# Observability Analysis of Nonlinear Systems Using Pseudo-Linear Transformation

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**Abstract:** In the linear control theory, the observability Popov-Belevitch-Hautus (PBH) test plays an important role in studying observability along with the observability rank condition and observability Gramian. The observability rank condition and observability Gramian have been extended to nonlinear systems and have found applications in the analysis of nonlinear systems. On the other hand, there is no observability criterion for nonlinear systems corresponding to the PBH test. In this study, we generalize the observability PBH test for nonlinear systems using pseudo-linear transformation.

Keywords: Nonlinear systems, observability, pseudo-linear transformation, PBH test

## 1. INTRODUCTION

For linear systems, there are several criteria for observability such as the PBH (Popov-Belevitch-Hautus) test, the observability rank condition, and the condition described by the observability Gramian. Every condition plays an important role in systems and control theory. For nonlinear systems, observability is also studied, and the rank condition and Gramian are generalized to the nonlinear system (Conte et al. [2007], Nijmeijer and van der Schaft [1990], Scherpen [1993], Fujimoto and Scherpen [2005]). The applications of the rank condition include the decomposition of an unobservable nonlinear system into an observable subsystem and an unobservable subsystem (Conte et al. [2007], Nijmeijer and van der Schaft [1990]), and the Gramian characterizes the balancing of nonlinear systems (Scherpen [1993], Fujimoto and Scherpen [2005]). Differently from the rank condition and Gramian, the PBH test has not been extended to nonlinear systems.

Pseudo-linear transformation (PLT) (Jacobson [1937], Leroy [1995], Bronstein and Petkovšek [1996]) helps in studying structures of nonlinear systems (Zheng et al. [2011], Lévine [2011], Halás [2008], Halás and Kotta [2007]). In particular, the concept of a transfer function of the nonlinear system is given using PLT (Halás [2008], Halás and Kotta [2007]). Halás [2008], reported that the PLT operates similarly to the Laplace transformation. The Laplace transformation plays a key role in analyzing linear systems. By using the Laplace transformation, not only structures but also stability can be studied. On the other hand, there is no application of PLT in analyses of stability for the nonlinear system.

For a linear system described by a state-space representation, the eigenvalues of the system matrix are important for analyzing the system, e.g., stability, observability and controllability analyses. For PLT, the eigenvalues and eigenvectors are defined (Leroy [1995], Lam et al. [2008]) and used in analyzing nonlinear systems (Aranda-Bricaire and Moog [2004]). Aranda-Bricaire and Moog [2004] exploits the eigenvalues and eigenvectors of PLT to study the existence of a coordinate transformation that transforms a system into its feed-forward form, and in the linear case showed that an eigenvalue of PLT is equivalent to an eigenvalue of the system matrix. Their results indicate that the eigenvalues and eigenvectors of PLT as well as the eigenvalues and eigenvectors of the system matrix of a linear system may be useful for analyzing nonlinear systems.

In this study, we derive two observability conditions: a necessary condition and a sufficient condition. In the linear case, each condition is equivalent to the observability PBH test. The observability PBH test on a linear system shows that the eigenvalues of the system matrix characterize observability. As in similarly the PBH test, our necessary condition is described using the eigenvalues of PLT. That is, our necessary condition shows that the eigenvalues of PLT as well as the eigenvalues of the system matrix of a linear system play important roles when testing observability. In summary, our necessary condition can be regarded as a generalization of the observability PBH test for a nonlinear system.

**Notation:** Let $\mathbf{N}$ be the set of non-negative integers and $\mathbf{C}$ be the field of complex numbers. Moreover, let $\mathcal{K}$ be the field of the complex meromorphic functions defined on $\mathbf{C}^n$ with the variables $x_1, x_2, \ldots, x_n$. For the matrix $A(x) \in \mathcal{K}^{n\times m}$, $\mathrm{rank}_{\mathcal{K}} A(x) = s$ means that the rank of $A(x)$ over the field $\mathcal{K}$ is $s$. Thus, $\mathrm{rank}_{\mathcal{K}} A(x) = s$ does not mean that $\mathrm{rank}_{\mathbf{C}} A(x) = s$ holds for all $x \in \mathbf{C}^n$, but $\mathrm{rank}_{\mathbf{C}} A(x) = s$ holds for almost all $x \in \mathbf{C}^n$. The Jacobian matrix of $\varphi(x) \in \mathcal{K}^n$ is denoted by $J_\varphi :=$

 

$(\partial\varphi(x)/\partial x) \in \mathcal{K}^{n\times n}$. Let $\mathrm{Diff}^n_{\mathcal{K}}(\mathbf{C}) \subset \mathcal{K}^n$ be the set of $\varphi \in \mathcal{K}^n$ such that $\mathrm{rank}_{\mathcal{K}} J_\varphi = n$. From the definition of $\mathrm{Diff}^n_{\mathcal{K}}(\mathbf{C})$, each $\varphi \in \mathrm{Diff}^n_{\mathcal{K}}(\mathbf{C})$ is a locally diffeomorphic mapping from an open and dense subset $M_\varphi \subset \mathbf{C}^n$ to $M_\varphi$, where $M_\varphi$ varies depending on $\varphi$. Let $\mathcal{X}$ be a vector space over $\mathcal{K}$ generated by the one-forms $dx_1, dx_2, \ldots, dx_n$, i.e., $\mathcal{X} := \mathrm{span}_{\mathcal{K}}\{dx_1, \ldots, dx_n\}$. Note that $\{dx_1, \ldots, dx_n\}$ is a basis of $\mathcal{X}$.

## 2. MOTIVATING EXAMPLES

Consider a continuous-time nonlinear system described by

$$\begin{cases} dx/dt = f(x), \\ y = h(x), \end{cases} \tag{1}$$

where $x \in \mathbf{C}^n$ and $y \in \mathbf{C}$ denote the state and output, respectively. The elements $f_i$ $(i = 1, 2, \ldots, n)$ and $h$ are complex meromorphic functions of $x$.

The observability PBH (Popov-Belevitch-Hautus) test is one of the criteria for observability of linear systems.

*Proposition 2.1.* (Observability PBH test) Suppose that $f = Ax$ and $h = c^{\mathrm{T}}x$ in (1), where $A \in \mathbf{C}^{n\times n}$ and $c \in \mathbf{C}^n$. System (1) is observable if and only if

$$\mathrm{rank}_{\mathbf{C}} \begin{bmatrix} \lambda I_n - A \\ c^{\mathrm{T}} \end{bmatrix} = n, \tag{2}$$

holds for all $\lambda \in \mathbf{C}$.

In fact, it suffices to check condition (2) only for all eigenvalues $\lambda \in \mathbf{C}$ of $A$.

Our aim is to generalize condition (2) to nonlinear system (1). First, in Examples 2.1 and 2.2 below, we consider the relations between observability and

$$\mathrm{rank}_{\mathcal{K}} \begin{bmatrix} \lambda I_n - \partial f(x)/\partial x \\ \partial h(x)/\partial x \end{bmatrix} = n \tag{3}$$

for all $\lambda \in \mathcal{K}$. Note that differently from that considered in condition (2), the field considered in condition (3) is the field of meromorphic functions. It is not required that condition (3) holds for all $x \in \mathbf{C}^n$.

We investigate the relations between observability and condition (3) in the following examples.

*Example 2.1.* Consider a nonlinear system described by

$$\begin{cases} \dot{x}_1 = x_1^2 + x_2, \\ \dot{x}_2 = x_1 x_2, \\ y = x_1. \end{cases} \tag{4}$$

It can be shown that the system is observable from Definition 3.1 below. Condition (3) for system (4) holds because we have

$$\begin{aligned} &\mathrm{rank}_{\mathcal{K}} \begin{bmatrix} \lambda - 2x_1 & 1 \\ -x_2 & \lambda - x_1 \\ 1 & 0 \end{bmatrix} \\ &= \mathrm{rank}_{\mathcal{K}} \begin{bmatrix} 0 & 1 \\ 0 & \lambda - x_1 \\ 1 & 0 \end{bmatrix} = \mathrm{rank}_{\mathcal{K}} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} = 2. \end{aligned}$$

Thus, in this example, an observable system satisfies condition (3).

*Example 2.2.* Consider a nonlinear system described by

$$\begin{cases} \dot{x}_1 = (x_1^2 - x_2^2)/2, \\ \dot{x}_2 = (x_1 - x_2)x_2, \\ y = x_1 - x_2. \end{cases} \tag{5}$$

It is possible to show that system (5) is not observable. Condition (2) for system (5):

$$\mathrm{rank}_{\mathcal{K}} \begin{bmatrix} \lambda - x_1 & x_2 \\ -x_2 & \lambda - x_1 + 2x_2 \\ 1 & -1 \end{bmatrix} = 2$$

does not hold when $\lambda = x_1 - x_2$. This $\lambda = x_1 - x_2$ is an eigenvalue of the PLT introduced in Section 3.2. In this example, an unobservable system does not satisfy condition (3).

These two examples demonstrate that condition (3) is potentially helpful for testing observability of a nonlinear system as well as for the observability PBH test on a linear system. From Example 2.2, an eigenvalue of PLT may play an important role in testing observability of a nonlinear system.

## 3. PRELIMINARIES

### 3.1 Observability of nonlinear system

In this paper, we consider the following observability (Conte et al. [2007]).

*Definition 3.1.* A system (1) is said to be observable if there exists an open and dense subset $M \subset \mathbf{C}^n$ such that system (1) is locally weakly observable (Hermann and Krener [1977]) at any initial state $x_0 \in M$.

The observability rank condition is a criterion for observability (Conte et al. [2007], Nijmeijer and van der Schaft [1990]).

*Proposition 3.1.* System (1) is observable if and only if the following observability rank condition holds:

$$\mathrm{rank}_{\mathcal{K}} \mathcal{O}_{n-1}(x) = n,$$
$$\mathcal{O}_i(x) := \begin{bmatrix} \partial h(x)/\partial x \\ \partial L_f h(x)/\partial x \\ \vdots \\ \partial L_f^i h(x)/\partial x \end{bmatrix}, \tag{6}$$

where $L_f^0 h(x) = h(x)$ and $L_f^{i+1} h(x) := (\partial L_f^i h(x)/\partial x) f(x)$ $(i \in \mathbf{N})$.

In general, the observability rank condition is a sufficient condition for local weak observability for all initial states in $\mathbf{C}^n$. By restricting $\mathbf{C}^n$ to an open and dense subset $M \subset \mathbf{C}^n$, the necessity is also guaranteed.

The observability rank condition has some applications in analyzing observability of nonlinear systems. For example, a system not satisfying rank condition (6) can be decomposed into an observable subsystem and an unobservable subsystem (Conte et al. [2007], Nijmeijer and van der Schaft [1990]).

*Proposition 3.2.* For system (1), let $\mathrm{rank}_{\mathcal{K}} \mathcal{O}_{n-1}(x) = r < n$. Thus, there exists the coordinate transformation $z = \varphi(x) \in \mathrm{Diff}^n_{\mathcal{K}}(\mathbf{C})$ such that

$$\begin{cases} dz_1/dt = \hat{f}_1(z_1, \ldots, z_r) \\ \quad \vdots \\ dz_r/dt = \hat{f}_r(z_1, \ldots, z_r) \\ dz_{r+1}/dt = \hat{f}_{r+1}(z) \\ \quad \vdots \\ dz_n/dt = \hat{f}_n(z) \\ y = h(z_1, \ldots, z_r) \end{cases} \tag{7}$$

holds.

### 3.2 PLT defined by the system

In this resarch, we study observability using PLT (Bronstein and Petkovšek [1996], Jacobson [1937], Leroy [1995]).

We give the definition of PLT. The derivation $\delta$ on the field $K$ is an additive mapping $\delta : K \to K$ such that

$$\delta(a + b) = \delta(a) + \delta(b), \forall a, b \in K, \tag{8}$$

$$\delta(ab) = a \cdot \delta(b) + b \cdot \delta(a), \forall a, b \in K. \tag{9}$$

A field $K$ is a differential field if $K$ is closed under a derivation $\delta$.

Let $V$ be a vector space over $K$.

*Definition 3.2.* A mapping $\theta : V \to V$ is called PLT if

$$\theta(u + v) = \theta(u) + \theta(v),$$

$$\theta(au) = a\theta(u) + \delta(a)u$$

hold for any $a \in K$ and $u, v \in V$.

We show a PLT defined by system (1). Let $\delta : \mathcal{K} \to \mathcal{K}$ be the mapping

$$\delta(a) := \sum_{i=1}^{n} \frac{\partial a}{\partial x_i} f_i, \ a \in \mathcal{K}. \tag{10}$$

Note that $\delta(a)$ is the Lie derivative of a function $a$ along $f$ in (1), which implies that $\delta$ depends on system (1). The mapping $\delta$ satisfies conditions (8) and (9). Thus, $\delta$ is a derivative of $\mathcal{K}$, and $\mathcal{K}$ is a differential field because of $\delta(a) \in \mathcal{K}$ for any $a \in \mathcal{K}$.

Next, let $d : \mathcal{K} \to \mathcal{X}$ be the mapping

$$da = \sum_{i=1}^{n} \frac{\partial a}{\partial x_i} dx_i.$$

Finally, we define the mapping $s : \mathcal{X} \to \mathcal{X}$.

$$s * \varepsilon := \sum_{i=1}^{n} \left( \delta(a_i) dx_i + a_i d\delta(x_i) \right), \ \varepsilon = \sum_{i=1}^{n} a_i dx_i. \tag{11}$$

For simplicity, we omit the symbol $*$ from (11). Note that $s * \varepsilon$ is the Lie derivative of the one-form $\varepsilon$ along $f$ in system (1). Thus, $s$ depends on system (1).

From (11), for any $a \in \mathcal{K}$ and $\varepsilon \in \mathcal{X}$, we have

$$s(a\varepsilon) = as\varepsilon + \delta(a)\varepsilon.$$

This equality implies that $s : \mathcal{X} \to \mathcal{X}$ is a PLT.

For the PLT $s : \mathcal{X} \to \mathcal{X}$ defined in (11), an eigenvalue and an eigenvector are defined as follows (Leroy [1995], Lam et al. [2008]).

*Definition 3.3.* $\lambda \in \mathcal{K}$ and $\varepsilon \in \mathcal{X}$ are called an eigenvalue and an eigenvector of the PLT $s : \mathcal{X} \to \mathcal{X}$ if $s\varepsilon = \lambda\varepsilon$ holds.

Since the PLT defined in (11) depends on system (1), an eigenvalue and eigenvector of the PLT $s : \mathcal{X} \to \mathcal{X}$ are determined by system (1).

*Example 3.1.* Consider the same system as Example 2.2. For instance, $x_1 - x_2$ and $d(x_1 - x_2)$ are an eigenvalue and eigenvector of the PLT defined by the system, respectively. Actually, we have

$$\begin{aligned} sd(x_1 - x_2) &= sdx_1 - sdx_2 \\ &= d\left(\frac{x_1^2 - x_2^2}{2}\right) - d\left((x_1 - x_2)x_2\right) \\ &= (x_1 dx_1 - x_2 dx_2) - (x_2 dx_1 + (x_1 - 2x_2)dx_2) \\ &= (x_1 - x_2)d(x_1 - x_2). \end{aligned}$$

In Example 2.2, an unobservable system does not satisfy condition (3) at an eigenvalue $\lambda = x_1 - x_2$ of PLT. In Section 4.1, we clarify the relations between observability and the eigenvalues of PLT.

## 4. OBSERVABILITY CONDITIONS

### 4.1 Nonlinear case

An eigenvalue of PLT (11) characterizes observability of the nonlinear system.

*Theorem 4.1.* If system (1) is observable, then

$$\begin{bmatrix} (s - \lambda)I_n \\ \lambda I_n - (\delta(J_\varphi) + J_\varphi(\partial f(x)/\partial x))J_\varphi^{-1} \\ (\partial h(x)/\partial x)J_\varphi^{-1} \end{bmatrix} v\varepsilon \neq 0 \tag{12}$$

holds for all $\varphi \in \mathrm{Diff}^n_{\mathcal{K}}(\mathbf{C})$, $\lambda \in \mathcal{K}$, $v \in \mathcal{K}^n \setminus \{0\}$ and $\varepsilon \in \mathcal{X} \setminus \{0\}$.

**Proof.** We prove this by contraposition. If there exist $\varphi \in \mathrm{Diff}^n_{\mathcal{K}}(\mathbf{C})$, $\lambda \in \mathcal{K}$, $v \in \mathcal{K}^n \setminus \{0\}$ and $\varepsilon \in \mathcal{X} \setminus \{0\}$ such that

$$\begin{bmatrix} (s - \lambda)I_n \\ \lambda I_n - (\delta(J_\varphi) + J_\varphi(\partial f(x)/\partial x))J_\varphi^{-1} \\ (\partial h(x)/\partial x)J_\varphi^{-1} \end{bmatrix} v\varepsilon = 0 \tag{13}$$

holds, then we have

$$\begin{bmatrix} sI_n - (\delta(J_\varphi) + J_\varphi(\partial f(x)/\partial x))J_\varphi^{-1} \\ (\partial h(x)/\partial x)J_\varphi^{-1} \end{bmatrix} v\varepsilon = 0.$$

Let $u \in \mathcal{K}^n$ be $J_\varphi^{-1}v$. The nonsingularity of $J_\varphi$ implies $u \neq 0$, and we have $v = J_\varphi u$. By substituting $v = J_\varphi u$ into the above equation, we obtain

$$\begin{bmatrix} J_\varphi(sI_n - (\partial f(x)/\partial x)) \\ (\partial h(x)/\partial x) \end{bmatrix} u\varepsilon = 0,$$

and consequently, from the nonsingularity of $J_\varphi$,

$$\begin{bmatrix} sI_n - (\partial f(x)/\partial x) \\ (\partial h(x)/\partial x) \end{bmatrix} u\varepsilon = 0. \tag{14}$$

Next, we show that equation (14) implies

 

$$\frac{\partial L_f^i h}{\partial x} u\varepsilon = 0, \ \forall i \in \mathbf{N} \tag{15}$$

by induction. Equation (14) yields

$$\left( sI_n - \frac{\partial f}{\partial x} \right) u\varepsilon = 0, \tag{16}$$

$$\frac{\partial h}{\partial x} u\varepsilon = 0. \tag{17}$$

When $i = 0$, equation (15) is nothing but (17).

Suppose that (15) holds when $i = k$, i.e.,

$$\frac{\partial L_f^k h}{\partial x} u\varepsilon = 0 \tag{18}$$

holds. By premultiplying $(\partial L_f^k h/\partial x)$ by (16), we have

$$\frac{\partial L_f^k h}{\partial x} \left( sI_n - \frac{\partial f}{\partial x} \right) u\varepsilon = 0. \tag{19}$$

Also, by premultiplying $s$ by (18), we have

$$s \frac{\partial L_f^k h}{\partial x} u\varepsilon = \left( \delta \left( \frac{\partial L_f^k h}{\partial x} \right) + \frac{\partial L_f^k h}{\partial x} s \right) u\varepsilon = 0. \tag{20}$$

By subtracting the left-hand side of (19) from the second left-hand side of (20), we obtain

$$\left( \delta \left( \frac{\partial L_f^k h}{\partial x} \right) + \frac{\partial L_f^k h}{\partial x} s - \frac{\partial L_f^k h}{\partial x} \left( sI_n - \frac{\partial f}{\partial x} \right) \right) u\varepsilon = 0$$

The left-hand side can be computed as follow

$$\begin{aligned} &\left( \delta \left( \frac{\partial L_f^k h}{\partial x} \right) + \frac{\partial L_f^k h}{\partial x} s - \frac{\partial L_f^k h}{\partial x} \left( sI_n - \frac{\partial f}{\partial x} \right) \right) u\varepsilon \\ &= \left( \frac{\partial^2 L_f^k h}{\partial x^2} f + \frac{\partial L_f^k h}{\partial x} \frac{\partial f}{\partial x} \right) u\varepsilon \\ &= \left( \frac{\partial}{\partial x} \left( \frac{\partial L_f^k h}{\partial x} f \right) \right) u\varepsilon = \frac{\partial L_f^{k+1} h}{\partial x} u\varepsilon. \end{aligned}$$

Thus, we have

$$\frac{\partial L_f^{k+1} h}{\partial x} u\varepsilon = 0.$$

Therefore, (15) holds.

Finally, we show that if (15) holds for $u \in \mathcal{K}^n \setminus \{0\}$ and $\varepsilon \in \mathcal{X} \setminus \{0\}$, then the observability rank condition does not hold. Equation (15) implies that

$$\mathcal{O}_{n-1} u\varepsilon = 0, \tag{21}$$

where $\mathcal{O}_i \in \mathcal{K}^{(i+1)\times n}$ is defined in (6). For any $\varepsilon \in \mathcal{X}$, there exists a vector $a \in \mathcal{K}^n$ such that

$$\varepsilon = a^{\mathrm{T}} dx \tag{22}$$

holds, where $\varepsilon \neq 0$ implies $a^{\mathrm{T}} \neq 0$ because $\{dx_1, \dots, dx_n\}$ is a basis of the $\mathcal{K}$-vector space $\mathcal{X}$. By substituting (22) into (21), we have

$$\mathcal{O}_{n-1} u a^{\mathrm{T}} dx = 0,$$

and consequently

$$\mathcal{O}_{n-1} u a^{\mathrm{T}} = 0,$$

where $u \neq 0$ and $a \neq 0$ imply that $(ua^{\mathrm{T}}) \in \mathcal{K}^{n\times n}$ is a nonzero matrix. Therefore, $\mathcal{O}_{n-1}$ is singular. That is, the observability rank condition does not hold. From Proposition 3.1, system (1) is not observable. □

From Theorem 4.1, if nonlinear system (1) is not observable, then (13) holds for some $\varphi \in \mathrm{Diff}^n_{\mathcal{K}}(\mathbf{C})$, $\lambda \in \mathcal{K}$, $v \in \mathcal{K}^n \setminus \{0\}$ and $\varepsilon \in \mathcal{X} \setminus \{0\}$. Condition (13) can be decomposed into

$$(s - \lambda) v\varepsilon = 0 \tag{23}$$

$$\begin{bmatrix} \lambda I_n - (\delta(J_\varphi) + J_\varphi(\partial f(x)/\partial x))J_\varphi^{-1} \\ (\partial h(x)/\partial x)J_\varphi^{-1} \end{bmatrix} v\varepsilon = 0. \tag{24}$$

Condition (23) implies that $\lambda \in \mathcal{K}$ and all $v_i\varepsilon \in \mathcal{X}$ $(i = 1, \dots, n)$ are eigenvalues and eigenvectors of the PLT $s : \mathcal{X} \to \mathcal{X}$. That is, Theorem 4.1 shows that the eigenvalues of PLT play important roles when testing observability of nonlinear systems. For linear systems, the observability PBH test shows that the eigenvalues, in the sense of linear algebra, of a system matrix characterize observability. Therefore, in Theorem 4.1, the eigenvalues of PLT operate like those of a system matrix.

In Section 4.2 below, it is shown that the condition of Theorem 4.1 is equivalent to the observability PBH test in the linear case. Thus, Theorem 4.1 can be viewed as a generalization of the observability PBH test on nonlinear systems.

Condition (24) is equivalent to

$$\begin{bmatrix} \lambda I_n - (\delta(J_\varphi) + J_\varphi(\partial f(x)/\partial x))J_\varphi^{-1} \\ (\partial h(x)/\partial x)J_\varphi^{-1} \end{bmatrix} v = 0. \tag{25}$$

Theorem 4.2 below shows that condition (25) also helps in testing the observability of nonlinear system (1). Condition (25) implies that $\lambda \in \mathcal{K}$ and $v \in \mathcal{K}^n \setminus \{0\}$ are an eigenvalue and right eigenvector of the matrix $(\delta(J_\varphi) + J_\varphi(\partial f(x)/\partial x))J_\varphi^{-1} \in \mathcal{K}^{n\times n}$, respectively, in the linear algebraic sense. Note that, the eigenvalues of $(\delta(J_\varphi) + J_\varphi(\partial f(x)/\partial x))J_\varphi^{-1}$ depend on the coordinate transformation $\varphi \in \mathrm{Diff}^n_{\mathcal{K}}(\mathbf{C}^n)$ due to the nonlinearity of system (1). On the other hand, the eigenvalues of the PLT $s : \mathcal{X} \to \mathcal{X}$ are invariant with respect to a coordinate transformation. Therefore, to check the condition of Theorem 4.1, we need to find a coordinate transformation $\varphi$ such that an eigenvalue of $(\delta(J_\varphi)+J_\varphi(\partial f(x)/\partial x))J_\varphi^{-1}$ becomes an eigenvalue of PLT.

Condition (25) is also important in its own right when testing observability of system (1).

*Theorem 4.2.* System (1) is observable if

$$\begin{bmatrix} \lambda I_n - (\delta(J_\varphi) + J_\varphi(\partial f(x)/\partial x))J_\varphi^{-1} \\ (\partial h(x)/\partial x)J_\varphi^{-1} \end{bmatrix} v \neq 0 \tag{26}$$

holds for all $\varphi \in \mathrm{Diff}^n_{\mathcal{K}}(\mathbf{C})$, $\lambda \in \mathcal{K}$ and $v \in \mathcal{K}^n \setminus \{0\}$.

**Proof.** We prove this by contraposition. That is, we show that if a system is not observable then there exist $\varphi \in \mathrm{Diff}^n_{\mathcal{K}}(\mathbf{C})$, $\lambda \in \mathcal{K}$ and $v \in \mathcal{K}^n \setminus \{0\}$ such that (25) holds.

From Proposition 3.1, if system (1) is not observable, then the observability rank condition does not hold. Let $\mathrm{rank}_{\mathcal{K}}\mathcal{O}_{n-1}(x) = r < n$. Proposition 3.2 shows that system (1) can be transformed into (7) by a coordinate transformation $z = \hat{\varphi}(x) \in \mathrm{Diff}^n_{\mathcal{K}}(\mathbf{C})$. By choosing $\varphi$ as $\hat{\varphi}$, condition (25) becomes

$$\begin{bmatrix} \lambda I_r - (\partial \bar{f}_1(\bar{z}_1)/\partial \bar{z}_1) & 0 \\ -(\partial \bar{f}_2(z)/\partial \bar{z}_1) & \lambda I_{n-r} - (\partial \bar{f}_2(z)/\partial \bar{z}_2) \\ (\partial \bar{h}(\bar{z}_1)/\partial \bar{z}_1) & 0 \end{bmatrix} v = 0, \tag{27}$$

where $\bar{z}_1 = [z_1, \ldots, z_r]^{\mathrm{T}} \in \mathcal{K}^r$, $\bar{z}_2 = [z_{r+1}, \ldots, z_n]^{\mathrm{T}} \in \mathcal{K}^{n-r}$, $\bar{f}_1 = [\hat{f}_1, \ldots, \hat{f}_r]^{\mathrm{T}} \in \mathcal{K}^r$ and $\bar{f}_2 = [\hat{f}_{r+1}, \ldots, \hat{f}_n]^{\mathrm{T}} \in \mathcal{K}^{n-r}$.

It suffices to show the existence of $\lambda \in \mathcal{K}$ and $v \in \mathcal{K}^n \setminus \{0\}$ satisfying condition (27). Let $\hat{\lambda} \in \mathcal{K}$ and $\hat{v} \in \mathcal{K}^{n-r} \setminus \{0\}$ be an eigenvalue and eigenvector of the matrix $(\partial \bar{f}_2(z)/\partial \bar{z}_2) \in \mathcal{K}^{n\times n}$ in the sense of linear algebra. Then, for $\lambda := \hat{\lambda} \in \mathcal{K}$ and $v := [0\ \hat{v}]^{\mathrm{T}} \in \mathcal{K}^n \setminus \{0\}$, condition (27) holds. □

The condition of Theorem 4.2 holds if and only if

$$\mathrm{rank}_{\mathcal{K}} \begin{bmatrix} \lambda I_n - (\delta(J_\varphi) + J_\varphi(\partial f(x)/\partial x))J_\varphi^{-1} \\ (\partial h(x)/\partial x)J_\varphi^{-1} \end{bmatrix} = n \tag{28}$$

holds for all $\varphi \in \mathrm{Diff}^n_{\mathcal{K}}(\mathbf{C})$ and $\lambda \in \mathcal{K}$. When $\varphi \in \mathrm{Diff}^n_{\mathcal{K}}(\mathbf{C})$ is the identity mapping, i.e., $J_\varphi \in \mathcal{K}^{n\times n}$ is the identity matrix, condition (28) is nothing but condition (3). Thus, condition (3) is a necessary condition for Theorem 4.2.

### *4.2 Linear Case*

In the linear case, we show that the conditions of Theorems 4.1 and 4.2 are equivalent. The condition of Theorem 4.2 is a sufficient condition for observability, and that of Theorem 4.1 is a necessary condition. Thus, the condition of Theorem 4.2 is a sufficient condition for that of Theorem 4.1. Here, the converse is shown in the linear case.

*Proposition 4.1.* Suppose that $f = Ax$ and $h = c^{\mathrm{T}}x$ in (1), where $A \in \mathbf{C}^{n\times n}$ and $c \in \mathbf{C}^n$. If condition (12) holds for all $\varphi \in \mathrm{Diff}^n_{\mathcal{K}}(\mathbf{C})$, $\lambda \in \mathcal{K}$, $v \in \mathcal{K}^n \setminus \{0\}$ and $\varepsilon \in \mathcal{X} \setminus \{0\}$, then condition (26) holds for all $\varphi \in \mathrm{Diff}^n_{\mathcal{K}}(\mathbf{C})$, $\lambda \in \mathcal{K}$ and $v \in \mathcal{K}^n \setminus \{0\}$.

**Proof.** We prove this by contraposition. That is, we show that if there exist $\varphi \in \mathrm{Diff}^n_{\mathcal{K}}(\mathbf{C})$, $\lambda \in \mathcal{K}$ and $v \in \mathcal{K}^n \setminus \{0\}$ satisfying condition (25), then there exists $\varepsilon \in \mathcal{X} \setminus \{0\}$ such that condition (13) holds for the same $\varphi$, $\lambda$ and $v$.

In condition (25), let $\varphi \in \mathrm{Diff}^n_{\mathcal{K}}(\mathbf{C})$ be the identity mapping. Then, $J_\varphi \in \mathcal{K}^{n\times n}$ is the identity matrix. In the linear case, $\lambda \in \mathbf{C} \subset \mathcal{K}$ and $v \in (\mathbf{C}^n \setminus \{0\}) \subset (\mathcal{K}^n \setminus \{0\})$ satisfying (25) are one of the eigenvalues and right eigenvectors of the matrix $A$ in the sense of linear algebra.

It suffices to show the existence of $\varepsilon \in \mathcal{X} \setminus \{0\}$ such that (13) holds for the above $\varphi$, $\lambda$ and $v$. Condition (13) can be decomposed into (23) and (25). Since (25) holds for the above $\varphi$, $\lambda$ and $v$, we show the existence of $\varepsilon \in \mathcal{X} \setminus \{0\}$ satisfying (23) for the same $\varphi$, $\lambda$ and $v$. Let $\varepsilon \in \mathcal{X} \setminus \{0\}$ be $a^{\mathrm{T}}dx$, where $a \in (\mathbf{C}^n \setminus \{0\}) \subset (\mathcal{K}^n \setminus \{0\})$ is a left eigenvector of matrix $A$ corresponding to the eigenvalue $\lambda$. By substituting $\varepsilon = a^{\mathrm{T}}dx$ into the left-hand side of (23), we have

$$(s - \lambda)va^{\mathrm{T}}dx = va^{\mathrm{T}}(A - \lambda I_n)dx = 0.$$

That is, condition (23) holds. □

In the linear case, the conditions of Theorems 4.1 and 4.2 are equivalent, and the condition of Theorem 4.2 is equivalent to the observability PBH test. Therefore, the condition of Theorem 4.1 is equivalent to the observability PBH test.

## 5. EXAMPLE

By using our results, we test the observability of the following system.

$$\begin{cases} \dfrac{dx}{dt} = \begin{bmatrix} x_2 + x_3^2 \\ x_1^2 x_3 \\ -x_1^2/2 \end{bmatrix}, \\ y = x_1. \end{cases}$$

We check the necessary condition of Theorem 4.1 for this system. Here, we consider finding $\varphi \in \mathrm{Diff}^n_{\mathcal{K}}(\mathbf{C})$, $\lambda \in \mathcal{K}$, $v \in \mathcal{K}^n \setminus \{0\}$ and $\varepsilon \in \mathcal{X} \setminus \{0\}$ such that condition (13) holds. First, we find an eigenvalue and an eigenvector of the PLT defined by the system. One of the eigenvalues and one of the eigenvectors of such PLT are $0 \in \mathcal{K}$ and $d(x_2 + x_3^2) \in \mathcal{X}$, respectively.

Next, we find a coordinate transformation $\varphi \in \mathrm{Diff}^n_{\mathcal{K}}(\mathbf{C})$ such that an eigenvalue of the matrix $(\delta(J_\varphi) + J_\varphi(\partial f/\partial x))J_\varphi^{-1}$ becomes $0 \in \mathcal{K}$. For instance, by choosing $\varphi(x)$ as $[x_1\ x_2 + x_3^2\ x_3]$, we have

$$J_\varphi := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2x_3 \\ 0 & 0 & 1 \end{bmatrix},\ \delta(J_\varphi) := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -x_1^2 \\ 0 & 0 & 1 \end{bmatrix},$$

and thus

$$\left(\delta(J_\varphi) + J_\varphi \frac{\partial f(x)}{\partial x}\right) J_\varphi^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -x_1 & 0 & 0 \end{bmatrix}.$$

Then, 0 is an eigenvalue of $(\delta(J_\varphi) + J_\varphi(\partial f/\partial x))J_\varphi^{-1}$, and one of its right eigenvectors is $v := [0\ 0\ 1]^{\mathrm{T}}$.

Finally, we check condition (12) for $\varphi := [x_1\ x_2 + x_3^2\ x_3]$, $\lambda := 0$, $v := [0\ 0\ 1]^{\mathrm{T}}$ and $\varepsilon := d(x_2 + x_3^2)$. We obtain

$$\begin{bmatrix} (s-\lambda)I_3 \\ \lambda I_3 - (\delta(J_\varphi) + J_\varphi(\partial f(x)/\partial x))J_\varphi^{-1} \\ (\partial h(x)/\partial x)J_\varphi^{-1} \end{bmatrix} v\varepsilon = \begin{bmatrix} sI_3 \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -x_1 & 0 & 0 \end{bmatrix} \\ [1\ 0\ 0] \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} d(x_2 + x_3^2) = 0.$$

Therefore, condition (12) does not hold. From Theorem 4.1, the system is not observable.

## 6. CONCLUSION

In this paper, we have derived two observability conditions for the nonlinear system: a necessary condition and a sufficient condition. Our necessary condition shows that observability of the nonlinear system is characterized by the eigenvalues of the PLT defined by the system. In the linear case, each eigenvalue of the PLT is nothing but an eigenvalue, in the sense of linear algebra, of the system matrix. Both our conditions are equivalent to the observability PBH (Popov-Belevitch-Hautus) test in the linear case. Therefore, our necessary condition can be viewed as a generalization of the observability PBH test for the nonlinear system.

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