Stabilization of generalized triangular form systems with dynamic uncertainties by means of small gain theorems ⋆

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Abstract: We prove that a nonlinear control system with periodic dynamics in the generalized triangular form (GTF) which is affected by external disturbances can be uniformly input-to-state stabilized by means of a periodic feedback and the gain can be chosen arbitrarily small in some sense. This allows us to stabilize such a system in presence of unmeasured dynamic uncertainties.

1. INTRODUCTION

The concept of input-to-state stability introduced in Sontag [1989] and the theory developed later within this framework has become a powerful tool for solving numerous problems of robust and adaptive nonlinear control. For instance, initially the idea of backstepping Coron et al. [1991], Kanellakopoulos et al. [1991], Krstic et al. [1995] was applied mainly to strict-feedback forms Freeman et al. [1998], Seto et al. [1994], which were introduced as early as Korobov [1973]. After the small-gain theorems had been proved Jiang et al. [1994], the above-mentioned recursive designs of robust and adaptive controllers became applicable to more general classes of nonlinear systems Jiang et al. [1994, 1997], Tsinias et al. [1999], Liu et al. [2012].

On the other hand a natural extension of the strict-feedback form is the so-called generalized (or general) triangular form, which possesses many properties of the strict-feedback form: global robust controllability Korobov et al. [2008], global asymptotic stabilizability Pavlichkov et al. [2009], and even uniform input-to-state stabilizability with respect to external disturbances Dashkovskiy et al. [2012].

However the generalized triangular form (GTF) describes an essentially larger class of systems in comparison with the strict-feedback form systems which leads to many differences in the corresponding properties. For example the GTF does not satisfy the well-known Respondek-Jakubczyk conditions and therefore is not feedback linearizable in general. In this context, one open problem is that of feedback triangulation of a nonlinear system in the "singular case", i.e., when the triangular canonical form is not feedback linearizable. Some promising results were obtained in Celikovsky et al. [1996], however they are local and are based on the assumption that the set of regular points is open and dense in the state space. Another challenging open problem is to extend all the above mentioned theory on the design of robust and adaptive controllers to the case of GTF systems. Since the "input-output maps" $x_{i+1}(\cdot) \mapsto x_i(\cdot)$ of a GTF system are not continuously invertible, the standard backstepping designs become not applicable, because they lead to getting discontinuous virtual controllers at each step of the backstepping algorithm whereas one needs to take their derivatives at the next step.

Following the latter line, the current paper addresses the same problem of partial state stabilizing feedback design for a nonlinear system with dynamic uncertainties as considered in Jiang et al. [1994], Tsinias et al. [1999]. In contrast to these works, we assume that the system under consideration is not in strict-feedback form but is of GTF. This makes us to develop and apply another technique in order to construct the desired controllers and to comply with the small gain condition.

2. PRELIMINARIES

First we recall that a function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{N}$ if it is continuous and nondecreasing; it is of class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$, and it is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and it is unbounded. A function continuous $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{KL}$ if for each fixed $t \geq 0$ the function $\beta(\cdot, t)$ is of class $\mathcal{K}$ and for each fixed $s \geq 0$, we have $\beta(s, t) \to 0$ as $t \to +\infty$ and $t \mapsto \beta(s, t)$ is decreasing.

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For any $x \in \mathbb{R}^{\mathrm{N}}$ with some $\mathrm{N} \in \mathbb{N}$, by $B_r(x)$ and $\overline{B_r(x)}$ we denote the open and closed balls $B_r(x) := \{y \in \mathbb{R}^{\mathrm{N}} | \, |y - x| < r\}$ and $\overline{B_r(x)} := \{y \in \mathbb{R}^{\mathrm{N}} | \, |y - x| \leq r\}$ respectively (if the value of N is not clear from the context, we will mention in which space we work).

Consider the nonlinear system

$$\dot{X}(t) = \Phi(t, X(t), D(t)), \qquad t \in \mathbb{R} \qquad (1)$$

with states $X \in \mathbb{R}^{\mathrm{n}}$, and external disturbances $D(\cdot)$ in $L_\infty(\mathbb{R}; \mathbb{R}^{\mathrm{l_0}})$. We assume that Φ is piecewise continuous w.r.t (t, X, D) and Lipschitz continuous w.r.t. (X, D).

For every $D(\cdot) \in L_\infty(\mathbb{R}; \mathbb{R}^{\mathrm{l_0}})$, we denote by $\|D(\cdot)\|$ its L_∞ - norm on $\mathbb{R}$, and for each $X^0 \in \mathbb{R}^{\mathrm{n}}$, each $t_0 \in \mathbb{R}$ by $t \mapsto X(t, t_0, X^0, D(\cdot))$ we denote the solution of the Cauchy problem $X(t_0) = X^0$ for system (1) with this $D(\cdot)$. The following definition can be found in Lin et al. [2005] (for the case of the equilibrium at $X^* = 0$) and is a natural extension of the original notion of the ISS introduced in Sontag [1989].

Definition 1 *System (1) is input-to-state stable (ISS) at point $X^* \in \mathbb{R}^{\mathrm{n}}$ if there are $\beta \in \mathcal{KL}$, $\Upsilon_0 \in \mathcal{N}$ and $\gamma \in \mathcal{K}$ such that for each t_0, each X^0 and each $D(\cdot) \in L_\infty$ we have*

$$|X(t, X^0, t_0, D(\cdot)) - X^*| \leq$$
$$\max\{\beta(\Upsilon_0(t_0)|X^0 - X^*|, t - t_0),\ \gamma(\|D(\cdot)\|_{L_\infty[t_0, +\infty[})\},\ t \geq t_0.$$

System (1) is semi-uniformly ISS at point $X^* \in \mathbb{R}^{\mathrm{n}}$ if it is ISS at point X^* and $\exists \Upsilon(\cdot) \in \mathcal{K}$ s.t. for each $X^0 \in \mathbb{R}^{\mathrm{n}}$, each $t \geq t_0$ and each $D(\cdot)$ in L_∞ we have

$$|X(t, X^0, t_0, D(\cdot)) - X^*| \leq \max\{\Upsilon(|X^0 - X^*|), \Upsilon(\|D(\cdot)\|)\}.$$

System (1) is said to be uniformly input-to-state stable (ISS) at point $X^* \in \mathbb{R}^{\mathrm{n}}$ if there are $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for each $t_0 \in \mathbb{R}$, each $X^0 \in \mathbb{R}^{\mathrm{n}}$ and each $D(\cdot) \in L_\infty(\mathbb{R}; \mathbb{R}^{\mathrm{l_0}})$ we have

$$|X(t, t_0, X^0, D(\cdot)) - X^*| \leq$$
$$\max\{\beta(|X^0 - X^*|, t - t_0),\ \gamma(\| D(\cdot)\|_{L_\infty[t_0, +\infty[})\},\ \ t \geq t_0.$$

Note that, if (1) is ISS at $X^* \in \mathbb{R}^{\mathrm{n}}$, then by this definition X^* is an equilibrium of (1) with $D(\cdot) = 0$. For any $\mathrm{N} \in \mathbb{N}$ by $\langle \cdot, \cdot \rangle$ we denote the scalar product in $\mathbb{R}^{\mathrm{N}}$ and for $\xi \in \mathbb{R}^{\mathrm{N}}$ let $|\xi|$ denote its Euclidean norm, i.e., $|\xi| = \langle \xi, \xi \rangle^{\frac{1}{2}}$.

In the current paper, we use the following immediate corollary of Theorem 3 from Lin et al. [2005] (see also earlier related result Jiang et al. [1997]).

Theorem 1 *Consider the following interconnected T - periodic time-varying system*

$$\begin{aligned} \dot{X}_1(t) &= \Phi_1(t, X_1(t), X_2(t), D(t)) \\ \dot{X}_2(t) &= \Phi_2(t, X_1(t), X_2(t), D(t)), \end{aligned} \qquad t \in \mathbb{R} \qquad (2)$$

(with Φ_1, Φ_2 of class C^1 and with some $T > 0$) composed of two subsystems $\dot{X}_1 = \Phi_1(t, X_1, X_2, D)$ with states $X_1 \in \mathbb{R}^{\mathrm{n_1}}$ and with inputs $[X_2, D] \in \mathbb{R}^{\mathrm{n_2}} \times \mathbb{R}^{\mathrm{N}}$ and $\dot{X}_2 = \Phi_2(t, X_1, X_2, D)$ with states $X_2 \in \mathbb{R}^{\mathrm{n_2}}$ and with inputs $[X_1, D] \in \mathbb{R}^{\mathrm{n_1}} \times \mathbb{R}^{\mathrm{N}}$. Suppose both the subsystems are uniformly ISS at $0 \in \mathbb{R}^{\mathrm{n_1}}$ and $0 \in \mathbb{R}^{\mathrm{n_2}}$, so that there are $\beta_i(\cdot, \cdot) \in \mathcal{KL}$, $\gamma_i(\cdot) \in \mathcal{K}$, $\gamma_{i,D}(\cdot) \in \mathcal{K}$, $i = 1, 2$, such that all their trajectories satisfy the following inequalities:

$$\begin{aligned} |X_1(t)| \leq \max\{&\beta(|X_1(t_0)|, t - t_0),\ \gamma_1(\| X_2(\cdot)\|_{L_\infty}), \\ &\gamma_{1,D}(\| D(\cdot)\|_{L_\infty})\}, \quad t \geq t_0, \end{aligned} \qquad (3)$$

and

$$\begin{aligned} |X_2(t)| \leq \max\{&\beta(|X_2(t_0)|, t - t_0),\ \gamma_2(\| X_2(\cdot)\|_{L_\infty}), \\ &\gamma_{2,D}(\| D(\cdot)\|_{L_\infty})\}, \quad t \geq t_0. \end{aligned} \qquad (4)$$

If the small gain condition $(\gamma_1 \circ \gamma_2)(r) < r$ holds true for all $r > 0$, then the interconnected system (2) is uniformly ISS w.r.t. the external disturbance $D(t)$.

(Our main result deals with T-periodic systems and we take into account that for the periodic systems semi-uniform ISS property implies uniform ISS property).

3. MAIN RESULT

We consider a control system of the following form

$$\begin{cases} \dot{\xi} = F(x_1, \xi, D(t)), \\ \dot{x}_1 = f_1(t, x_1, x_2) + \varphi_1(t, x_1, \xi, D(t)), \\ \dot{x}_2 = f_2(t, x_1, x_2, x_3) + \varphi_2(t, x_1, x_2, \xi, D(t)), \\ \dots \\ \dot{x}_\nu = f_\nu(t, x_1, \dots, x_\nu, u) + \varphi_\nu(t, x_1, \dots, x_\nu, \xi, D(t)), \end{cases} \qquad (5)$$

where $u \in \mathbb{R}^m = \mathbb{R}^{m_{\nu+1}}$ is the control, $x = [x_1^T, \dots, x_\nu^T]^T \in \mathbb{R}^n$ are *measured* components of the state vector with $x_i \in \mathbb{R}^{m_i}$, $m_i \leq m_{i+1}$, $n = m_1 + \ldots + m_\nu$ and $\xi = [\xi_1, \dots, \xi_N]^T \in \mathbb{R}^N$ are *unmeasured* components of the state vector and $D(\cdot) \in L_\infty(\mathbb{R}; \mathbb{R}^{\mathrm{l_0}})$ are external disturbances.

As regards the x - subsystem of the overall system (5), we assume that the following conditions hold true:

- A1: $f = [f_1, \dots, f_\nu]^T$ and $\varphi = [\varphi_1, \dots, \varphi_\nu]^T$ are of class $C^{\nu+1}(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n)$ and $C^{\nu+1}(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^{\mathrm{l_0}}; \mathbb{R}^n)$ respectively and are T-periodic, i.e., there is $T > 0$ such that $f(t + T, x, u) = f(t, x, u)$ and $\varphi(t + T, x, \xi, D) = \varphi(t, x, \xi, D)$ for all $[t, x, u, \xi, D]$ in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^N \times \mathbb{R}^{\mathrm{l_0}}$.
- A2: For each $i = 1, \dots, \nu$ and each $[t, x_1, \dots, x_i]$ in $[0, T] \times \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_i}$ we have $f_i(t, x_1, \dots, x_i, \mathbb{R}^{m_{i+1}}) = \mathbb{R}^{m_i}$, i.e., $f_i(t, x_1, \dots, x_i, \cdot)$ is a surjection.
- A3: There are $x_i^* \in \mathbb{R}^{m_i}$, $1 \leq i \leq \nu$, and $u^* = x_{\nu+1}^* \in \mathbb{R}^{\mathrm{m}}$ such that $\mathrm{rank} \frac{\partial f_i}{\partial x_{i+1}}(t, x_1^*, \dots, x_{i+1}^*) = m_i$, and $f_i(t, x_1^*, ..., x_{i+1}^*) = \varphi_i(t, x_1^*, ..., x_i^*, 0, 0) = 0 \in \mathbb{R}^{m_i}$ for all $t \in [0, T]$, $i = 1, \dots, \nu$.

Note that A1-A3 are standard conditions characterizing GTF when solving a stabilization problem - see Pavlichkov et al. [2009], Dashkovskiy et al. [2012], Tsinias [1995]. In our case the x-subsystem is of GTF and is interconnected with the ξ-subsystem. We also assume that the latter satisfies the following conditions:

- A4: F is of class C^1, $F(x_1^*, 0, 0) = 0 \in \mathbb{R}^N$ and $\frac{\partial F}{\partial \xi}(x_1^*, 0, 0)$ is an asymptotically stable matrix.
- A5: The system
$$\dot{\xi} = F(x_1, \xi, D) \qquad (6)$$
is ISS with known gain w.r.t. $[x_1(\cdot) - x_1^*, D(\cdot)]$ as the input.

Note that A4 and A5 are also quite natural assumptions - see, for instance Jiang et al. [1994].

While $\xi \in \mathbb{R}^N$ is assumed to be *unmeasured*, we suppose that the local and global gains of system (6) obtained from Assumptions A4, A5 respectively are *known* as well. Our goal is to find a feedback $u = u(t,x)$, which depends on the measured states x only and globally input-to-state stabilizes the overall system (5) w.r.t. the external disturbances $D(\cdot)$. The main result of the current paper is as follows.

Theorem 2 *Assume that (5) satisfies conditions A1-A5. Then there exists a T-periodic feedback law $u = u(t,x)$ of class $C^1(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^m)$ such that $u^* = u(t,x_1^*,...,x_\nu^*)$ and such that the closed-loop system (5) with $u = u(t,x)$ is uniformly ISS at point $[x^*,0] = [x_1^*,...,x_\nu^*,0] \in \mathbb{R}^n\times\mathbb{R}^N$ w.r.t. the external disturbances $D(t)$. Furthermore, if $D(t) = 0$ a.e. on $\mathbb{R}$, then system (5) is not only GAS but also locally exponentially stable at point $[x^*,0] \in \mathbb{R}^n\times\mathbb{R}^N$.*

Remark 1 Let us note that work Tsinias et al. [1999] deals with the same problem but for the case of strict-feedback systems. Our Theorem 2 addresses the class of GTF systems, and, in this sense, Theorem 2 is an extension of the result of Tsinias et al. [1999]. However, we consider the T-periodic systems only, while Tsinias et al. [1999] deals with time-varying systems which are not necessarily T-periodic but satisfy some other restrictions, e.g., bounded growth in time - see [Tsinias et al. , 1999, Assumptions A1),A2)]

The rest of the paper is devoted to the proof of our main result.

4. PROOF OF THEOREM 2

The proof is based on application of the same technique as in Pavlichkov et al. [2009], Dashkovskiy et al. [2012] and since the detailed proof of the main results of Dashkovskiy et al. [2012] takes more than 15 pages, we provide a sketched proof of our main results and refer to these works when necessary. Without loss of generality we assume that x^*=0 and u^*=0.

Given any $t_0 \in \mathbb{R}$, any $\xi_0 \in \mathbb{R}^N$, any $x_1(\cdot)$ in $L_\infty(\mathbb{R};\mathbb{R}^{m_1})$ and any $D(\cdot)$ in $L_\infty(\mathbb{R};\mathbb{R}^{l_0})$, let $t \mapsto \xi(t,t_0,\xi^0,x_1(\cdot),D(\cdot))$ denote the trajectory, of system (6), that is defined by the input $[x_1(\cdot),D(\cdot)]$ and by the initial condition $\xi(t_0) = \xi^0$.

From Assumptions A4, A5 it follows that there are $\bar\beta(\cdot,\cdot) \in \mathcal{KL}$, $\bar\gamma(\cdot) \in \mathcal{K}_\infty$ and $\bar\gamma_D(\cdot) \in \mathcal{K}_\infty$ such that

$$\forall t_0{\in}\mathbb{R}\ \ \forall \xi^0{\in}\mathbb{R}^N\ \ \forall x_1(\cdot){\in}L_\infty(\mathbb{R};\mathbb{R}^{m_1})\ \ \forall D(\cdot){\in}L_\infty(\mathbb{R};\mathbb{R}^{l_0})$$
$$\forall t \geq t_0\quad |\xi(t,t_0,\xi^0,x_1(\cdot),D(\cdot))| \leq \max\{\bar\beta(|\xi^0|,t{-}t_0), \bar\gamma(\| x_1(\cdot)\|_{L_\infty[t_0,+\infty[}),\bar\gamma_D(\| D(\cdot)\|_{L_\infty[t_0,+\infty[})\}. \quad (7)$$

and such that for some $\bar\varepsilon > 0$ and some $c_1 > 0$, $c_2 > 0$, $c_3 > 0$ we have

$$\bar\gamma(\varepsilon) = c_1\varepsilon, \qquad \text{whenever } \varepsilon \in [0,\bar\varepsilon[; \quad (8)$$
$$\bar\beta(s,t) = e^{-c_2 t}c_3 s, \qquad \text{whenever } s \in [0,\bar\varepsilon[\text{ and } t \geq 0 \quad (9)$$

4.1 Plan of the Proof of Theorem 2

To prove Theorem 2, we consider the control system

$$\begin{cases} \dot x_1 = f_1(t,x_1,x_2) + \varphi_1(t,x_1,\xi,D(t)), \\ \dot x_2 = f_2(t,x_1,x_2,x_3) + \varphi_2(t,x_1,x_2,\xi,D(t)), \\ \dots \\ \dot x_\nu{=}f_\nu(t,x_1,\dots,x_\nu,u){+}\varphi_\nu(t,x_1,\dots,x_\nu,\xi,D(t)), \end{cases} \quad (10)$$

with states $x = [x_1^T,\dots,x_\nu^T]^T \in \mathbb{R}^n$, controls $u \in \mathbb{R}^{\mathrm{m}} = \mathbb{R}^{m_{\nu+1}}$ and with external disturbances $\eta(\cdot) := [\xi(\cdot),D(\cdot)]^T \in L_\infty(\mathbb{R};\mathbb{R}^N\times\mathbb{R}^{l_0})$.

Our goal is to find a global state transformation of the form

$$\begin{aligned} &Z_1 = x_1 \\ &Z_2 = x_2 - \alpha_1(t,x_1) \\ &\dots \\ &Z_\nu = x_\nu - \alpha_{\nu-1}(t,x_1,\dots,x_{\nu-1}), \end{aligned} \quad (11)$$

where $\alpha_i \in C^{\nu-i+1}(\mathbb{R}\times\mathbb{R}^{m_1+\dots+m_i};\mathbb{R}^{m_{i+1}})$ are T -periodic w.r.t. the time variable t and $\alpha_i(t,0,...,0) = 0$, $t \in \mathbb{R}$ and to find a T -periodic feedback $u = u(t,x_1,...,x_\nu)$ of class $C^1(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^m)$ such that $u(t,0,...,0) = 0$, $t \in \mathbb{R}$ and such that the trajectories $t \mapsto Z(t,t_0,Z^0,u(\cdot,\cdot),\xi(\cdot),D(\cdot))$, of system (10) written in the new coordinates $Z = [Z_1,...,Z_\nu]$ that are defined by (11), satisfies the inequality

$$\forall t{\geq}t_0\ |Z(t,t_0,Z^0,u(\cdot,\cdot),\xi(\cdot),D(\cdot))|{\leq}\max\{\beta(|Z^0|,t{-}t_0), \gamma_{\mathrm{sm}}(\|\xi(\cdot)\|_{L_\infty([t_0,+\infty[;\mathbb{R}^N)}),\gamma_D(\|D(\cdot)\|_{L_\infty([t_0,+\infty[;\mathbb{R}^{l_0})})\}, \quad (12)$$

with $\beta \in \mathcal{KL}$, $\gamma_D \in \mathcal{K}_\infty$, $\gamma_{\mathrm{sm}} \in \mathcal{K}_\infty$ such that the following small gain condition holds:

$$(\bar\gamma \circ \gamma_{\mathrm{sm}})(r) < r \qquad \text{for all } r > 0. \quad (13)$$

Since $Z_1 = x_1$, we combine (7) and (13) and apply Theorem 1. Then system (5) with $u = u(t,x)$ becomes uniformly ISS w.r.t. $D(\cdot)$ as the input.

To do this, we will design the desired controller which satisfies the small gain condition for the system

$$\begin{cases} \dot x_1 = f_1(t,x_1,x_2) + \varphi_1(t,x_1,\xi,D(t)), \\ \dot x_2 = f_2(t,x_1,x_2,x_3) + \varphi_2(t,x_1,x_2,\xi,D(t)), \\ \dots \\ \dot x_i{=}f_i(t,x_1,\dots,x_i,x_{i+1}){+}\varphi_i(t,x_1,\dots,x_i,\xi,D(t)), \end{cases} \quad (14)$$

with states $[x_1,...,x_i]{\in}\mathbb{R}^{m_1+\dots+m_i}$, controls $x_{i+1}{\in}\mathbb{R}^{m_{i+1}}$ and disturbances $[\xi,D]{\in}\mathbb{R}^{N+l_0}$ by induction on $i = 1,\dots,\nu$.

4.2 The Base Case: $i = 1$

Define

$$\gamma := \frac{1}{4}(2\bar\gamma)^{-1}. \quad (15)$$

Then we have

$$\forall r > 0 \quad (\gamma\circ(2\bar\gamma))\,(r){<}\frac{1}{2}r \ \text{ and } \ ((2\bar\gamma)\circ(2\gamma))\,(r){<}r \quad (16)$$

$$\text{and} \qquad \forall r \in [0,c_1\bar\varepsilon[\qquad \gamma(r) = \frac{1}{8c_1}r \quad (17)$$

Our goal is to design a controller which satisfies the small gain condition (13) with $\gamma_{\mathrm{sm}}(r) = \gamma(2r)$ for all $r \geq 0$.

Take and fix any $\lambda_0 \geq 1$ and any $\gamma_0 > 0$ such that

$$4\sqrt{\frac{3\gamma_0}{\lambda_0}} < \frac{1}{8c_1} \quad \text{and} \quad 4\sqrt{\frac{3\gamma_0}{\lambda_0}} < \frac{1}{8} \quad (18)$$

$$\text{and} \qquad 64\gamma_0 c_1^2 < \frac{\lambda_0 e^{-3\lambda_0 T}}{2}. \qquad (19)$$

Let us remark that the goal of inequalities (18) is to provide the desired small gain condition (13) with $\gamma_{\rm sm}(r) = \gamma(2r)$ locally, around the origin by using (8), (17) (and the goal of (19) is to achieve below (26) for all $q \in \mathbb{Z}$).

Fix any $D_{\max} > 0$. Then, using the implicit function theorem and Assumption A3, we obtain the existence of $\rho(\lambda_0, \gamma_0) > 0$ and a T - periodic feedback $\omega_1(\cdot,\cdot)$ of class $C^\nu(\mathbb{R} \times \overline{B_{\rho(\lambda_0,\gamma_0)}(0)}; \mathbb{R}^{m_2})$, where $\overline{B_{\rho(\lambda_0,\gamma_0)}(0)}$ denotes the closed ball in $\mathbb{R}^{m_1}$ with its center at $0 \in \mathbb{R}^{m_1}$ and radius $\rho(\lambda_0, \gamma_0)$, such that

$$\begin{aligned} &\langle 2x_1, f_1(t, x_1, \omega_1(t, x_1)) + \varphi_1(t, x_1, \xi, D)\rangle \le -\lambda_0 \langle x_1, x_1\rangle \\ &\quad + \gamma_0\langle \xi, \xi\rangle + \gamma_0 \langle D, D\rangle, \qquad \text{whenever} \\ &[t, x_1, \xi, D] \in \mathbb{R} \times \mathbb{R}^{m_1} \times \mathbb{R}^{N+l_0} \qquad \text{satisfies} \\ &|x_1| \le \rho(\lambda_0, \gamma_0) \quad \text{and} \quad |\xi|^2 + |D|^2 \le D_{\max}^2 \qquad (20)\end{aligned}$$

Without loss of generality we assume that

$$\rho(\lambda_0, \gamma_0) < \gamma(D_{\max}) e^{-\frac{\lambda_0 T}{2}} \quad \text{and} \quad 8\rho(\lambda_0,\gamma_0) < \bar\varepsilon \quad \text{and} \quad 8\rho(\lambda_0, \gamma_0) < c_1 \bar\varepsilon \qquad (21)$$

(otherwise we take a smaller $\rho(\lambda_0, \gamma_0) > 0$). Take any sequence $\{\rho_q\}_{q=-\infty}^{+\infty} \subset]0, +\infty[$ such that

$$0 < \rho_4 < \rho(\lambda_0, \gamma_0) \quad \text{and} \quad \rho_q = e^{\frac{\lambda_0 T}{8}} \rho_{q-1} \text{ for all } q \in \mathbb{Z} \qquad (22)$$

Then we define the sequences $\{r_q\}_{q=-\infty}^{+\infty} \subset]0, +\infty[$ and $\{D_q\}_{q=-\infty}^{+\infty} \subset]0, +\infty[$ by

$$r_q = \rho_q e^{-\frac{\lambda_0 T}{16}} \quad \text{for all } q \in \mathbb{Z} \qquad (23)$$

(in particular $0 < \rho_q < r_{q+1} < \rho_{q+1}$) and by

$$D_q := \gamma^{-1}(\rho_{q+4}) \ \Leftrightarrow\ (\gamma(D_q))^2 = \rho_{q+4}^2 \quad \text{for all } q \in \mathbb{Z} \qquad (24)$$

Consider the system

$$\dot x_1 = f_1(t, x_1, x_2) + \varphi_1(t, x_1, \xi, D), \qquad t \in \mathbb{R} \qquad (25)$$

where $x_1 \in \mathbb{R}^{m_1}$ is the state, $x_2 \in \mathbb{R}^{m_2}$ is the control and $\eta := [\xi, D] \in \mathbb{R}^N \times \mathbb{R}^{l_0}$ is the external disturbance.

For each $t_0 \in \mathbb{R}$, each $x_1^0 \in \mathbb{R}^{m_1}$, each $\eta(\cdot)$ in $L_\infty(\mathbb{R}; \mathbb{R}^{N+l_0})$ and each controller $v(\cdot,\cdot)$ of class $C^1(\mathbb{R} \times \mathbb{R}^{m_1}; \mathbb{R}^{m_2})$, by $t \mapsto x_1(t, t_0, x_1^0, \eta(\cdot), v(\cdot,\cdot))$ denote the trajectory, of system (25), that is defined by these initial condition $x_1(t_0) = x_1^0$, disturbance $\eta = \eta(t)$ and control $x_2 = v(t, x_1)$.

Then, arguing as in Pavlichkov et al. [2009], Dashkovskiy et al. [2012], we obtain the existence of a T - periodic feedback $v_1(\cdot,\cdot)$ of class $C^\nu(\mathbb{R} \times \mathbb{R}^{m_1}; \mathbb{R}^{m_2})$ such that

(a) $v_1(t, x_1) = \omega_1(t, x_1)$ for all $[t, x_1] \in \mathbb{R} \times \overline{B_{\rho_4}(0)}$
(b) for each $x_1^0 \in \mathbb{R}^{m_1}$, each $t_0 \in [0, T]$, each $\eta(\cdot) = [\xi(\cdot), D(\cdot)]^T$ in $L_\infty([t_0, t_0 + T]; \mathbb{R}^N \times \mathbb{R}^{l_0})$ and each $q \in \mathbb{N}$ we have:

$$\begin{aligned} &\big(|x_1^0|^2 \le r_{q+2}^2 \text{ and } |\xi(t)|^2 + |D(t)|^2 \le D_q^2 \\ &\text{a.e. on } [t_0, t_0 + T]\big) \Rightarrow \big(|x_1(t, t_0, x_1^0, \eta(\cdot), v_1(\cdot,\cdot))|^2 \\ &\le \rho_{q+2}^2 - \frac{t - t_0}{T}(\rho_{q+2}^2 - \rho_q^2) \text{ for all } t \in [t_0, t_0 + T]\big) \qquad (26)\end{aligned}$$

Using (19), (20) and condition (a), we easily obtain that (26) holds for all $q \in \mathbb{Z}$.

For every $q \in \mathbb{Z}$ define

$$\beta_{q+1}(\tau) := \Big(\rho_{(q-\kappa)+2}^2 - \frac{\tau - \kappa T}{T}(\rho_{(q-\kappa)+2}^2 - \rho_{q-\kappa}^2)\Big)^{\frac{1}{2}}$$

$$\text{and} \quad \gamma_q(\tau) := \Big(\rho_{q+2}^2 - \frac{\tau - \kappa T}{T}(\rho_{q+2}^2 - \rho_q^2)\Big)^{\frac{1}{2}} \quad \text{and}$$

$$\beta(r_{q+1}, \tau) := \Big(\rho_{(q-\kappa)+2}^2 - \frac{\tau - \kappa T}{T}(\rho_{(q-\kappa)+2}^2 - \rho_{(q-\kappa)+1}^2)\Big)^{\frac{1}{2}},$$
$$\text{whenever } \tau \in [\kappa T, (\kappa+1)T[, \quad \kappa \in \mathbb{Z}_+ \qquad (27)$$

and then define

$$\beta(r, \tau) := \beta(r_q, \tau) + \frac{r - r_q}{r_{q+1} - r_q}(\beta(r_{q+1}, \tau) - \beta(r_q, \tau)),$$
$$\text{whenever } r \in [r_q, r_{q+1}[, \quad \text{with } q \in \mathbb{Z} \quad \text{for all } \tau \ge 0, \qquad (28)$$

and

$$\hat\gamma(D) := \Big(\rho_{q+2}^2 + \frac{D - D_{q-1}}{D_q - D_{q-1}}(\rho_{q+3}^2 - \rho_{q+2}^2)\Big)^{\frac{1}{2}},$$
$$\text{whenever } D \in]D_{q-1}, D_q)], \ q \in \mathbb{Z} \qquad (29)$$

Then we define functions $\beta(\cdot,\cdot) \in \mathcal{KL}$ and $\hat\gamma \in \mathcal{K}_\infty$ by $\beta(0, \tau) := 0$, $\tau \ge 0$ and $\hat\gamma(0) := 0$ for $r = 0$, $D = 0$ and by (28), (29) for $r > 0$, $D > 0$. Note that by the construction, more specifically by (24), (29), we have

$$\hat\gamma(D) < \gamma(D) \quad \text{for all } D > 0. \qquad (30)$$

Take and fix arbitrary $\kappa \in \mathbb{Z}_+$, $t_0 \in \mathbb{R}$, $x_1^0 \in \mathbb{R}^{m_1}$ and $\eta(\cdot)$ in $L_\infty(\mathbb{R}; \mathbb{R}^{N+l_0})$. Without loss of generality assume that $x_1^0 \neq 0 \in \mathbb{R}^{m_1}$ and $\eta(\cdot) \neq 0 \in L_\infty(\mathbb{R}; \mathbb{R}^{N+l_0})$ (the cases $x_1^0 = 0$ or $\eta(\cdot) = 0$ are studied similarly).

Define $\bar q \in \mathbb{Z}$ and $\hat q \in \mathbb{Z}$ by $r_{\bar q+1}^2 < |x_1^0|^2 \le r_{\bar q+2}^2$ and by $D_{\hat q-1} < \| \eta(\cdot) \|_{L_\infty([t_0, t_0+\kappa T]; \mathbb{R}^{N+l_0})} \le D_{\hat q}$.

If $\hat q \le \bar q$, then for all $t \in [t_0, t_0 + \kappa T]$ we have

$$\begin{aligned} &|x_1(t, t_0, x_1^0, \eta(\cdot), v_1(\cdot,\cdot))|^2 \le \max\{\beta_{\bar q+1}^2(t - t_0),\ \gamma_{\hat q}^2(t - t_0)\} \\ &\le \max\{\beta^2(|x_1^0|, t - t_0),\ \rho_{\hat q+2}^2\} \le \max\{\beta^2(|x_1^0|, t - t_0), \\ &\hat\gamma^2(\| \eta(\cdot) \|_{L_\infty([t_0, t_0+\kappa T]; \mathbb{R}^{N+l_0})})\} \le \\ &\le \max\{\beta^2(|x_1^0|, t - t_0), \gamma^2(\| \eta(\cdot) \|_{L_\infty([t_0, t_0+\kappa T]; \mathbb{R}^{N+l_0})})\}.\end{aligned}$$

If $\hat q > \bar q$, then for all $t \in [t_0, t_0 + \kappa T]$ we have

$$\begin{aligned} &|x_1(t, t_0, x_1^0, \eta(\cdot), v_1(\cdot,\cdot))|^2 \le \gamma_{\hat q}^2(t - t_0) \le \rho_{\hat q+2}^2 \\ &\le \max\{\beta^2(|x_1^0|, t - t_0),\ \rho_{\hat q+2}^2\} \\ &\le \max\{\beta^2(|x_1^0|, t - t_0), \hat\gamma^2(\| \eta(\cdot) \|_{L_\infty([t_0, t_0+\kappa T]; \mathbb{R}^{N+l_0})})\} \le \\ &\le \max\{\beta^2(|x_1^0|, t - t_0), \gamma^2(\| \eta(\cdot) \|_{L_\infty([t_0, t_0+\kappa T]; \mathbb{R}^{N+l_0})})\}.\end{aligned}$$

Since $\kappa \in \mathbb{Z}_+$ is chosen arbitrarily, we obtain

$$\begin{aligned} &\forall t \ge t_0 \quad (t, t_0, x_1^0, \eta(\cdot), v_1(\cdot,\cdot))|^2 \le \max\{\beta(|x_1^0|, t - t_0), \\ &\gamma(\| \eta(\cdot)\|_{L_\infty([t_0, +\infty[; \mathbb{R}^{N+l_0})})\} \le \max\{\beta(|x_1^0|, t - t_0), \\ &\gamma(\| 2\xi(\cdot)\|_{L_\infty([t_0, +\infty[; \mathbb{R}^N)}), \gamma(\| 2D(\cdot)\|_{L_\infty([t_0, +\infty[; \mathbb{R}^{l_0})})\},\end{aligned}$$

Thus for the Base Case $i = 1$, i.e., for system (25), we have designed the controller $v_1(\cdot,\cdot)$ that satisfies (13) with $\gamma_{\rm sm}(r) := \gamma(2r)$, $r \ge 0$ (this is because of (16)).

4.3 Inductive Step: Adding an Integrator

To design the desired controller for system (14) by induction over $i = 1, \dots, \nu$, we need the following Theorem 3 given below.

Consider a control system of the form

$$\begin{cases} \dot{z} = g(t, z, w) + \varphi(t, z, \eta(t)) \\ \dot{w} = h(t, z, w, v) + \phi(t, z, w, \eta(t)), \end{cases} \qquad t \in \mathbb{R} \qquad (31)$$

with states $y = [z, w]^T \in \mathbb{R}^k \times \mathbb{R}^{\mathrm{q}}$, controls $v \in \mathbb{R}^{\mathrm{m}}$, where $\mathrm{q} \le \mathrm{m}$, and with external disturbances $\eta(\cdot) = [\xi(\cdot), D(\cdot)]^T \in L_\infty(\mathbb{R}; \mathbb{R}^{N+\mathrm{l}_0})$. We assume that the following conditions hold:

(C1) $g \in C^{\mu+1}(\mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^{\mathrm{q}}; \mathbb{R}^k)$; $\varphi \in C^{\mu+1}(\mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^{N+\mathrm{l}_0}; \mathbb{R}^k)$; $h \in C^{\mu+1}(\mathbb{R} \times \mathbb{R}^{k+\mathrm{q}} \times \mathbb{R}^{\mathrm{m}}; \mathbb{R}^{\mathrm{q}})$; $\phi \in C^{\mu+1}(\mathbb{R} \times \mathbb{R}^{k+\mathrm{q}} \times \mathbb{R}^{N+\mathrm{l}_0}; \mathbb{R}^{\mathrm{q}})$ with some $\mu \in \mathbb{N}$ and there is $T > 0$ such that $g(t+T, z, w) = g(t, z, w)$, $\varphi(t+T, z, \eta) = \varphi(t, z, \eta)$, $h(t+T, z, w, v) = h(t, z, w, v)$, $\phi(t+T, z, w, \eta) = \phi(t, z, w, \eta)$ for all $t \in \mathbb{R}$, $z \in \mathbb{R}^k$, $w \in \mathbb{R}^{\mathrm{q}}$, $v \in \mathbb{R}^{\mathrm{m}}$, $\eta \in \mathbb{R}^{N+\mathrm{l}_0}$.

(C2) $g(t, 0, 0) = \varphi(t, 0, 0) = 0 \in \mathbb{R}^k$ and $h(t, 0, 0, 0) = \phi(t, 0, 0, 0) = 0 \in \mathbb{R}^{\mathrm{q}}$ for all $t \in \mathbb{R}$; and rank$\frac{\partial h(t,0,0,0)}{\partial v} = \mathrm{q}$ for all $t \in \mathbb{R}$.

(C3) $h(t, z, w, \mathbb{R}^{\mathrm{m}}) = \mathbb{R}^{\mathrm{q}}$ for every $[t, z, w]$ in $[0, T] \times \mathbb{R}^{k+\mathrm{q}}$.

Given a controller $v(\cdot, \cdot) : \mathbb{R} \times \mathbb{R}^{k+\mathrm{q}} \to \mathbb{R}^{\mathrm{m}}$, a disturbance $\eta(\cdot) \in L_\infty(\mathbb{R}; \mathbb{R}^{N+\mathrm{l}_0})$ and an initial state $y^0 \in \mathbb{R}^{k+\mathrm{q}}$ for system (31), by $t \mapsto y(t, t_0, y^0, v(\cdot, \cdot), \eta(\cdot))$ denote the trajectory, of (31), defined by $y(t_0) = y^0$, $v = v(t, y)$, $\eta = \eta(t)$ on the maximal possible interval (by definition, we deal only with the controllers that provide the existence and the uniqueness of the solution of the Cauchy problem; in our Theorem 3 given below, the constructed controllers are of class C^1 at least; if within the proof, which is a modification of those of [Dashkovskiy et al. , 2012, Theorem 2] and [Pavlichkov et al. , 2009, Theorem 3.1], one obtains discontinuous controllers, then the existence and uniqueness are provided anyway).

Similarly, for the "subsystem"

$$\dot{z} = g(t, z, w) + \varphi(t, z, \eta(t)), \qquad t \in \mathbb{R}, \qquad (32)$$

where $z \in \mathbb{R}^k$ is treated as the state, $w \in \mathbb{R}^{\mathrm{q}}$ is the control, and $\eta(\cdot)$ is the disturbance signal, we denote by $t \mapsto z(t, t_0, z^0, w(\cdot, \cdot), \eta(\cdot))$ its (maximal) trajectory defined by the initial condition $z(t_0) = z^0$, by the controller $w = w(t, z)$ and by $\eta = \eta(t)$ for every $z^0 \in \mathbb{R}^k$, every $\eta(\cdot) \in L_\infty(\mathbb{R}; \mathbb{R}^{N+\mathrm{l}_0})$ and every suitable (in sense of existence and uniqueness) $w(\cdot, \cdot) : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^{\mathrm{q}}$.

Define the Lyapunov functions for (31) and (32) by

$$\forall y{=}[z, w]^T{\in}\mathbb{R}^k{\times}\mathbb{R}^{\mathrm{q}} \;\; V(y){:=}\langle y, y\rangle \text{ and } W(z){:=}\langle z, z\rangle$$

The last Condition that we assume is as follows (it means the same property for system (32) with $w = 0$ as that which we achieved in the Base Case for system (25) with $x_2 = v_1(t, x_1)$; thus, roughly speaking, this condition is the inductive hypothesis).

(C4) There are sequences $\{r_q^\star\}_{q=4}^{+\infty} \subset]0, +\infty[$ and $\{\rho_q^\star\}_{q=3}^{+\infty} \subset]0, +\infty[$ with $0 < \rho_q^\star < r_{q+1}^\star < \rho_{q+1}^\star \to +\infty$ as $q \to +\infty$ such that $\rho_4^\star < \rho(\alpha_0, \gamma_0)$ and $\rho_{q+3}^\star < \gamma(D_q) \le \rho_{q+4}^\star$ for all $q \in \mathbb{N}$ and there are $\lambda^\star \in]0, \lambda_0[$ and $\gamma^\star \ge \gamma_0$ such that

(i) The following inequalities hold:

$$4\sqrt{\frac{3\gamma^\star}{\lambda^\star}} < \frac{1}{8c_1} \quad \text{and} \quad 4\sqrt{\frac{3\gamma^\star}{\lambda^\star}} < \frac{1}{8} \quad \text{and}$$

$$64\gamma^\star c_1^2 < \frac{\lambda^\star e^{-3\lambda^\star T}}{2}$$

(ii) For every $t \in [0, T]$, every $z \in \mathbb{R}^k$ and every $\eta \in \mathbb{R}^{N+\mathrm{l}_0}$, if $|z| \le r_4^\star$ and $\gamma(|\eta|) \le \rho_8^\star$, then

$$\frac{\partial W(z)}{\partial z}(g(t, z, 0) + \varphi(t, z, \eta)) \le -\lambda^\star W(z) + \gamma^\star \langle \eta, \eta\rangle$$

(iii) For every $q{\ge}3$, $q{\in}\mathbb{Z}$, every $z^0{\in}\mathbb{R}^k$, every $t_0{\in}[0, T]$, and every $\eta(\cdot) \in L_\infty([t_0, t_0 + T]; \mathbb{R}^{N+\mathrm{l}_0})$ the inequalities $|z^0|{\le}r_{q+2}^\star$ and $\|\eta(\cdot)\|_{L_\infty([t_0,t_0+T];\mathbb{R}^{N+\mathrm{l}_0})} \le D_q$ imply

$$|z(t, t_0, z^0, 0, \eta(\cdot))|^2 \le {\rho_{q+2}^\star}^2 - \frac{t - t_0}{T}({\rho_{q+2}^\star}^2 - {\rho_q^\star}^2)$$

$$\text{for all} \quad t \in [t_0, t_0 + T].$$

Define:

$$r_3^\star := \rho_3^\star e^{-\frac{\lambda^\star T}{16}}; \quad \text{and} \quad r_q^\star := r_{q+1}^\star e^{-\frac{\lambda^\star T}{8}},$$

$$\rho_q^\star := \rho_{q+1}^\star e^{-\frac{\lambda^\star T}{8}} \qquad \text{for all } q \le 2, \;\; q \in \mathbb{Z} \qquad (33)$$

The Inductive Step is based on the following Theorem on "adding an integrator". As we noted above, it is a modification of [Dashkovskiy et al. , 2012, Theorem 2] and [Pavlichkov et al. , 2009, Theorem 3.1] and for better understanding we underlined the words with the principal distinction that allows us to provide arbitrarily small gains.

Theorem 3 *Suppose (C1)-(C4) hold true. Then, for each sequence $\{R_q\}_{q=-\infty}^{+\infty}$ such that $\rho_q^\star < R_q < r_{q+1}^\star$ and $R_{q+3} < \gamma(D_q) \le R_{q+4}$ for all $q \in \mathbb{Z}$ and such that $R_4 \le \rho(\lambda_0, \gamma_0)$, there is $q_0 \in \mathbb{N}$, there is $\varepsilon \in]0, \lambda^*[$ such that*

$$4\sqrt{\frac{3(\gamma^\star + \varepsilon)}{\lambda^\star - \varepsilon}} < \frac{1}{8c_1} \quad \text{and} \quad 4\sqrt{\frac{3(\gamma^\star + \varepsilon)}{\lambda^\star - \varepsilon}} < \frac{1}{8} \quad \text{and}$$

$$64(\gamma^\star + \varepsilon)c_1^2 < \frac{(\lambda^\star - \varepsilon)e^{-3(\lambda^\star - \varepsilon)T}}{2} \qquad (34)$$

and there is a feedback $v(\cdot, \cdot)$ of class $C^\mu(\mathbb{R} \times \mathbb{R}^{k+1}; \mathbb{R}^{\mathrm{m}})$ such that $v(t, 0) = 0 \in \mathbb{R}^{\mathrm{m}}$ and $v(t + T, y) = v(t, y)$ for all $[t, y] \in \mathbb{R} \times \mathbb{R}^{k+\mathrm{q}}$ and such that the following conditions hold true:

(I) For every $t \in [0, T]$, every $y = [z, w]^T \in \mathbb{R}^{k+\mathrm{q}}$ and every $\eta \in \mathbb{R}^{N+\mathrm{l}_0}$, the inequalities $|z| \le r_{-q_0+4}^\star$ and $\gamma(|\eta|) \le R_{-q_0+8}$ imply

$$\frac{\partial V(z, w)}{\partial z}(g(t, z, w) + \varphi(t, z, \eta))$$

$$+\frac{\partial V(z, w)}{\partial w}(h(t, z, w, v(t, z, w)) + \phi(t, z, w, \eta)) \le$$

$$-(\lambda^\star - \varepsilon)V(z, w) + (\gamma^\star + \varepsilon)\langle \eta, \eta\rangle$$

(II) For each $q{\ge}{-q_0}{+}3$, $q{\in}\mathbb{Z}$, each $y^0{\in}\mathbb{R}^{k+\mathrm{q}}$, each $t_0{\in}[0, T]$, and each $\eta(\cdot)$ in $L_\infty([t_0, t_0{+}T]; \mathbb{R}^{N+\mathrm{l}_0})$ the inequalities $|y^0|{\le}r_{q+2}^\star$ and $\|\eta(\cdot)\|_{L_\infty([t_0,t_0+T];\mathbb{R}^{N+\mathrm{l}_0})} \le D_q$ imply

$$|y(t, t_0, y^0, v(\cdot, \cdot), \eta(\cdot))|^2 \le R_{q+2}^2 - \frac{t - t_0}{T}(R_{q+2}^2 - R_q^2)$$

$$\textit{for all} \quad t \in [t_0, t_0 + T]$$

Indeed, having designed the controller $v_1(\cdot,\cdot)$ for system (25), we define:

$$k{:=}m_1,\ \ q{:=}m_2,\ \ m{:=}m{+}3,\ \ z{:=}x_1,\ \ w{:=}x_2{-}v_1(t,x_1),$$
$$v{:=}x_3,\quad \text{where}\quad t{\in}\mathbb{R},\ \ x_1{\in}\mathbb{R}^{m_1},\ \ x_2{\in}\mathbb{R}^{m_2},\ \ x_3{\in}\mathbb{R}^{m_3}; \quad (35)$$
$$g(t,z,w) := f_1(t,z,w+v_1(t,z));\quad \varphi(t,z,\eta) := \varphi_1(t,z,\eta);$$
$$h(t,z,w,v) := f_2(t,z,w+v_1(t,z),v) - \frac{\partial v_1(t,z)}{\partial t}$$
$$-\frac{\partial v_1(t,z)}{\partial z} f_1(t,z,w+v_1(t,z));\quad \phi(t,z,w,\eta) :=$$
$$\varphi_2(t,z,w+v_1(t,z),\eta) - \frac{\partial v_1(t,z)}{\partial z}\varphi_1(t,z,\eta); \quad (36)$$
$$\lambda^* := \lambda_0;\ \ \gamma^* := \gamma_0;\quad \text{and}$$
$$\rho^*_q := \rho_q,\ \ r^*_{q+1} := r_{q+1}\quad \text{for all}\ \ q \geq 3,\ q \in \mathbb{Z}. \quad (37)$$

To apply Theorem 3, we take $\{R_q\}_{q=-\infty}^{+\infty}$ such that $\rho^\star_q < R_q < r^\star_{q+1}$ and $R_{q+3} < \gamma(D_q) \leq R_{q+4}$ for all $q \in \mathbb{Z}$ and such that $R_4 \leq \rho(\lambda_0,\gamma_0)$. Then we apply Theorem 3 and find a controller $v(\cdot,\cdot)$, which satisfies (I),(II) with $\varepsilon > 0$ such that (34) holds true. If $\{R_q\}_{q=-\infty}^{+\infty}$ is chosen so that the positive numbers $(R_q - \rho^*_q)$ are small enough, then the $\mathcal{K}_\infty$ function γ_{ind} defined by $\gamma_{\text{ind}}(D) := 0$ if $D = 0$ and

$$\gamma_{\text{ind}}(D) := \left(R^2_{q+2} + \frac{D - D_{q-1}}{D_q - D_{q-1}}(R^2_{q+3} - R^2_{q+2})\right)^{\frac{1}{2}},$$
$$\text{whenever}\ \ D \in]D_{q-1}, D_q)],\ q \in \mathbb{Z}$$

satisfies the condition $\gamma_{\text{ind}}(D) < \gamma(D)$ for all $D > 0$ (this follows from (24), (29) and (30)).

Then, arguing as in the Base Case and replacing $\hat{\gamma}$ with γ_{ind}, we obtain the following estimate for the trajectories of system (31) that is defined by (35), (36):

$$\forall t \geq t_0\ \ |y(t,t_0,y^0,v(\cdot,\cdot),\eta(\cdot))| \leq \max\{\beta_{\text{ind}}(|y^0|, t-t_0),$$
$$\gamma(2 \parallel \xi(\cdot)\|_{L_\infty([t_0,+\infty[;\mathbb{R}^N)}), \gamma(2 \parallel D(\cdot)\|_{L_\infty([t_0,+\infty[;\mathbb{R}^{l_0})})\},$$

for all $t_0 \in \mathbb{R}$, $y^0 \in \mathbb{R}^{m_1+m_2}$, $\eta(\cdot) := [\xi(\cdot), D(\cdot)]^T \in L_\infty(\mathbb{R};\mathbb{R}^{N+l_0})$, where $\beta_{\text{ind}}(\cdot)$ is some $\mathcal{KL}$ - function and $\gamma(\cdot)$ is defined by (15). Arguing similarly by induction over $i = 1,2,3,\ldots,\nu$, we obtain the desired estimate (13) with $\gamma_{\text{sm}}(r) := \gamma(2r)$, whenever $r \geq 0$. This completes the proof of our main Theorem 2.

Remark 2 The proof of Theorem 3 is similar to that of Theorem 2 from Dashkovskiy et al. [2012]. The only difference is the underlined text. In fact, $\{R_q\}$ can be chosen arbitrarily close to $\{\rho_q\}$, which is easily seen from [Dashkovskiy et al. , 2012, p. 161, Fig.1]. For this, $\varrho_q > 0$, $q \geq -q_0$, $q \in \mathbb{Z}$ should be chosen small enough. We omit the proof of Theorem 3 due to space limits.

5. CONCLUSION

We have proved that a generalized triangular form system with unmeasured dynamic uncertainties can be globally uniformly stabilized with respect to external L_∞ disturbances. In contrast to the previous related work Dashkovskiy et al. [2012], we proved that it is possible to build a controller which provides gains as small as we need in order to comply with the small gain condition and to ensure the desired uniform ISS property.

REFERENCES

S. Celikovsky, and H. Nijmeijer. Equivalence of nonlinear systems to triangular form: the singular case. *Syst. Control Lett.*, 27:135–144, 1996.

J.-M. Coron, and L. Praly. Adding an integrator for the stabilization problem. *Syst. Control Lett.*, 17:89–104, 1991.

S. Dashkovskiy, and S. Pavlichkov. Global uniform input-to-state stabilization of large-scale interconnections of MIMO generalized triangular form switched systems. *Math. Control, Signals, Syst.*, 24:135–168, 2012.

H. Edwards, Y.Lin, and Y.Wang. On input-to-state stability for time varying nonlinear systems. *Proc. 39th IEEE Conf. Decision and Control (CDC'00), Sydney*, 3501–3506, 2000.

R.A. Freeman, M. Krstic, and P.V. Kokotovic. Robustness of adaptive nonlinear control to bounded uncertainties. *Automatica*, 34:1227–1230, 1998.

Z.P. Jiang, A.R. Teel, and L. Praly. Small-gain theorem for ISS systems and applications. *Math. Control, Signals, Syst.*, 7:95–120, 1994.

Z.P. Jiang, and I.M.Y. Mareels. A small gain control method for nonlinear cascaded systems with dynamic uncertainties. *IEEE Trans. Automat. Control*, 42:292–308, 1997.

I. Kanellakopoulos, P.V. Kokotovic, and A.S. Morse. Systematic design of adaptive controllers for feedback linearizable systems. *IEEE Trans. Automat. Control*, 38: 1241–1253, 1991.

V.I. Korobov. Controllability and stability of certain nonlinear systems. *Differential Equations*, 9:614–619, 1973.

V.I. Korobov, and S.S. Pavlichkov. Global properties of the triangular systems in the singular case. *J. Math. Anal. Appl.*, 342:1426–1439, 2008.

M. Krstic, I. Kanellakopoulos, and P. Kokotovic. *Nonlinear and adaptive control design* Wiley, New York, 1995.

Y.Lin, Y.Wang, and D.Cheng. On non-uniform and semi-uniform input-to-state stability for time-varying systems. *Proc. 16th IFAC World Congress*, 2005.

T. Liu, Z.P. Jiang, and D.J. Hill. Quantized stabilization of strict-feedback nonlinear systems based on ISS cyclic-small-gain theorem. *Math. Control, Signals, Syst.*, 24: 75–110, 2012.

S.S. Pavlichkov, and S.S. Ge. Global stabilization of the generalized MIMO triangular systems with singular input-output links. *IEEE Trans. Automat. Control*, 54: 1794–1806, 2009.

D. Seto, A.M. Annaswamy, and J. Baillieul. Adaptive control of nonlinear systems with a triangular structure. *IEEE Trans. Automat. Control*, 39:1411–1428, 1994.

E.D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Trans. Automat. Control*, 34:435–443, 1989.

J. Tsinias. Partial-state global stabilization for general triangular systems. *Syst. Contr. Lett.*, 24:139–145, 1995.

J. Tsinias, and I. Karafyllis. ISS property for time-varying systems and application to partial-state feedback stabilization and asymptotic tracking. *IEEE Trans. Automat. Control*, 44:2179–2184, 1999.