4.1 and for constant supplies in Section 4.2.

**Preliminaries:** The notation we employ is standard. The set of (non-negative) real numbers is denoted by  $\mathbb{R}$  ( $\mathbb{R}_{\geq}$ ). The distance of a point  $q$  from a set  $\mathcal{A}$  is defined as  $\text{dist}_{\mathcal{A}} q = \inf_{p \in \mathcal{A}} \|p - q\|$ . The range-space and null-space of a matrix  $B$  are denoted by  $\mathcal{R}(B)$  and  $\mathcal{N}(B)$ , respectively. A *graph*  $\mathcal{G} = (V, E)$  is an object consisting of a finite set of nodes, with  $|V| = n$ , and edges, with  $|E| = m$ . The incidence matrix  $B \in \mathbb{R}^{n \times m}$  of the graph  $\mathcal{G}$  with arbitrary orientation, is a  $\{0, \pm 1\}$  matrix with  $[B]_{ik}$  having value ‘+1’ if node  $i$  is the initial node of edge  $k$ , ‘-1’ if it is the terminal node, and ‘0’ otherwise. We refer sometimes to the flow space of  $\mathcal{G}$  as the null space  $\mathcal{N}(B^T)$ . Additionally,  $\mathcal{N}(B)$  is named the circulation space of  $\mathcal{G}$ , and  $\mathcal{R}(B^T)$  the differential space.

### 2. PROBLEM FORMULATION AND NECESSARY CONDITIONS

We consider a network of dynamical systems defined on a connected, undirected graph  $\mathcal{G} = (V, E)$ . Each node represents a nonlinear system

$$\begin{aligned} \dot{x}_i &= f_i(x_i, u_i, w_i) \\ y_i &= h_i(x_i, w_i), \quad i = 1, 2, \dots, n, \end{aligned} \quad (1)$$

where  $x_i \in \mathbb{R}^{r_i}$  is the state,  $u_i, y_i \in \mathbb{R}^p$  are the input and the output, respectively. Each system (1) is driven by the time-

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varying signal  $w_i \in \mathbb{R}^{q_i}$ , representing, e.g., a disturbance or reference. We assume that the exogenous signal  $w_i$  is generated by a dynamical system of the form  $\dot{w}_i = s_i(w_i)$ ,  $w_i(0) \in \mathcal{W}_i$ , where  $\mathcal{W}_i$  is a set whose properties are specified below. The vector field  $s_i(w_i)$  satisfies for all  $w_i, w'_i$

$$(w_i - w'_i)^T (s_i(w_i) - s_i(w'_i)) \leq 0. \quad (2)$$

As an example, consider the linear function with skew-symmetric matrix  $s_i(w_i) = S_i w_i$ ,  $S_i^T + S_i = 0$ .

We stack together the signals  $w_i$ , for  $i = 1, 2, \dots, n$ , and obtain the vector  $w$ , which satisfies the equation  $\dot{w} = s(w)$ . In what follows, whenever we refer to the solutions of  $\dot{w} = s(w)$ , we assume that the initial condition is chosen in a compact set  $\mathcal{W} = \mathcal{W}_1 \times \dots \times \mathcal{W}_n$ . The set  $\mathcal{W}$  is assumed to be forward invariant for the system  $\dot{w} = s(w)$ .

Similarly, let  $x$ ,  $u$ , and  $y$  be the stacked vectors of  $x_i, u_i$ , and  $y_i$ , respectively. Using this notation, the totality of all systems is given by

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{x} &= f(x, u, w) \\ y &= h(x, w) \end{aligned} \quad (3)$$

with state space  $\mathcal{W} \times \mathcal{X}$  and  $\mathcal{X}$  a compact subset of  $\mathbb{R}^{r_1} \times \dots \times \mathbb{R}^{r_n}$ . The control objective is to reach output agreement of all nodes in the network, independent of the exact representation of the time-varying external signals. Therefore, a dynamic controller

$$\begin{aligned} \dot{\xi}_k &= F_k(\xi_k, v_k) \\ \lambda_k &= H_k(\xi_k), \quad k = 1, 2, \dots, m, \end{aligned} \quad (4)$$

with state  $\xi_k \in \mathbb{R}^{v_k}$  and input  $v_k \in \mathbb{R}^p$  is placed between any pair of neighboring nodes. When stacked together, the controllers (4) give rise to the overall controller

$$\begin{aligned} \dot{\xi} &= F(\xi, v) \\ \lambda &= H(\xi), \end{aligned} \quad (5)$$

where  $\xi \in \Xi$ , a compact subset of  $\mathbb{R}^{v_1} \times \dots \times \mathbb{R}^{v_m}$ . Throughout the paper the following interconnection structure between the plants, placed on the nodes of  $\mathcal{G}$ , and the controllers, placed on the edges of  $\mathcal{G}$  is considered. A controller (4), associated with edge  $k$  connecting nodes  $i, j$ , has access to the relative outputs  $y_i - y_j$ . In vector notation, the relative outputs of the systems are

$$z = (B \otimes I_p)^T y. \quad (6)$$

The controllers are then driven by the systems via the interconnection condition

$$v = -z, \quad (7)$$

where  $v$  are the stacked inputs of the controllers. Additionally, the output of a controller  $\lambda_k$  influences its two incident systems. Thus, the stacked vector of controller's output  $\lambda$  drives the process (3) via the interconnection<sup>1</sup>

$$u = (B \otimes I_p) \lambda. \quad (8)$$

We are now ready to introduce the output agreement problem.

**Definition 1.** The *output agreement problem* is solvable for the process (3) under the interconnection relations (6), (7), (8), if there exists controllers (5), such that every solution  $(w(t), x(t), \xi(t))$  originating from  $\mathcal{W} \times \mathcal{X} \times \Xi$  is bounded and satisfies  $\lim_{t \rightarrow \infty} (B^T \otimes I_p) y(t) = \mathbf{0}$ .

The first step is to investigate *necessary conditions* for the output agreement problem to be solvable. To this purpose, we strengthen the requirement on the convergence of the regulation error to the origin, requiring that  $\lim_{t \rightarrow \infty} (B^T \otimes I_p) y(t) = \mathbf{0}$

<sup>1</sup> The interconnection structure (6), (8) naturally represents a canonical structure for distributed control laws. This structure is often considered in the context of passivity-based cooperative control, see e.g., Arcaj [2007], Bai et al. [2011], van der Schaft and Maschke [2012], De Persis and Jayawardhana [2012b], Bürger et al. [2013b].

*uniformly* in the initial condition (Isidori and Byrnes. [2008]). The closed-loop system (3), (5), (6), (7), (8) can be written as

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{x} &= f(x, (B \otimes I_p) H(\xi), w) \\ \dot{\xi} &= F(\xi, -(B \otimes I_p)^T h(x, w)). \end{aligned} \quad (9)$$

If the output agreement problem is solvable, then<sup>2</sup> the  $\omega$ -limit set  $\Omega(\mathcal{W} \times \mathcal{X} \times \Xi)$  is nonempty, compact, invariant and uniformly attracts  $\mathcal{W} \times \mathcal{X} \times \Xi$  under the flow of (9). Furthermore, the  $\omega$ -limit set  $\Omega(\mathcal{W} \times \mathcal{X} \times \Xi)$  must be a subset of the set of all pairs  $(w, x)$  for which  $(B \otimes I_p)^T h(x, w) = 0$ . This set is the graph of a map defined on the whole  $\mathcal{W}$  and is invariant for the closed-loop system. By the invariance, for any solution  $w$  of the exosystem originating from  $\mathcal{W}$ , there exists  $(x^w, u^w, \xi^w)$  such that

$$\begin{aligned} \dot{x}^w &= f(x^w, u^w, w) \\ 0 &= (B \otimes I_p)^T h(x^w, w) \end{aligned} \quad (10)$$

and

$$\begin{aligned} \dot{\xi}^w &= F(\xi^w, 0) \\ u^w &= (B \otimes I_p) H(\xi^w). \end{aligned} \quad (11)$$

We summarize the necessary condition as follows.

**Proposition 1.** If the output agreement problem is solvable, then, for every  $w$  solution to  $\dot{w} = s(w)$  originating in  $\mathcal{W}$ , there must exist solutions  $(x^w, u^w, \xi^w)$  such that the equations (10), (11) are satisfied.

The constraint (10) ensures that there exists a feed-forward control input  $u^w$  that keeps the systems in output agreement. The second constraint (11) ensures that the controller (5) is able to generate this feed-forward input signal.

The constraints (11) can be rewritten independently of the controller (Isidori and Marconi [2010]). Let in the following  $\lambda^w$  be some trajectory satisfying  $u^w = (B \otimes I_p) \lambda^w$ , where  $u^w$  is a solution to (10). Note that  $\lambda^w$  is only then uniquely defined if the graph  $\mathcal{G}$  has no cycles. Otherwise, it can be varied in the circulation space of  $\mathcal{G}$ . Bearing in mind the structure of the controllers (5), it descends from the constraints (11) that

$$\begin{aligned} \dot{\xi}^w &= F(\xi^w, 0) \\ \lambda^w &= H(\xi^w). \end{aligned} \quad (12)$$

Suppose now, that there exists an integer  $d$  and maps  $\tau : \mathcal{W} \mapsto \mathbb{R}^d$ ,  $\phi : \mathbb{R}^d \mapsto \mathbb{R}^d$  and  $\psi : \mathbb{R}^d \mapsto \mathbb{R}^{mp}$  satisfying

$$\begin{aligned} \frac{\partial \tau}{\partial w} s(w) &= \phi(\tau(w)) \\ \lambda^w &= \psi(\tau(w)). \end{aligned} \quad (13)$$

Observe that  $\tau, \phi, \psi$  do not depend on the controller since  $\lambda^w$  depends on  $B$  and  $u^w$ , the latter being dependent only on the process to control and  $B$  on the topology of the underlying graph. Now, the dynamical system

$$\begin{aligned} \dot{\eta} &= \phi(\eta), \quad \eta \in \mathbb{R}^d \\ \lambda &= \psi(\eta). \end{aligned} \quad (14)$$

has the property that if  $\eta_0 = \tau(w(0))$ , then the solution  $\eta(t)$  to (14) starting from  $\eta_0$  is such that  $\lambda(t) = \lambda^w(t)$  for all  $t \geq 0$ .

For designing a controller with the structure (5), i.e., that decomposes into controllers on the edges of  $\mathcal{G}$ , we introduce a vector  $\eta_k \in \mathbb{R}^d$  for each edge  $k = 1, \dots, m$ , and denote with  $\psi_k$  the entries of the vector valued function  $\psi$  corresponding to the edge  $k$ . Each edge is now assigned a controller of the form

$$\begin{aligned} \dot{\eta}_k &= \phi(\eta_k) \\ \lambda_k &= \psi_k(\eta_k), \quad k = 1, 2, \dots, m. \end{aligned} \quad (15)$$

With the stacked vector  $\eta = [\eta_1^T, \dots, \eta_m^T]^T$  and the vector valued functions

<sup>2</sup> By  $\omega$ -limit set  $\Omega(\mathcal{W} \times \mathcal{X} \times \Xi)$  it is meant the set of points  $(w, x, \xi)$  for which there exists a sequence of pairs  $(t_k, (w_k, x_k, \xi_k))$  with  $t_k \rightarrow +\infty$  and  $(w_k, x_k, \xi_k) \in \mathcal{W} \times \mathcal{X} \times \Xi$  such that  $\phi(t_k, (w_k, x_k, \xi_k)) \rightarrow (w, x, \xi)$  as  $k \rightarrow +\infty$ , where  $\phi(\cdot, \cdot)$  is the flow of (9).

$$\bar{\phi}(\eta) = \begin{bmatrix} \phi(\eta_1) \\ \vdots \\ \phi(\eta_m) \end{bmatrix}, \quad \bar{\psi}(\eta) = \begin{bmatrix} \psi_1(\eta_1) \\ \vdots \\ \psi_m(\eta_m) \end{bmatrix} \quad (16)$$

the overall controller is given as

$$\begin{aligned} \dot{\eta} &= \bar{\phi}(\eta) \\ \lambda &= \bar{\psi}(\eta). \end{aligned} \quad (17)$$

Note that if the initial condition is chosen as  $\eta_0 = I_m \otimes \tau(w(0))$  then the solution  $\eta(t)$  to (14) starting from  $\eta_0$  is such that  $\lambda(t) = \lambda^w(t)$  for all  $t \geq 0$ , which is the same property as (12).

*Remark 1.* If the functions  $\phi(\eta)$  and  $\psi(\eta)$  are linear, that is  $\phi(\eta) = \Phi\eta$  ( $\Phi \in \mathbb{R}^{d \times d}$ ), and  $\psi(\eta) = \Psi\eta$  ( $\Psi \in \mathbb{R}^{m \times d}$ ), then  $\psi_k = \Psi_k^T$  ( $\Psi_k^T \in \mathbb{R}^{p \times d}$ ) are the rows of the matrix  $\Psi$  corresponding to the edge  $k$ , and the global functions (16) are given by  $\bar{\phi}(\eta) = (I_m \otimes \Phi)\eta$ , and  $\bar{\psi}(\eta) = \text{block.diag}(\Psi_1^T, \dots, \Psi_m^T)\eta$  (see also De Persis [2013]).

In summary, the overall controller (17) can be interpreted as internal-model-based controllers placed at the edges of the graph  $\mathcal{G}$ , where all controllers have the same global internal model. The role of the internal model in coordination problems has been investigated in Wieland et al. [2011] for linear systems and in Wieland [2010], Chapter 5 for nonlinear systems. The result above adds up to these results.

*Remark 2.* Suppose that the  $\omega$ -limit set can be expressed as  $\Omega(\mathcal{W} \times \mathcal{X} \times \Xi) = \{(w, x, \xi) : x = \pi(w), \xi = \pi_c(w)\}$ . Then  $x^w = \pi(w)$  and the so-called regulator equations (10) express the existence of an invariant manifold where the ‘‘regulation error’’  $(B^T \otimes I_p)y$  is identically zero provided that the control input  $u^w$  is applied. The conditions (12) express the existence of a controller able to provide  $u^w$ . Moreover, (10), (12) take the familiar expressions, see e.g. Isidori and Byrnes [1990]:

$$\begin{aligned} \frac{\partial \pi}{\partial w} s(w) &= f(\pi(w), (B \otimes I_p)H(\pi_c(w)), w) \\ 0 &= (B \otimes I_p)^T h(\pi(w), w) \end{aligned} \quad (18)$$

and

$$\begin{aligned} \frac{\partial \pi_c}{\partial w} s(w) &= F(\pi_c(w), 0) \\ \lambda(w) &= H(\pi_c(w)). \end{aligned} \quad (19)$$

### 3. OUTPUT AGREEMENT UNDER TIME-VARYING DISTURBANCES

In this section we highlight sufficient conditions that lead to a solution of the problem for a special class of systems (1), namely incrementally passive systems, see e.g., Pavlov and Marconi [2008], to which we refer the reader for the definition of a *regular* storage function.

*Definition 2.* The system (1) is said to be incrementally passive if there exists a  $C^1$  regular storage function  $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^r \times \mathbb{R}^r \mapsto \mathbb{R}_{\geq 0}$  such that for any two inputs  $u_i, u'_i$  and any two solutions  $x_i, x'_i$ , corresponding to these inputs, the respective outputs  $y_i, y'_i$  satisfy

$$\frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial x_i} f_i(x_i, u_i, w_i) + \frac{\partial V_i}{\partial x'_i} f_i(x'_i, u'_i, w_i) \leq (y_i - y'_i)^T (u_i - u'_i). \quad (20)$$

*Example 1.* Linear systems of the form

$$\begin{aligned} \dot{x}_i &= A_i x_i + G_i u_i + P_i w_i \\ y_i &= C_i x_i \end{aligned} \quad (21)$$

that are passive from the input  $u_i$  to the output  $y_i$  satisfy the assumption above, with  $V_i = \frac{1}{2}(x_i - x'_i)^T Q_i (x_i - x'_i)$  and  $Q_i = Q_i^T > 0$  the matrix such that  $A_i^T Q_i + Q_i A_i \leq 0$  and  $Q_i G_i = C_i^T$ .

*Example 2.* Nonlinear systems of the form

$$\begin{aligned} \dot{x}_i &= f_i(x_i) + G_i u_i + P_i w_i \\ y_i &= C_i x_i \end{aligned}$$

with  $f_i(x_i) = \nabla F_i(x_i)$ ,  $F_i(x_i)$  twice continuously differentiable and concave, and  $G_i = C_i^T$  are incrementally passive. In fact, by concavity of  $F_i(x_i)$ ,  $(x_i - x'_i)^T (f_i(x_i) - f_i(x'_i)) \leq 0$ , and  $V_i = \frac{1}{2}(x_i - x'_i)^T (x_i - x'_i)$  is the incremental storage function.

In the previous section, it was shown that the controllers at the edge have to take the form (15). Now, they must be completed by considering additional control inputs that guarantee the achievement of the steady state. While we require the internal model to be identical for all edges, i.e.,  $\phi(\eta_k)$ , the new augmented systems might be different. Then, the controllers modify as

$$\begin{aligned} \dot{\eta}_k &= \phi_k(\eta_k, v_k) \\ \lambda_k &= \psi_k(\eta_k), \quad k = 1, 2, \dots, m, \end{aligned} \quad (22)$$

where all controllers reduce to the common internal model, i.e., (15), if no external forcing is applied, i.e.,  $\phi_k(\eta_k, 0) = \phi(\eta_k)$ . The following is the main standing assumption that the controllers must satisfy to solve the output agreement problem for the class of incrementally passive systems:

*Assumption 1.* For each  $k = 1, 2, \dots, m$ , there exists regular functions  $W_k(\eta_k, \eta'_k)$ , with  $W_k : \mathbb{R}^{q_k} \times \mathbb{R}^{q_k} \rightarrow \mathbb{R}_{>0}$  such that

$$\frac{\partial W_k}{\partial \eta_k} \phi_k(\eta_k, v_k) + \frac{\partial W_k}{\partial \eta'_k} \phi_k(\eta'_k, v'_k) \leq (\lambda_k - \lambda'_k)^T (v_k - v'_k). \quad (23)$$

Assumption 1 is the critical assumption in this paper and restricts the possible solutions of (13), i.e., an incrementally passive internal model has to be designed, which is in general a difficult task. A first simple example when the design is possible is when the feedforward control input is linear, that is (13) is satisfied with  $\tau = \text{Id}$ ,  $\phi = s$  and  $\psi$  is a linear function of its argument. In this case, we let

$$\phi_k(\eta_k, 0) = s(\eta_k), \quad \psi_k(\eta_k) = M_k \eta_k$$

and define

$$\phi_k(\eta_k, v_k) = s(\eta_k) + M_k^T v_k. \quad (24)$$

Under the standing assumption on  $s(w)$ , i.e., (2), the storage function  $W_k(\eta_k, \eta'_k) = \frac{1}{2}(\eta_k - \eta'_k)^T (\eta_k - \eta'_k)$  satisfies (23), i.e.,

$$\begin{aligned} \frac{\partial W_k}{\partial \eta_k} \phi_k(\eta_k, v_k) + \frac{\partial W_k}{\partial \eta'_k} \phi_k(\eta'_k, v'_k) &= \\ (\eta_k - \eta'_k)^T (s(\eta_k) - s(\eta'_k)) + (\eta_k - \eta'_k)^T M_k^T (v_k - v'_k) & \\ \leq (\psi_k(\eta_k) - \psi_k(\eta'_k))^T (v_k - v'_k). & \end{aligned} \quad (25)$$

We state below the main result that, while extending to networked systems the results of Pavlov and Marconi [2008], provides a solution to the output agreement problem in the presence of time-varying disturbances. It is a slightly more general statement than Theorem 2 in De Persis [2013].

*Theorem 1.* Consider the system (3) and let the regulator equations (10) hold. Consider the controllers

$$\begin{aligned} \dot{\eta} &= \bar{\phi}(\eta, v) \\ \lambda &= \bar{\psi}(\eta) + \mu \end{aligned} \quad (26)$$

where  $\mu$  is an extra feedback to design, and let  $\bar{\phi}$  and  $\bar{\psi}$  be the stacked functions of  $\phi_k(\eta_k, v_k)$  and  $\psi_k(\eta_k)$  with the internal model property being satisfied. Consider the interconnection conditions

$$\begin{aligned} u &= (B \otimes I_p)\lambda, \\ v &= -(B^T \otimes I_p)y. \end{aligned} \quad (27)$$

If Assumption 1 holds and  $\mu = -z$ , with  $z = (B^T \otimes I_p)y$ , then the output agreement problem is solvable, that is every solution

starting from  $\mathcal{W} \times \mathcal{X} \times \Xi$  is bounded<sup>3</sup> and

$$\lim_{t \rightarrow +\infty} z(t) = \lim_{t \rightarrow +\infty} (B \otimes I_p)^T y(t) = \mathbf{0}.$$

*Proof:* By the incremental passivity property of the  $x$  subsystem in (3) and (10), it is true that

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x, u, w) + \frac{\partial V}{\partial x^w} f(x^w, u^w, w) \leq (y - y^w)^T (u - u^w),$$

where  $V = \sum_i V_i$ . Similarly by Ass. 1, system (26) satisfies

$$\frac{\partial W}{\partial \eta} \bar{\phi}(\eta, v) + \frac{\partial W}{\partial \eta^w} \bar{\phi}(\eta^w) \leq (\lambda - \lambda^w)^T v - (\mu - \mu^w) v,$$

with  $W = \sum_k W_k$  and  $\bar{\phi}(\eta^w) = I_m \otimes \phi(\eta^w)$ . Bearing in mind the interconnection constraints  $u = (B \otimes I_p) \lambda$ ,  $u^w = (B \otimes I_p) \lambda^w$ , and  $v = -(B^T \otimes I_p) y$ , and letting  $U((x, x^w), (\eta, \eta^w)) = V(x, x^w) + W(\eta, \eta^w)$  we obtain

$$\begin{aligned} \dot{U}((x, x^w), (\eta, \eta^w)) &:= \dot{V}(x, x^w) + \dot{W}(\eta, \eta^w) \\ &= (y - y^w)^T (u - u^w) + (\lambda - \lambda^w)^T v - (\mu - \mu^w)^T v \\ &= (y - y^w)^T (B \otimes I_p) (\lambda - \lambda^w) \\ &\quad - (\lambda - \lambda^w)^T (B^T \otimes I_p) y + (\mu - \mu^w)^T (B^T \otimes I_p) y. \end{aligned}$$

By definition of output agreement,  $(B \otimes I_p)^T y^w = 0$  and the previous equality becomes

$$\dot{U}((x, x^w), (\eta, \eta^w)) = \mu^T (B^T \otimes I_p) y = -\|(B^T \otimes I_p) y\|^2 = -z^T z,$$

by definition of  $\mu = -z$  and  $\mu^w = 0$ . Since  $U$  is non-negative and non-increasing, then  $U(t)$  is bounded. As  $x^w, \eta^w$  are bounded and  $U$  is regular, then  $x, \eta$  are bounded as well. Hence the solutions exist for all  $t$ . Integrating the latter inequality we obtain

$$\int_0^{+\infty} z^T(s) z(s) ds \leq U(0).$$

By Barbalat's lemma, if one proves that  $\frac{d}{dt} z^T(t) z(t)$  is bounded then one can conclude that  $z^T(t) z(t) \rightarrow 0$ . Now,  $z(t) = (B^T \otimes I_p) y = (B^T \otimes I_p) h(x, w)$  is bounded because  $x, w$  are bounded. If  $h$  is continuously differentiable and  $\dot{x}, \dot{w}$  are bounded, then  $\dot{z}$  is bounded and one can infer that  $\frac{d}{dt} z^T(t) z(t)$  is bounded. By assumption,  $w$  is the solution of  $\dot{w} = s(w)$  starting from a forward invariant compact set. Hence, both  $w$  and  $\dot{w}$  are bounded. On the other hand,  $\dot{x}$  satisfies

$$\dot{x} = f(x, (B \otimes I_p) \bar{\psi}(\eta) - z, w)$$

which proves that it is bounded because  $x, \eta, z$  were proven to be bounded, while  $w$  is bounded by assumption. Therefore,  $\dot{x}, \dot{w}$  are bounded and this implies that  $\frac{d}{dt} z^T(t) z(t)$  is bounded. Then by Barbalat's Lemma we have  $\lim_{t \rightarrow +\infty} z(t) = \mathbf{0}$  as claimed.

### 3.1 Linear systems and distribution networks

We investigate next the output agreement problem for linear dynamical systems and focus on a routing control problem in inventory systems under time-varying demand and supply. Consider an inventory system with  $n$  inventories and  $m$  transportation lines and let  $B$  be the incidence matrix of the transportation network. The dynamics of the inventory system is given as

$$\dot{x} = B\lambda + Pw, \quad (28)$$

where  $x \in \mathbb{R}^n$  represents the storage level,  $\lambda \in \mathbb{R}^m$  the flow along one line, and  $Pw$  an external in-/outflow of the inventories, i.e., the supply or demand. We assume that the time varying supply/demand is generated by a linear dynamics

$$\dot{w} = Sw \quad (29)$$

<sup>3</sup> Notice that in this statement the requirement on the compactness of  $\mathcal{X}, \Xi$  can be relaxed and the two sets can be taken equal to the whole state space.

and that it is balanced at any time, i.e.,  $\mathbf{1}^T Pw(t) = 0$  for all  $t \geq 0$ . The control problem is to find a distributed control law

$$\begin{aligned} \eta_k &= \Phi_k \eta_k + \Lambda_k v_k \\ \lambda_k &= \Psi_k \eta_k + \Upsilon_k v_k \end{aligned} \quad (30)$$

such that for each initial condition  $(w_0, x_0, \eta_0)$  the solution of the closed loop system remains bounded and a balancing of the inventory levels is achieved, i.e.,  $\lim_{t \rightarrow \infty} B^T x = 0$ .

The closed-loop system (28), (29), (30) can be understood as feedback interconnection (6), (7), (8), of the controller (30) and the linear system

$$\begin{aligned} \dot{w} &= Sw \\ \dot{x} &= u + Pw, \end{aligned} \quad (31)$$

i.e., with  $A = \mathbf{0}$  and  $G = I_n$ . Thus, the distribution problem can be understood as an output agreement problem and solved following the general treatment discussed above. One can prove<sup>4</sup> that there exists a matrix  $H$  such that  $\lambda^w = Hw$ , and the steady-state solution is such that  $x^w = x_*^w \mathbf{1}$ . Also, the steady-state solution  $x^w$  must be a constant vector and the steady-state routing  $\lambda^w$  satisfies  $B\lambda^w + Pw = 0$ . Thus, any controller (30) solving the output agreement problem, solves at the same time the exact routing problem of the time-varying supply/demand. The condition (13) is satisfied with  $\tau = Id$ ,  $\phi = S$  and  $\psi = H$ . Following our previous discussion, it remains to design the incrementally passive local controllers. Let in the following  $H_k^T$  denote the  $k$ -th row of the matrix  $H$ . We know from (24), that the internal model controller on the edge  $k$  of the form

$$\begin{aligned} \dot{\eta}_k &= S\eta_k + H_k v_k \\ \lambda_k &= H_k^T \eta_k \end{aligned} \quad (32)$$

is incrementally passive. Finally, it follows directly from Theorem 1 that the distributed internal model controller

$$\begin{aligned} \dot{\eta} &= \bar{S}\eta - \bar{H}B^T x \\ \lambda &= \bar{H}\eta - B^T x \end{aligned} \quad (33)$$

where  $\bar{S} = I_n \otimes S$  and  $\bar{H} = \text{block.diag}(H_1^T, \dots, H_m^T)$ , solves the output agreement problem and, additionally, achieves an exact routing of the time-varying supply through the network.

### 3.2 Output agreement in the case of constant disturbances

The critical assumption in the derivation presented above is the incremental passivity of the internal model, i.e., Assumption 1. However, the proof above exploits Assumption 1 only in a weaker form, since it requires the incremental passivity property (23) not to hold with respect to any two trajectories, but only with respect to the real and the steady-state trajectory, i.e., with  $\eta'_k = \eta_k^w, v'_k = 0, \lambda'_k = \lambda^w$ . Bearing in mind this observation, it is possible to find a storage function  $W_k$  that fulfills (23) in the case of *constant disturbances* and nonlinear coupling functions. In fact in this case  $(\eta_k^w, \lambda_k^w)$  satisfy  $\lambda_k^w = \psi_k(\eta_k^w)$  for some constant  $\eta^w$ , i.e.,  $\eta_k^w = 0$ .

Suppose now that  $\psi_k$  is a *strongly monotone* function, and consider the following storage function (Jayawardhana et al. [2007], Burger et al. [2013b]):

$$W_k(\eta_k, \eta_k^w) = \Psi_k(\eta_k) - \Psi_k(\eta_k^w) - \nabla \Psi_k^T(\eta_k^w)(\eta_k - \eta_k^w), \quad (34)$$

where  $\Psi_k : \mathbb{R}^q \rightarrow \mathbb{R}$  is a twice continuously differentiable function such that  $\nabla \Psi_k(\eta_k) = \psi_k(\eta_k)$ . Now if  $\psi_k$  is monotone,  $\Psi_k$  is convex and, by the global under-estimator property of the gradient, we have

$$\Psi_k(\eta_k) \geq \Psi_k(\eta_k^w) + \nabla \Psi_k^T(\eta_k^w)(\eta_k - \eta_k^w)$$

for each  $\eta_k, \eta_k^w$ . If  $\Psi_k$  is strictly convex, i.e.,  $\psi_k$  is strongly monotone, then the previous inequality holds if and only if

<sup>4</sup> We refer the reader to (De Persis [2013]) for more details.

$\eta_k = \eta_k^w$ , and  $W_k$  is regular (Jayawardhana et al. [2007]). Furthermore,

$$\frac{\partial W_k}{\partial \eta_k} \phi_k(\eta_k, v_k) = (\psi_k(\eta_k) - \psi_k(\eta_k^w))^T v_k.$$

Hence, in the case of constant disturbances and monotonic nonlinear couplings, Assumption 1 is always fulfilled and the output agreement problem is solved by

$$\begin{aligned} \dot{\eta} &= v = -(B \otimes I_p)y \\ \lambda &= \bar{\psi}(\eta) - (B^T \otimes I_p)y. \end{aligned} \quad (35)$$

Control laws of the form (35) have been studied in the context of network clustering in Bürger et al. [2013a], Bürger et al. [2011] and in a more general network optimization framework in Bürger et al. [2013b].

#### 4. TIME-VARYING OPTIMAL DISTRIBUTION PROBLEMS

We revisit now the distribution problem of inventory systems discussed in Section 3.1, i.e.,

$$\dot{x} = B\lambda + Pw, \quad (36)$$

with a time-varying external demand/supply. Let for now the supply/demand vectors be generated by a possibly nonlinear dynamics  $\dot{w} = s(w)$ . The control objective is now not only to balance the inventory levels, but additionally to achieve an *optimal* routing in the network. We associate therefore to each transportation line a convex and continuously differentiable cost function

$$\mathcal{P}_k(\lambda_k), \quad k = 1, 2, \dots, m.$$

Before moving to the dynamic control problem, we briefly review the *static optimal distribution problem*. Consider a fixed constant supply vector  $w$ . The (static) optimal distribution problem is to find a routing  $\lambda^w \in \mathbb{R}^m$  such that

$$\begin{aligned} \lambda^w &= \arg \min \sum_{k=1}^m \mathcal{P}_k(\lambda_k) \\ 0 &= B\lambda + Pw. \end{aligned} \quad (37)$$

In the following, the notation  $\mathcal{P}(\lambda) = \sum_{k=1}^m \mathcal{P}_k(\lambda_k)$  will be used. The Lagrangian function associated to (37) with multiplier  $\zeta$  is

$$\mathcal{L}(\lambda, v) = \mathcal{P}(\lambda) + \zeta^T (-B\lambda - Pw).$$

From the Lagrangian, one obtains directly the optimality conditions (KKT-conditions). In particular,  $(\lambda^w, \zeta^w)$  is an optimal primal/dual solution pair to (37) if the following nonlinear equations hold

$$\begin{aligned} \nabla \mathcal{P}(\lambda^w) - B^T \zeta^w &= 0 \\ B\lambda^w + Pw &= 0. \end{aligned} \quad (38)$$

Note that the first condition simply implies  $\nabla \mathcal{P}(\lambda^w) \in \mathcal{R}(B^T)$ . We define the optimal routing/supply pairs as

$$\Gamma = \{(\lambda, w) \in \mathbb{R}^m \times \mathcal{W} : \nabla \mathcal{P}(\lambda) \in \mathcal{R}(B^T), B\lambda + Pw = 0\}.$$

In particular,  $(\lambda^w, w) \in \Gamma$  if and only if  $\lambda^w$  is an optimal solution to the static optimal distribution problem (37) with the supply vector  $w$ . The main difficulty associated with the set  $\Gamma$  relates to the constraint  $\nabla \mathcal{P}(\lambda) \in \mathcal{R}(B^T)$ . This constraint can be avoided if the optimality conditions are expressed in terms of the *dual solutions*  $\zeta$ . In what follows, we impose the condition that  $\nabla \mathcal{P}$  is invertible. The two optimality conditions (38) can now be expressed as the following single nonlinear expression

$$B\nabla \mathcal{P}^{-1}(B^T \zeta^w) + Pw = 0.$$

Bearing this in mind, we define the set of optimal dual solutions as  $\Gamma_D = \{(\zeta, w) \in \mathbb{R}^n \times \mathcal{W} : B\nabla \mathcal{P}^{-1}(B^T \zeta) + Pw = 0\}$ . We want to emphasize two properties of  $\Gamma_D$ . First, if  $(\zeta^w, w) \in \Gamma_D$  then  $(\zeta^w + c\mathbf{1}, w) \in \Gamma_D$  for any  $c \in \mathbb{R}$ . Second,  $(\zeta^w, w) \in \Gamma_D$  if and only if the corresponding routing strategy  $\lambda^w = \nabla \mathcal{P}^{-1}(B^T \zeta^w)$  satisfies  $(\lambda^w, w) \in \Gamma$ . We can now formalize the dynamic problem.

*Definition 3.* The *time-varying optimal distribution problem* is solvable for the system (36) if there exists a controller (5) such that any solution originating from  $\mathcal{W} \times \mathcal{X} \times \Xi$  is bounded and (i)  $\lim_{t \rightarrow \infty} B^T x(t) = 0$  and (ii)  $\lim_{t \rightarrow \infty} \text{dist}_T(\lambda(t), w(t)) = 0$ .

As in the static case, it is also in the dynamic case advantageous to consider the dual solutions  $\zeta$ . Therefore, we restrict our attention to *dynamic dual controllers* of the form

$$\begin{aligned} \dot{\eta} &= \phi(\eta, v) \\ \lambda &= \nabla \mathcal{P}^{-1}(B^T \psi(\eta)). \end{aligned} \quad (39)$$

For the time-varying optimal distribution problem to be feasible, it is necessary that the manifold

$$H(w, x, \eta) = \begin{bmatrix} B^T x \\ B\nabla \mathcal{P}^{-1}(B^T \psi(\eta)) + Pw \end{bmatrix} = 0 \quad (40)$$

is invariant under the closed-loop dynamics

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{x} &= B\nabla \mathcal{P}^{-1}(B^T \psi(\eta)) + Pw \\ \dot{\eta} &= \phi(\eta, B^T x). \end{aligned} \quad (41)$$

Note that, in contrast to the original output regulation problem, the “output” function  $H$  depends explicitly on the state of the controller  $\eta$ . However, at this point the advantage of the internal model controller design for the dual variables becomes obvious. Let  $(w, x^w, \eta^w)$  be a solution to (41) starting in  $\Omega(\mathcal{W} \times \mathcal{X} \times \Xi)$ , satisfying in particular

$$h(x) := B^T x^w = 0. \quad (42)$$

Now, by the structure of the inventory dynamics follows  $\mathbf{1}^T \dot{x} = 0$  at any time, and consequently  $\dot{x}^w = 0$ , i.e.,

$$\dot{x}^w = 0 = B\nabla \mathcal{P}^{-1}(B^T \psi(\eta^w)) + Pw.$$

Thus, the corresponding routing strategy  $\lambda^w = \nabla \mathcal{P}^{-1}(B^T \psi(\eta^w))$  is optimal at any point in time, i.e.,  $(\lambda^w(t), w(t)) \in \Gamma$ . Thus, by restricting the discussion to the “dual” controller structure (39), we transformed the time-varying optimal distribution problem into an output agreement problem.

We show next, that for two important problem representations, i.e., if the objective functions are quadratic and if the supply is constant, the optimal routing controllers are incrementally passive, and therefore, the time-varying optimal distribution problem can be solved using the established theory.

##### 4.1 The Linear-Quadratic Case

Suppose the supply is generated by a linear system  $\dot{w} = Sw$ , with  $S + S^T = 0$ , and the cost functions are quadratic

$$\mathcal{P}(\lambda) = \frac{1}{2} \lambda^T Q \lambda,$$

for  $Q = \text{diag}(q_1, \dots, q_m)$  and  $q_k > 0$ . Since the exo-system is linear, we choose simply  $\phi(\eta) = S\eta$ , such that the steady state solution satisfies  $\eta^w = w$ . It remains to design  $\psi(\eta)$ . We assume a linear structure, i.e.,  $\psi(\eta) = H\eta$  and observe that, in order to satisfy the internal model property,  $H$  must satisfy

$$BQ^{-1}B^T Hw + Pw = 0. \quad (43)$$

Note that  $L_Q = BQ^{-1}B^T$  is a weighted Laplacian matrix of the network. As  $L_Q$  has one eigenvalue at zero, with the corresponding eigenvector  $\mathbf{1}$ , it is not invertible. However, one possible solution to (43) is  $H = -L_Q^\dagger P$ , where  $L_Q^\dagger$  pseudo-inverse of the weighted Laplacian  $L_Q$ , see e.g., Gutman and Xiao [2004].<sup>5</sup> We assign now an internal model controller to

<sup>5</sup> By the properties of  $L_Q^\dagger$ , it is promptly verified that  $BHw + Pw = -BQ^{-1}B^T L_Q^\dagger Pw + Pw = -(I - \frac{\mathbf{1}\mathbf{1}^T}{n})Pw + Pw = \mathbf{0}$ .

each node and introduce the variable  $\eta_i \in \mathbb{R}^d$ , satisfying the dynamics

$$\begin{aligned}\dot{\eta}_i &= S\eta_i \\ \zeta_i &= H_i^T \eta_i, \quad i = 1, \dots, n,\end{aligned}\quad (44)$$

where  $H_i^T$  is the  $i$ -th row of  $H$ . The routing at one edge computes then simply as  $\lambda_k = q_k^{-1}(\zeta_j - \zeta_i)$ , where  $j, i$  are the nodes incident to edge  $k$ . After introducing  $\eta$  and  $\zeta$  as the stacked vectors of  $\eta_i$  and  $\zeta_i$ , respectively, and the matrices  $\bar{S} = (I_n \otimes S)$ ,  $\bar{H} = \text{block.diag}(H_1^T, \dots, H_n^T)$ , we can express the overall controller as

$$\begin{aligned}\dot{\eta} &= \bar{S}\eta \\ \lambda &= Q^{-1}B^T\zeta = Q^{-1}B^T\bar{H}\eta.\end{aligned}\quad (45)$$

This distributed controller has the desired internal model property for the optimal distribution problem. It remains to render this controller incrementally passive.

**Theorem 2.** Consider the inventory system (28) with the supply generated by the linear dynamics  $\dot{w} = Sw$ . Consider the controller

$$\begin{aligned}\dot{\eta} &= \bar{S}\eta + \bar{H}^T B Q^{-1}v \\ \lambda &= Q^{-1}B^T\bar{H}\eta + v.\end{aligned}$$

with the interconnection condition  $v = -B^T x$ . Then, every solution of the closed-loop system is bounded and (i)  $\lim_{t \rightarrow +\infty} B^T x = 0$ , and (ii)  $\lim_{t \rightarrow +\infty} \text{dist}_\Gamma(\lambda(t), w(t)) = 0$ .

The proof of this result follows directly along the same lines as the proof of Theorem 1, taking into account that  $V(x, x^w) = \frac{1}{2}\|x - x^w\|^2$  and  $W = \frac{1}{2}\|\eta - \eta^w\|^2$  are regular incremental storage functions. Additionally, optimality follows directly since the steady state routing  $\lambda^w$  satisfies at every time instant the optimality conditions (38).

#### 4.2 Optimal distribution with constant supply

A second version of the optimal distribution problem, that can be solved by the internal model approach, relates to problems with constant supply and demand, i.e.,  $s(w) = 0$ . In this case, we can consider general strictly convex cost functions  $\mathcal{P}_k$ . As discussed in Section 3.2, the output agreement problem with a static reference signal is feasible for any internal model controller nonlinear, but strongly monotone output function. We consider now the dual variables  $\sigma = B^T \zeta$  instead of  $\zeta$  as introduced in (38), and the corresponding controller

$$\begin{aligned}\dot{\sigma} &= v, \quad \sigma(0) \in \mathcal{R}(B^T) \\ \lambda &= \nabla \mathcal{P}^{-1}(\sigma) + v.\end{aligned}\quad (46)$$

Note that  $\mathcal{P}^{-1}(\cdot)$  is strongly monotone since  $\mathcal{P}$  is strictly convex and continuously differentiable. Clearly,  $\lambda^w$ , as generated by this controller is an optimal routing. The controller (46) it is incrementally passive with respect to any constant input signal, as discussed in Section 3.2. A storage function can therefore be found in the structure (34).<sup>6</sup> We can now immediately conclude that the controller (46) with the interconnection condition  $v = -B^T x$  solves the optimal distribution problem.

### 5. CONCLUSIONS

We proposed an internal model control design approach for output agreement of incrementally passive nonlinear systems with time-varying external disturbances. Building upon these results, we studied the optimal distribution problem in inventory

<sup>6</sup> The storage function here takes the form  $W_k = \mathcal{P}^*(\sigma) - \mathcal{P}^*(\sigma^w) - \nabla \mathcal{P}^*(\sigma^w)(\sigma - \sigma^w)$ , where  $\mathcal{P}^*$  is the convex conjugate of  $\mathcal{P}$ , see Rockafellar [1997], Bürger et al. [2013b].

systems with time-varying supply and demand. These problems can be cast as output agreement problems if their dual formulation is considered. We showed how the optimal distribution problem can be solved by internal model controllers for the dual variables. The specific solution to the distribution problem is discussed for the linear-quadratic case and for problems with constant supplies.

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